Lecture 20: Master Equation: Examples

Last lecture we found the Master Equation for a two-level atom interacting with a bath of harmonic oscillators at temperature T (i.e. a Black-body radiation environment)

$$\Rightarrow \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}\left[\hat{H}_A, \hat{\rho}\right] + \mathcal{L}_{relax}\left[\hat{\rho}\right]$$

$$\mathcal{L}_{relax}[\hat{\rho}] = -\frac{\Gamma}{2}(\bar{n}+1)\left(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_- - 2\hat{\sigma}_-\rho\hat{\sigma}_+\right)$$
$$-\frac{\Gamma}{2}\bar{n}\left(\hat{\sigma}_-\hat{\sigma}_+\hat{\rho} + \hat{\rho}\hat{\sigma}_-\hat{\sigma}_+ - 2\hat{\sigma}_+\hat{\rho}\hat{\sigma}_-\right)$$

where $\bar{n} = \dfrac{1}{e^{\hbar\omega_{eg}/k_BT} - 1}$ = Thermal occupation of Bath

$\hat{H}_A = \hbar\omega_{eg}\,|e\rangle\langle e|$ (Free atomic Hamiltonian)

including any level shifts

<u>Note:</u> Trace preserving

$\frac{d}{dt}\mathrm{Tr}(\hat{\rho}) = 0$ using

cyclic property of trace

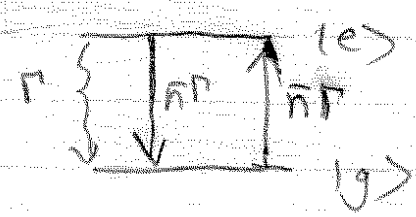
Let us consider matrix elements

With $\hat{\sigma}_+ = |e\rangle\langle g|$ $\quad \hat{\sigma}_- = |g\rangle\langle e|$

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_A, \hat{\rho}] - \frac{\Gamma}{2}(\bar{n}+1)\left(\{|e\rangle\langle e|, \hat{\rho}\} - 2|g\rangle\langle g|\, \rho_{ee}\right)$$
$$- \frac{\Gamma}{2}\bar{n}\left(\{|g\rangle\langle g|, \hat{\rho}\} - 2|e\rangle\langle e|\, \rho_{gg}\right)$$

$$\Rightarrow \frac{d}{dt}\rho_{ee} = \frac{d}{dt}\langle e|\hat{\rho}|e\rangle$$
$$= -\Gamma(\bar{n}+1)\rho_{ee} + \Gamma\bar{n}\rho_{gg}$$

$$\frac{d}{dt}\rho_{gg} = \frac{d}{dt}\langle g|\hat{\rho}|g\rangle$$
$$= -\Gamma\bar{n}\rho_{gg} + \Gamma(\bar{n}+1)\rho_{ee} = -\frac{d}{dt}\rho_{ee}$$



Steady state $\Rightarrow$ Detailed balance

$$\Rightarrow \frac{d}{dt}\rho = 0 \Rightarrow \frac{\rho_{ee}}{\rho_{gg}} = \frac{\bar{n}}{\bar{n}+1} = \frac{\frac{1}{e^{\hbar\omega/kT}-1}}{\frac{1}{e^{\hbar\omega/kT}-1}+1}$$

Boltzmann! $\quad = e^{-\hbar\omega_{eg}/kT}$

Thus, in steady state the atom come to equilibrium with the bath, as expected

In fact, Einstein derived the spontaneous emission rate to get thermal equilibrium (see Einstein A/B coefficients)

Decay of coherences (in the absence of coherent driving)

$$\frac{d}{dt}\rho_{eg} = \frac{d}{dt}\langle e|\hat{\rho}|g\rangle$$

$$= -i\omega_{eg}\,\rho_{eg} - \frac{\Gamma}{2}(2\bar{n}+1)\;\rho_{eg}$$

Decay of coherences $\gamma = \dfrac{\Gamma_e + \Gamma_g}{2}$
(in the absence of collisions)

