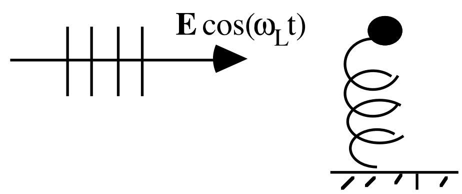
Physics 566: Quantum Optics
Problem Set #5
Due Monday, Oct. 6, 2008

Problem 1: Lorentz Classical Model of Absorption and Emission (15 points)
Suppose we were to model an atom as an electron on a spring - i.e. a damped simple harmonic oscillator of mass m, with resonance frequency ω_0, and damping constant Γ. Consider driving the oscillator with a monochromatic plane wave, of frequency ω_L.



(a) Show that rate at which the dipole absorbs energy from the field, given by the rate at which the field does work on the charge averaged over one period, is

$$\frac{dW_{abs}}{dt} = \frac{\pi e^2 |\mathbf{E}|^2}{4m} g(\omega_L), \text{ where } g(\omega) = \frac{\Gamma/(2\pi)}{(\omega-\omega_{eg})^2 + \Gamma^2/4} \text{ is the line shape.}$$

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) The absorption cross section, σ_{abs}, is defined as the rate at which energy is absorbed by an atom, divided by the flux, Φ, of photons incident on the atom, $\Phi \equiv I/\hbar\omega_L$ (i.e. the rate of photons incident on the atom per unit area), where $I = \frac{c}{8\pi}|\mathbf{E}_0|^2$ is the incident intensity (CGS units). Show that the classical model of absorption gives,

$$\sigma_{classical} = \frac{2\pi^2 e^2}{mc} g(\omega_L),$$

Evaluate this on-resonance, for a the parameters associated with Na, where the excitation wavelength is 589 nm and the linewidth (Full width at half-maximum) is 10 MHz.

The ratio of the integrated cross section an atomic transition and that given by the classical model to the quantum mechanical expression with equal resonance frequency and line width is known as the *oscillator* strength of the transition.

(c) From standard texts we have $\sigma_{quantum} = 4\pi^2 \frac{e^2}{\hbar c}\left|\langle e|\mathbf{x}|g\rangle\right|^2 \omega_L\, g(\omega_L)$, where $\langle e|\mathbf{x}|g\rangle$ is the matrix element of the electron position relative to the nucleus for the resonant transition. **Show** that on resonance,

$$f = \frac{\sigma_{quantum}}{\sigma_{classical}} = \frac{2m\omega_0}{\hbar}\left|\langle e|x|g\rangle\right|^2 .$$

Note that $\hbar / 2m\omega_0$ is the square of the characteristic length scale of a quantum simple harmonic oscillator . Thus, the oscillator strength measures the ratio of dipole oscillation amplitude for a two level atom as compared to a simple harmonic oscillator.

Let us now assume that our spring has no *intrinsic* damping associated with it. Consider the scattering of an electromagnetic wave by this oscillating charge. As the charge radiates, the electromagnetic field will carry away energy. This energy must come from the kinetic energy of the accelerating charge. Thus the very act of radiating should "damp" the motion of the charge. This is known as radiation reaction, and will determine a classical decay rate Γ_{class} for the oscillator. In steady state the power radiated by the charge (given by the classical Larmor formula) is equal to the power absorbed.

(d) Assume that the oscillator is damped as Γ_{class}, and show that

$P_{abs} = P_{radiated} \Rightarrow \Gamma_{class} = \frac{2}{3}\frac{e^2}{mc^3}\omega^2 = \frac{2}{3}\, k r_c\, \omega$,where r_c is the classical electron radius.

(e) Show that the quantum mechanical decay rate is related to the classical formula by

$\Gamma_{quantum} = f\,\Gamma_{class}$, where f is the oscillator strength.

