Physics 566: Quantum Optics
Problem Set #6
Due Monday Oct. 20, 2008

Problem 1: Boson Algebra (10 Points)
This problem is to give you some practice manipulating the boson algebra. A great source is the classic "Quantum Statistical Properties of Radiation", by W. H. Louisell, reprinted by "Wiley Classics Library", ISBN 0-471-52365-8.

(a) Gaussian integrals in phase-space are used all the time. Show that

$$\int \frac{d^2\beta}{\pi}\, e^{-A|\beta|^2} e^{\alpha\beta^* - \beta\alpha^*} = \frac{1}{A}\, e^{-|\alpha|^2/A}.$$

(b) Prove the completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1}$$ (Hint: Expand in number states).

(c) The "quadrature" operators in optics are the analogs of Q and P, $\hat{a} = \hat{X}_1 + i\hat{X}_2$. Show

$$\begin{aligned} \hat{U}^\dagger(\theta)\hat{X}_1\hat{U}(\theta) &= \cos\theta\, \hat{X}_1 + \sin\theta\, \hat{X}_2 \\ \hat{U}^\dagger(\theta)\hat{X}_2\hat{U}(\theta) &= \cos\theta\, \hat{X}_2 - \sin\theta\, \hat{X}_1 \end{aligned}, \quad \text{where} \quad \hat{U}(\theta) = e^{-i\theta\hat{a}^\dagger\hat{a}}.$$

Interpret in phase space.

(d) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\left\{i\,\mathrm{Im}(\alpha\beta^*)\right\}$$

(e) Show that the displacement operators has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle \alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2} e^{i\,\mathrm{Im}\left(\alpha\alpha_2^* - \alpha_1\alpha^* - \alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\mathsf{L}_n\left(|\alpha|^2\right)$, where L_n is the Laguerre polynomial of order n

Problem 1: Properties of the Wigner function (10 Points)

A standard form of the Wigner function (in accordance with what Wigner wrote down originally) is (see Scully for derivation)

$$W(X_1,X_2) = \int \frac{dX'}{\pi} e^{-2iX'X_2} \langle X_1 - X' | \hat{\rho} | X_1 + X' \rangle = \int \frac{dX'}{\pi} e^{-2iX'X_2}\, \psi(X_1 - X')\psi^*(X_1 + X')$$

where the last form applies only for a pure state.

(a) In standard statistics, given a joint probability distribution on many random variables, one defines the "marginal distributions" by integrating out the others.

$$P(X_1) = \int dX_2\, W(X_1,X_2) \quad P(X_2) = \int dX_1\, W(X_1,X_2)\,.$$

Show that these marginal are in fact the *correct* true marginals predicted by quantum mechanics (here QM gives true probability distributions).

(b) Suppose we have operators which are functions only of the quadratures $\hat{f}_1 = f_1(\hat{X}_1)$, $\hat{f}_2 = f_2(\hat{X}_1)$. Show that

$$\langle \hat{f}_1 \rangle = \int dX_1 dX_2 W(X_1,X_2) f_1(X_1),\ \langle \hat{f}_2 \rangle = \int dX_1 dX_2 W(X_1,X_2) f_2(X_2),$$

and thus show, for example, the quantum uncertainties ΔX_1 and ΔX_2 are the respective rms widths of the Wigner function.

(c) Generalize these results to the case of the rotated quadratures,

$$\begin{aligned} \hat{X}_1(\theta) &= \cos\theta\, \hat{X}_1 + \sin\theta\, \hat{X}_2 \\ \hat{X}_2(\theta) &= \cos\theta\, \hat{X}_2 - \sin\theta\, \hat{X}_1 \end{aligned}.$$

Epilog: Given the marginals, we can invert to find the joint distribution. This process is well known in medical imaging – tomography. It turns out one can perform the same procedure here to use the marginals to obtain the *Wigner function*. This process is known as "quantum tomography", see e.g. M. G. Raymer *et al*., Phys. Rev. Lett. **72,** 1137 (1994).

Problem 3: A "Schrödinger cat" state. (10 Points)

Consider a superposition state of two "macroscopically" distinguishable coherent states, $|\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle)$, $|\alpha_1 - \alpha_2| >> 1$, where $N = \left[2\left(1+\exp\{-|\alpha_1 - \alpha_2|^2\}\right)\right]^{-1/2}$ is normalization. This state is often referred to as a "Schrodinger cat", and is very nonclassical.

(a) Calculate the Wigner function, for the simpler case $|\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle)$, with α real, and plot it for different values of $|\alpha_1 - \alpha_2| = 2\alpha$. Comment please.
(b) Calculate the marginals in X_1 and X_2 and show they are what you expect.

Problem 4: Thermal Light (15 points)

Consider a single mode field in thermal equilibrium at temperature *T*, Boltzmann factor $\beta = 1/k_B T$. The state of the field is described by the "canonical ensemble",
$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$, $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$ is the Hamiltonian and $Z = Tr\left(e^{-\beta \hat{H}}\right)$ is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- $\langle n \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$ (the Planck spectrum)
- $P_n = \frac{\langle n \rangle^n}{(1+\langle n \rangle)^{n+1}}$ (the Bose-Einstein distribution). Plot a histogram for various $\langle n \rangle$.
- $\Delta n^2 = \langle n \rangle + \langle n \rangle^2$. How does this compare to a coherent state?
- $\langle \hat{a} \rangle = 0 \Rightarrow \langle \vec{E} \rangle = 0$. How does this compare to a coherent state?

(b) Find the *P*. *Q,* and *W* distributions for this field, and show they are *Gaussian* functions. For example, you should find $P(\alpha) = \frac{1}{\pi\langle n \rangle} \exp\left(-\frac{|\alpha|^2}{\langle n \rangle}\right)$. Show that these three distributions give the proper functions in the limit, $\langle n \rangle \to 0$, i.e. the vacuum.

(c) Calculate $\Delta n^2, (\Delta X_1(\theta))^2, (\Delta X_2(\theta))^2$ using an appropriate quasi-probability distribution. Interpret Δn^2 as having a "particle" and a "wave" component.