Note: Without coherent drives
Separate equations of coherences and populations

<u>Another example: Damped SHO</u>

Given oscillator @ freq ω_0 coupled to a bath of <u>thermal</u> oscillators

E.g. mode in leaky cavity

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}(\bar{n}+1)\left[\{\hat{a}^{\dagger}\hat{a},\hat{\rho}\} - 2\hat{a}\hat{\rho}\hat{a}^{\dagger}\right]$$

$$-\frac{\Gamma}{2}\bar{n}\left[\{\hat{a}\hat{a}^{\dagger},\hat{\rho}\} - 2\hat{a}^{\dagger}\hat{\rho}\hat{a}\right]$$

Derived in same Born-Markov approx with

$$\hat{H} = \hbar\omega_0\hat{a}^{\dagger}\hat{a} + \sum_k \hbar\omega_k \hat{a}_k^{\dagger}\hat{a}$$

$$+ \sum_k \left(g_k \hat{b}_k^{\dagger}\hat{a} + g_k^{*}\hat{b}_k\hat{a}^{\dagger}\right)$$

linear coupling

In condensed matter literature

known as "Caldera-Legget" model)

Consider evolution of mean excitation of oscillator $\langle \hat{n} \rangle = \langle \hat{a}^\dagger \hat{a} \rangle$

$$\frac{d}{dt}\langle \hat{n} \rangle = \mathrm{Tr}\left(\hat{n}\,\frac{d\rho}{dt}\right)$$

$$= -\frac{\Gamma}{2}(\bar{n}+1)\left[\mathrm{Tr}\left(\hat{n}^2\hat{\rho} + \hat{n}\hat{\rho}\hat{n}\right) - 2\mathrm{Tr}\left(\hat{n}\hat{a}\hat{\rho}\hat{a}^\dagger\right)\right.$$

$$-\frac{\Gamma}{2}\bar{n}\left[\mathrm{Tr}\left(\hat{n}^2\hat{\rho} + \hat{n}\hat{\rho}\hat{n}\right) + 2\mathrm{Tr}\left(\hat{n}\hat{\rho}\right)\right.$$

$$\left. - 2\mathrm{Tr}\left(\hat{n}a^\dagger\hat{\rho}\hat{a}\right)\right]$$

$$= -\frac{\Gamma}{2}(2\bar{n}+1)\langle n^2\rangle - \Gamma\bar{n}\langle n\rangle$$

$$+\Gamma(\bar{n}+1)\langle \hat{a}^\dagger\hat{n}\hat{a}\rangle + \Gamma\bar{n}\langle \hat{a}\hat{n}a^\dagger\rangle$$

Aside:

$$\hat{a}^\dagger\hat{n}\hat{a} = \hat{a}^\dagger\hat{a}\hat{n} + a^\dagger[\hat{n},\hat{a}]$$

$$= \hat{n}^2 - \hat{n}$$

$$\hat{a}\hat{n}\hat{a}^\dagger = \hat{n}\hat{a}\hat{a}^\dagger + [\hat{a},\hat{n}]\hat{a}^\dagger$$

$$= \hat{n}^2 + \hat{n} + \hat{a}\hat{a}^\dagger$$

$$= \hat{n}^2 + 2\hat{n} + \underline{1}$$

$$\Rightarrow \boxed{\frac{d}{dt}\langle \hat{n}\rangle = -\Gamma\langle \hat{n}\rangle + \Gamma\bar{n}}$$

Steady state: $\frac{d}{dt} P_n = 0$

$$\Rightarrow \left[n(\bar{n}+1)+\bar{n}(n+1)\right] P_n = (n+1)(\bar{n}+1) P_{n+1} + n\bar{n} P_{n-1}$$

$$n=0 \Rightarrow \quad \bar{n} P_0 = (\bar{n}+1) P_1$$

$$\Rightarrow \frac{P_1}{P_0} = e^{-\hbar\omega_0/k_B T} \quad \checkmark$$

Boltzmann

$$n=1 \Rightarrow (3\bar{n}+1) P_1 = 2(\bar{n}+1) P_2 + \underbrace{\bar{n} P_0}_{(\bar{n}+1) P_1}$$

$$\Rightarrow 2\bar{n} P_1 = (\bar{n}+1) P_2$$

$$\Rightarrow \frac{P_2}{P_1} = \frac{\bar{n}}{n+1} = e^{-\frac{\hbar\omega_0}{k_B T}} \quad \checkmark$$