Problem 2: Radiation reaction and decay of the quantum oscillator (15 points)
Radiation reaction can be shown to be lead to the decay of the quantum mechanical oscillator as well if we work in the *Heisenberg picture*. Start with the total Hamitonian for a two level atom interacting with the quantized field, as discussed in class,

$$\hat{H} = \frac{1}{2}\hbar\omega_{eg}\,\hat{\sigma}_z + \sum_{\mathbf{k}\lambda}\hbar\omega_k \hat{a}^\dagger_{\mathbf{k}\lambda}\hat{a}_{\mathbf{k}\lambda} - \sum_{\mathbf{k}\lambda}\hbar\left(g_{\mathbf{k}\lambda}\,\hat{\sigma}_+\hat{a}_{k\lambda} + g^*_{\mathbf{k}\lambda}\,\hat{a}^\dagger_{k\lambda}\hat{\sigma}_-\right),$$

$$\text{where } g_{\mathbf{k}\lambda} = i\sqrt{\frac{2\pi\hbar\omega_k}{V}}\,\vec{\varepsilon}_{\mathbf{k}\lambda}\cdot\mathbf{d}_{eg} .$$

Note: We have expressed the interaction operator in "normally ordered" form, so that annihilation operators are to the right and creation operators are to the left. We have complete freedom to do this since field and atomic operators commute *at equal times*.

(a) Show that the Heisenberg equations of motion are:

$$\frac{d}{dt}\hat{a}_{k\lambda} = -i\omega_k \hat{a}_{k\lambda} + ig^*_{\mathbf{k}\lambda}\hat{\sigma}_-$$

$$\frac{d}{dt}\hat{\sigma}_- = -i\omega_{eg}\hat{\sigma}_- - i\sum_{\mathbf{k},\lambda} g_{\mathbf{k},\lambda}\hat{\sigma}_z \hat{a}_{\mathbf{k}\lambda}$$

$$\frac{d}{dt}\hat{\sigma}_z = 2i\sum_{k,\lambda}\left(g_{k\lambda}\hat{\sigma}_+\hat{a}_{k\lambda} - g^*_{k\lambda}\hat{a}^\dagger_{k\lambda}\hat{\sigma}_-\right)$$

(b) The first equation is linear in the operators, and so can be formally integrated. Show that

$$\hat{a}_{k\lambda}(t) = \underbrace{\hat{a}_{k\lambda}(0)e^{-i\omega_k t}}_{\hat{a}^{vac}_{k\lambda}(t)} + \underbrace{ig^*_{\mathbf{k}\lambda}\int_0^t dt'\,\hat{\sigma}_-(t')e^{-i\omega_k(t-t')}}_{\hat{a}^{source}_{k\lambda}(t)}$$

is a solution. The first term $\hat{a}^{vac}_{k\lambda}(t)$ is known as the vacuum field operator and $\hat{a}^{source}_{k\lambda}(t)$ is known as the source component due to dipole radiation by the atom.

(c) Show that, in general, given $[\hat{a}_{k\lambda}(t), \hat{a}^\dagger_{k'\lambda'}(t)] = \delta_{\lambda\lambda'}\delta_{kk'}$, unitary evolution implies $\left[\hat{a}_{k\lambda}(0), \hat{a}^\dagger_{k\lambda}(0)\right] = \delta_{\lambda\lambda'}\delta_{kk'}$. Show that the source part alone does not satisfy these relations.

(d) Plug the solution (b) back into the equation for $\hat{\sigma}_z$. Take the Heisenberg state of the system (initial state in the Schrodinger picture) to be $|\Psi\rangle = |\psi\rangle_{atom} \otimes |0\rangle_{field}$, i.e., arbitrary state of the atom plus field in the vacuum. Shown that expectation value satisfies

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -2\sum_{\mathbf{k}\lambda}|g_{\mathbf{k}\lambda}|^2\int_0^t \langle\hat{\sigma}_+(t)\hat{\sigma}_-(t')\rangle e^{-i\omega_k(t-t')} + c.c$$

(e) Now let us make the Markov approximation. Assume that $\hat{\sigma}_-(t) = \hat{\Sigma}_-(t)e^{-i\omega_{eg}t}$, where $\hat{\Sigma}_-(t)$ is a slowly varying operator on the scale of ω_{eg}. Under this assumption $\left\langle\hat{\Sigma}_+(t)\,\hat{\Sigma}_-(t')\right\rangle \approx \left\langle\hat{\Sigma}_+(t)\hat{\Sigma}_-(t)\right\rangle = \left\langle\hat{\sigma}_+(t)\hat{\sigma}_-(t)\right\rangle$. Use this approximation to show,

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -\Gamma\left(1+\langle\hat{\sigma}_z\rangle\right),$$

where $\Gamma = \sum_{\mathbf{k}\lambda} 2\pi |g_{\mathbf{k}\lambda}|^2 \delta(\omega_k - \omega_{eg})$ is the Einstein A coefficient!

This is the expected decay! Note that vacuum fluctuations play no role in determining the *rate* of emission. They just initiate the process if the initial dipole moment is zero.