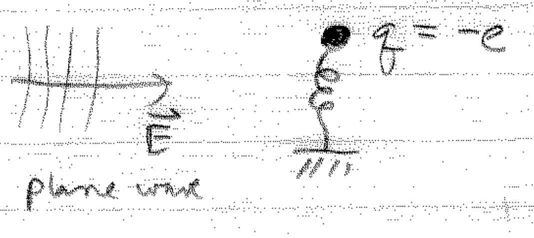
Physics 566 - Quantum Optics

Problem Set #5 Solutions

Problem 1: Lorentz Classical Model



charge on spring

Eq of motion $m\ddot{\vec{x}} = -m\Gamma\dot{\vec{x}} - m\omega_0^2\vec{x} - e\vec{E}(t)$

Rate at which field does work

$$\frac{dW}{dt} = \dot{\vec{x}}\cdot\vec{F} = -e\dot{\vec{x}}\cdot\vec{E}$$

Need to solve for $\dot{\vec{x}}$ in steady state.

Use complex representation: $\vec{E} = \vec{E}_0 e^{-i\omega t}$

$\Rightarrow$ steady state $\vec{x} = \vec{x}_0 e^{-i\omega t}$

$$\ddot{\vec{x}} + \Gamma\dot{\vec{x}} + \omega_0^2\vec{x} = -\frac{e}{m}\vec{E}_0 e^{-i\omega t}$$

$$\Rightarrow \left(-\omega^2 + \omega_0^2 - i\omega\Gamma\right)\vec{x}_0 = -\frac{e}{m}\vec{E}_0$$

$$\Rightarrow \vec{x}_0 = \left(\frac{-e/m}{-\omega^2 + \omega_0^2 - i\omega\Gamma}\right)\vec{E}_0$$

Go to near resonance limit $\omega - \omega_0 \equiv \Delta$

$$\Rightarrow \omega_0 = \omega - \Delta \qquad \omega_0^2 - \omega^2 = (\omega-\Delta)^2 - \omega^2$$

$$\approx -2\omega\Delta \text{ to } \mathcal{O}(\Delta)$$

$$\Rightarrow \vec{x}_0 \approx \left(\frac{e/2m\omega}{\Delta + i\frac{\Gamma}{2}} \right) \vec{E}_0$$

$\therefore$ Time averaged over a period

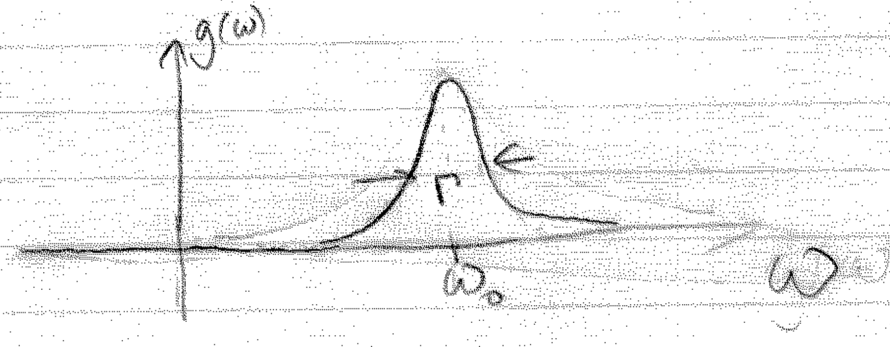
$$\frac{dW}{dt} = \frac{-e}{2} \mathrm{Re}\left(-i\omega \vec{x}_0 \cdot \vec{E}_0\right)$$

$$= \frac{-e\omega}{2} \mathrm{Im}\left(\vec{x}_0 \cdot \vec{E}_0\right)$$

$$= -\frac{e\omega}{2} \left(\frac{e}{2m\omega}\right) \frac{-\frac{\Gamma}{2}}{\Delta^2 + \frac{\Gamma^2}{4}} |\vec{E}_0|$$

$$= \frac{\pi e^2}{4m} \underbrace{\left(\frac{\frac{\Gamma}{2\pi}}{\Delta^2 + \frac{\Gamma^2}{4}} \right)}_{g(\omega)} |\vec{E}_0|^2$$