Coherences

$$\frac{d}{dt}\langle a \rangle = \langle \hat{a} \frac{d\hat{\rho}}{dt} \rangle$$

$$= \underbrace{-\frac{i}{\hbar}\langle [\hat{a},\hat{H}]\hat{\rho} \rangle}_{-i\omega_0 \hat{a}} + \mathrm{Tr}\left(\mathcal{L}[\hat{\rho}]\,\hat{a}\right)$$

$$\frac{d}{dt}\langle\hat{n}\rangle = -\Gamma\left(\langle\hat{n}\rangle - \bar{n}\right)$$

$$\boxed{\langle\hat{n}\rangle(0) = \langle\hat{n}\rangle(0)\, e^{-\Gamma t} + \bar{n}\left(1 - e^{-\Gamma t}\right)}$$

Steady state

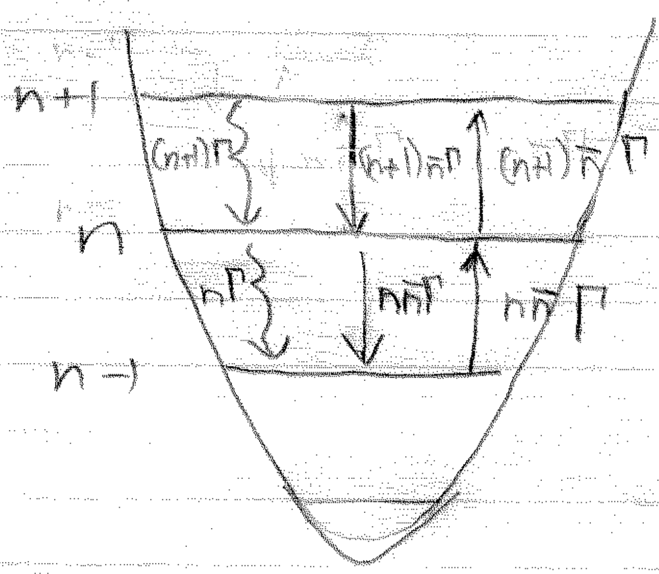
$\langle\hat{n}\rangle = \bar{n}$ Thermal equilibrium

What about probability distribution? $P_n = \langle n|\hat{\rho}|n\rangle$

$$\frac{d}{dt}P_n = \frac{d}{dt}\langle n|\hat{\rho}|n\rangle$$

$$= -n\Gamma(\bar{n}+1)\,P_n + (n+1)\Gamma(\bar{n}+1)\,P_{n+1}$$

$$-(n+1)\Gamma\bar{n}\,P_n + \Gamma n\bar{n}\,P_{n-1}$$



After tedious algebra

$$\mathrm{Tr}\left(\mathcal{L}[\hat{\rho}]\,\hat{a}\right) = -\frac{\Gamma}{2}\langle \hat{a}\rangle$$

$$\Rightarrow \frac{d}{dt}\langle \hat{a}\rangle = \left(-i\omega_0 - \frac{\Gamma}{2}\right)\langle \hat{a}\rangle$$

$$\langle \hat{a}\rangle = \langle \hat{a}\rangle(0)\; e^{-i\omega_0 t}\; e^{-\Gamma t/2}$$

Decay amplitude

Note: Decay rate Γ independent of $\langle \hat{a}\rangle$

How is this possible? the levels decay $\propto$ to occupation # n?

Look at coherences of density operator

$$\frac{d}{dt}\rho_{n+1,n} = \frac{d}{dt}\langle n+1|\hat{\rho}|n\rangle$$

$$= \left(-i\omega_0 - \left[(n+\tfrac{1}{2}) + 2(n+1)\bar{n}\right]\Gamma\right)\rho_{n+1,n}$$

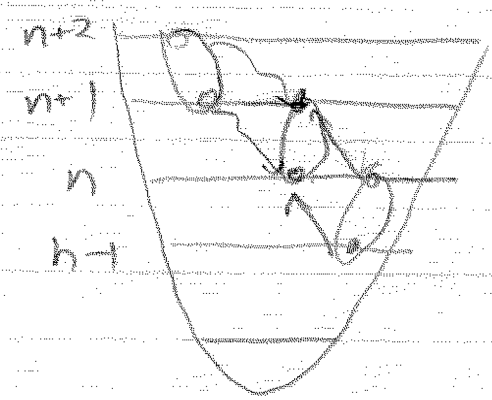
$$+ \sqrt{(n+1)(n+2)}\,(\bar{n}+1)\,\Gamma\,\rho_{n+2,n+1}$$

$$+ \sqrt{n(n+1)}\;\bar{n}\;\rho_{n,n-1}$$

Note: Coherence **decay** at rate depending on n

BUT there are also **feeding** terms

Transfer of coherence



Coherent superposition of $|n+2\rangle$ and $|n+1\rangle$ **transferred** to superposition of $|n+1\rangle$ and $|n\rangle$

This is only possible because the two decay paths are **indistinguishable**

This is true only for **harmonic ladder**

Where the spacing between levels is equal.

Linblad Form

The master equation we derived in of a generic type know as "Linblad form". Consider the zero temperature case

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \mathcal{L}_{LB}[\hat{\rho}]$$

where $\mathcal{L}_{LB}[\hat{\rho}] = \sum_{\mu}\left[-\frac{1}{2}\left(\hat{L}^{\dagger}_{\mu}\hat{L}_{\mu}\hat{\rho} + \hat{\rho}\hat{L}^{\dagger}_{\mu}\hat{L}_{\mu}\right) + \hat{L}_{\mu}\hat{\rho}\hat{L}^{\dagger}_{\mu}\right]$

The set of operators $\{\hat{L}_{\mu}\}$ are known as the "Linbled opertors" or "Jump operators". The latter rotation will become clearer as we study the relationship between the master equation and "quantum trajectories".

For the two cases we looked at, the damped 2-level atom and the damped SHO, the system had one degree of freedom and thus one Linblad op.

Damped 2-lev atom: $\hat{L} = \sqrt{\Gamma}\,\hat{\sigma}_{-}$

Damped SHO: $\hat{L} = \sqrt{\Gamma}\,\hat{a}$

For system with multiple degrees of freedom, there can be multiple Linblad operators.

To get a basic understanding of the Lindblad operators, consider that the relaxation terms in the Master equation donot Couple populations (diagonal matrix elements) and coherences (off-diagonal matrix elements). Thus we can look at rate equations for the populations, ignoring the off-diagonal elements of the density operator.

Consider the basis of energy eigenstates $\{|g\rangle\}$

Let $\hat{\rho} = \sum_g P_g |g\rangle\langle g|$ (ignoring off-diagonal)

$$\Rightarrow \dot{P}_g = \langle g| \frac{d\hat{\rho}}{dt} |g\rangle$$

$$= -\gamma_g P_g + \sum_{g'} \gamma_{g \leftarrow g'} P_{g'}$$

where $\gamma_{g \leftarrow g'} = \sum_\mu |\langle g| \hat{L}_\mu |g'\rangle|^2$

$$\text{and } \gamma_g = \sum_\mu |\langle g| \hat{L}_\mu^\dagger \hat{L}_\mu |g\rangle|^2$$

$$= \sum_{g'} \left(\sum_\mu |\langle g'| \hat{L}_\mu |g\rangle|^2 \right)$$

$$= \sum_{g'} \gamma_{g' \leftarrow g}$$

Thus the terms $-\frac{1}{2}\sum_\mu (\hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho} + \hat{\rho}\hat{L}_\mu^\dagger \hat{L}_\mu)$

lead to decay out of level j, γ_j

whereas the terms $\sum_\mu \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$ lead

to feeding of population into level j

from all other levels j', $\gamma_{j \leftarrow j'}$.

these two different terms will be seen to have different physical signifance in terms of "quantum trajectories" to be studied at the end of the class.