$g(\omega)$ ← atomic line-shape



(b) Absorption cross-section

$$P_{abs} = \sigma_{abs} I_{inc} = \sigma_{abs} \frac{c}{8\pi} |\vec{E}_0|^2$$

$$\| $$

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2}{4m} g(\omega) |\vec{E}_0|$$

$$\Rightarrow \boxed{\sigma_{abs} = \frac{2\pi^2 e^2}{mc} g(\omega)}$$

Evaluate for $\frac{\Gamma}{2\pi} = 10 \text{ MHz}$ $\quad \lambda = 589 \text{ nm}$

on resonance $\Delta = 0$ $\quad g(\omega = \omega_0) = \frac{2}{\pi} \frac{1}{\Gamma}$

$$\Rightarrow \sigma_{abs} = 2\left(\frac{2\pi}{\Gamma}\right) \frac{e^2}{mc} = 2\left(\frac{2\pi}{\Gamma}\right) c \left(\frac{e^2}{mc^2}\right)$$

classical electron radius $= r_e$

$$= 2\left(10^{-7} s^{+1}\right)\left(3 \times 10^{10} \frac{cm}{s}\right)\left(2.8 \times 10^{-13} cm\right)$$

$$\boxed{\sigma_{abs} = 1.68 \times 10^{-9} \text{ cm}^2}$$

Note: On resonance $\sigma_{abs} \sim \lambda_0^2 = 3.5 \times 10^{-9} \text{ cm}^2$

(c) $\sigma_{quantum} = 4\pi^2 \frac{e^2}{\hbar c} |\langle e|\vec{x}|g\rangle|^2 \omega\, g(\omega)$

Oscillator strength $f = \left.\frac{\sigma_{quantum}}{\sigma_{classical}}\right|_{resonance}$

$$\Rightarrow \boxed{f = \frac{2m\omega_0}{\hbar} |\langle e|\vec{x}|g\rangle|^2}$$

(d) The driven oscillator will radiate electromagnetically

$$P_{radiated} = \frac{ck^4}{3} |\vec{d}_0|^2 \quad \text{Larmor formula}$$

$$= \frac{ck^4 e^2}{3} |\vec{x}_0|^2$$

$$P_{abs} = \frac{e\omega}{2} \operatorname{Im}(\vec{x}_0 \cdot \vec{E}_0)$$

Now $\vec{x}_0 = \alpha \vec{E}_0$ where $\alpha = \dfrac{e/2m\omega}{\Delta + i\frac{\Gamma}{2}}$

$$\Rightarrow \frac{ck^4 e^2}{3} |\vec{E}_0|^2 \underbrace{|\alpha|^2}_{\downarrow\ \frac{\frac{e^2}{4m^2\omega^2}}{\Delta^2 + \frac{\Gamma^2}{4}}} = \frac{e\omega}{2} |\vec{E}_0|^2 \underbrace{\operatorname{Im}(\alpha)}_{\downarrow\ \frac{-i\frac{\Gamma}{2}\, e/2m\omega}{\Delta^2 + \frac{\Gamma^2}{4}}}$$

$$\therefore \boxed{\Gamma_{class} = \frac{2}{3}\frac{ck^4}{\omega^2}\frac{e^2}{m} = \frac{2}{3}\frac{e^2}{mc^3}\omega^2}$$

(e) We have quantum mechanically,

$$\Gamma_{quantum} = \frac{4}{3}\frac{k^3}{\hbar}|\langle e|\vec{d}_0|g\rangle|^2$$

$$= \frac{4}{3}\frac{e^2}{\hbar}|\langle e|\vec{x}_0|g\rangle|^2\frac{\omega^3}{c^3}$$

$$\rightarrow \frac{\Gamma_{quant}}{\Gamma_{class}} = \frac{2m\omega}{\hbar}|\langle e|\vec{x}_0|g\rangle|^2 = f \checkmark$$

the oscillator strength

Problem 2: Radiation Reaction + decay

Two-level atom + Vacuum (RWA)

$$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\hat{\sigma}_z + \sum_{\vec{k},\lambda}\hbar\omega_k \hat{a}^\dagger_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda}$$
$$- \sum_{\vec{k},\lambda}\hbar\left(g_{\vec{k}\lambda}\hat{\sigma}_+\hat{a}_{\vec{k}\lambda} + g^*_{\vec{k}\lambda}\hat{\sigma}_-\hat{a}^\dagger_{\vec{k}\lambda}\right)$$

where $g_{\vec{k},\lambda} = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\vec{\epsilon}_{\vec{k}\lambda}\cdot\vec{d}_{eg}$ (Vacuum Rabi freq)

(a) Heisenberg equations of motion

$$\frac{d}{dt}\hat{a}_{\vec{k},\lambda} = \frac{-i}{\hbar}[\hat{a}_{\vec{k},\lambda}, \hat{H}]$$

Notes: $[\hat{a}_{\vec{k}\lambda}, \hat{a}^\dagger_{\vec{k}'\lambda'}] = \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}$

all other operators commute with $\hat{a}_{\vec{k},\lambda}$

$$\Rightarrow \boxed{\frac{d}{dt}\hat{a}_{\vec{k},\lambda} = -i\omega_k\hat{a}_{\vec{k},\lambda} + i g^*_{\vec{k},\lambda}\hat{\sigma}_-}$$

$$\frac{d}{dt}\hat{\sigma}_- = -\frac{i}{\hbar}[\hat{\sigma}_-, \hat{H}], \quad \frac{d}{dt}\hat{\sigma}_z = -\frac{i}{\hbar}[\hat{\sigma}_z, \hat{H}]$$

Notes: $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$ $\qquad [\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$

(Next Page)

$$\Rightarrow \boxed{\begin{aligned} \frac{d}{dt}\hat{\sigma}_- &= -i\omega_{eg}\hat{\sigma}_- - i\sum_{\vec{k},\lambda} g_{\vec{k},\lambda}\,\hat{\sigma}_z \\ \frac{d}{dt}\hat{\sigma}_z &= 2i\sum_{\vec{k},\lambda}\left(g_{\vec{k}\lambda}\hat{\sigma}_+ a_{\vec{k}\lambda} - g_{\vec{k}\lambda}\hat{\sigma}_-\hat{a}^\dagger_{\vec{k}\lambda}\right) \end{aligned}}$$

(b) Solve $\underbrace{\frac{d}{dt}\hat{a}_{\vec{k}\lambda} + i\omega_k a_{\vec{k}\lambda}}_{\text{Homogeneous eq}} = \underbrace{ig^*_{\vec{k}\lambda}\hat{\sigma}_-}_{\text{Source}}$

Use Green's function for homogeneous eq

$$\Rightarrow \hat{a}_{\vec{k}\lambda}(t) = \underbrace{\hat{a}_{\vec{k}\lambda}(0)e^{-i\omega_k t}}_{\text{vacuum}} + \underbrace{ig^*_{\vec{k},\lambda}\int_0^t \hat{\sigma}_-(t')\,e^{-i\omega_k(t-t')}dt'}_{\text{radiation from source}}$$

(c) Equal-time commutator

$$[\hat{a}_{\vec{k},\lambda}(t), \hat{a}^\dagger_{\vec{k}',\lambda'}(t)] = [\hat{U}^\dagger(t)\hat{a}_{\vec{k}\lambda}(0)\hat{U}(t), \hat{U}^\dagger(t)\hat{a}^\dagger_{\vec{k},\lambda}(0)\hat{U}(t)]$$

$$= U^\dagger(t)\hat{a}_{\vec{k}\lambda}(0)\overbrace{\hat{U}(t)\hat{U}^\dagger(t)}^{=1}a^\dagger_{\vec{k}\lambda}(0)\hat{U}(t)$$

$$- U^\dagger(t)\hat{a}^\dagger_{\vec{k},\lambda}(0)\underbrace{\hat{U}(t)\hat{U}^\dagger(t)}_{=1}\hat{a}_{\vec{k}\lambda}(0)\hat{U}(t)$$

$$\Rightarrow [\hat{a}_{\vec{k}\lambda}(t), \hat{a}^\dagger_{\vec{k}'\lambda'}(t)] = \hat{U}^\dagger(t) \underbrace{[\hat{a}_{\vec{k}\lambda}(0), \hat{a}^\dagger_{\vec{k}'\lambda'}(0)]}_{\delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}} \hat{U}(t)$$

$$\Rightarrow \boxed{[\hat{a}_{\vec{k}\lambda}(t), \hat{a}^\dagger_{\vec{k}'\lambda'}(t)] = \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'} = [\hat{a}_{\vec{k}\lambda}(0), \hat{a}^\dagger_{\vec{k}'\lambda'}(0)]}$$

$\Rightarrow$ Equal-time commutators preserved by time evolution, but non-equal time are not.

Note: $$[\hat{a}_{\substack{source\\ \vec{k},\lambda}}(t), \hat{a}^\dagger_{\substack{source\\ \vec{k}',\lambda'}}(t)] = g^*_{\vec{k}\lambda} g_{\vec{k}'\lambda'} \int_0^t dt' \int_0^t dt'' \overbrace{[\hat{\sigma}_-(t'), \hat{\sigma}_-(t'')]}^{\neq 0} e^{-i\omega_k(t'-t'')}$$

$$\neq \delta_{\vec{k}\vec{k}'}\delta_{\lambda\lambda'}$$

(d) Plug solution for $\hat{a}_{\vec{k}\lambda}(t)$ into $\frac{d\hat{\sigma}_z}{dt} \Rightarrow$ Integro-differential equation

$$\Rightarrow \frac{d}{dt}\hat{\sigma}_z(t) = 2\left[\sum_{\vec{k},\lambda}\left(i g_{\vec{k}\lambda} \hat{\sigma}_+(t)\hat{a}_{\vec{k}\lambda}(0) e^{-i\omega_k t} + h.c.\right)\right.$$
$$\left. - \sum_{\vec{k},\lambda}\left(|g_{\vec{k}\lambda}|^2 \int_0^t dt' \hat{\sigma}_+(t)\hat{\sigma}_-(t') e^{-i\omega_k(t-t')} + h.c.\right)\right]$$

Now take the expectation value in the state $|\Psi\rangle = |\psi_{atom}\rangle \otimes |Vac\rangle$

(constant in Heisenberg picture)

Note $\hat{a}_{\vec{k},\lambda}(0)|vac\rangle = 0$ $\quad\quad \langle vac|\hat{a}^\dagger_{k,\lambda} = 0$

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{\sigma}_z(t)\rangle = -2\sum_{\vec{k},\lambda}|g_{\vec{k}\lambda}|^2\int_0^t dt\left(\langle\hat{\sigma}_+(t)\,\hat{\sigma}_-(t')\rangle e^{-i\omega_k(t-t')} + c.c.\right)}$$

(e) Under Markov approx $\quad \hat{\sigma}_-(t) = \hat{\Sigma}_-(t)\,e^{-i\omega_{eg}t}$

$$\Downarrow$$

$\hat{\Sigma}_-$: Slowly varying envelope

$$\langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t')\rangle \approx \langle\hat{\Sigma}_+(t)\,\hat{\Sigma}_-(t)\rangle = \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle$$

$$\therefore \langle\hat{\sigma}_+(t)\,\hat{\sigma}_-(t')\rangle = \langle\hat{\Sigma}_+(t)\,\hat{\Sigma}_-(t')\rangle\, e^{i\omega_{eg}(t-t')}$$

$$\approx \langle\hat{\Sigma}_+(t)\,\hat{\Sigma}_-(t)\rangle e^{+i\omega_{eg}(t-t')} = \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle e^{+i\omega_{eg}(t-t')}$$

$$\therefore \frac{d}{dt}\langle\hat{\sigma}_z(t)\rangle \approx -2\sum_{\vec{k},\lambda}|g_{\vec{k}\lambda}|^2 \underbrace{\int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')}\,dt'}_{= f(\omega_{eg}-\omega_k)} + c.c.$$

$\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\rangle$

$$f(\omega_{eg}-\omega_k) \longrightarrow \pi\,\delta(\omega_{eg}-\omega_k) - i\,P\left[\frac{1}{\omega_{eg}-\omega_k}\right]$$

in usual Markov approx $\quad t \to \infty$

$$\Rightarrow \frac{d}{dt}\langle\hat{\sigma}_z\rangle = -2\underbrace{\sum_{\vec{k}\lambda}|g_{\vec{k}\lambda}|^2\, 2\pi\delta(\omega_{\vec{k}}-\omega_{eg})}_{=\Gamma \text{ (spontaneous emission rate)}}\langle\hat{\sigma}_+\hat{\sigma}_-\rangle$$

Note: $\hat{\sigma}_+(t)\,\hat{\sigma}_-(t) = \dfrac{1+\hat{\sigma}_z(t)}{2}$ (projector)

$$\Rightarrow \boxed{\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -\frac{\Gamma}{2}\left(1+\langle\hat{\sigma}_z\rangle\right)}$$

Moral of the story: One can see spontaneous emission as arising from radiation reaction, i.e. the oscillator decays because energy is carried away by the field. The vacuum part of $\hat{a}(t)$ played no role. It's role is to induce the initial dipole with a random phase, that then radiates.