Problem 1: Boson Algebra (10 Points)


(a) Gaussian integrals in phase-space are used all the time. Show that
\[
\int \frac{d^2\beta}{\pi} e^{-A|\beta|^2} \epsilon^{ab\beta^* - \beta a^*} = \frac{1}{A} e^{-|a|^2/A}.
\]

(b) Prove the completeness integral for coherent states
\[
\int \frac{d^2\alpha}{\pi} \langle \alpha | \alpha \rangle = 1 \text{ (Hint: Expand in number states).}
\]

(c) The “quadrature” operators in optics are the analogs of \(Q\) and \(P\). \(\hat{a} = \hat{X}_1 + i\hat{X}_2\). Show
\[
\hat{U}^\dagger(\theta) \hat{X} \hat{U}(\theta) = \cos \theta \hat{X}_1 + \sin \theta \hat{X}_2, \quad \text{where} \quad \hat{U}(\theta) = e^{-i\theta \hat{a}^\dagger \hat{a}}.
\]
Interpret in phase space.

(d) Prove the group property of the displacement operator
\[
\hat{D}(\alpha) \hat{D}(\beta) = \hat{D}(\alpha + \beta) \exp\{i \text{Im}(\alpha \beta^*)\}
\]

(e) Show that the displacement operators has the following matrix elements

Vacuum: \( \langle 0 | \hat{D}(\alpha) | 0 \rangle = e^{-|\alpha|^2/2} \)

Coherent states: \( \langle \alpha_1 | \hat{D}(\alpha) | \alpha_2 \rangle = e^{-|\alpha_1 + \alpha_2 - \alpha_1|^2/2} e^{i \text{Im}(\alpha_2 \alpha_2^* - \alpha_2 \alpha_1^* - \alpha_1 \alpha_2^*)} \)

Fock states: \( \langle n | \hat{D}(\alpha) | n \rangle = e^{-|\alpha|^2/2} L_n(|\alpha|^2) \), where \( L_n \) is the Laguerre polynomial of order \( n \).
Problem 1: Properties of the Wigner function (10 Points)

A standard form of the Wigner function (in accordance with what Wigner wrote down originally) is (see Scully for derivation)

\[
W(X_1, X_2) = \int \frac{dX'}{\pi} e^{-2iX'X_2} \langle X_1 - X' | \hat{\rho} | X_1 + X' \rangle = \int \frac{dX'}{\pi} e^{-2iX'X_2} \psi(X_1 - X')\psi^*(X_1 + X')
\]

where the last form applies only for a pure state.

(a) In standard statistics, given a joint probability distribution on many random variables, one defines the “marginal distributions” by integrating out the others.

\[
P(X_1) = \int dX_2 W(X_1, X_2) \quad P(X_2) = \int dX_1 W(X_1, X_2)
\]

Show that these marginal are in fact the correct true marginals predicted by quantum mechanics (here QM gives true probability distributions).

(b) Suppose we have operators which are functions only of the quadratures

\[
\hat{f}_1 = f_1(\hat{X}_1), \quad \hat{f}_2 = f_2(\hat{X}_1)
\]

Show that

\[
\langle \hat{f}_1 \rangle = \int dX_1 dX_2 W(X_1, X_2) f_1(X_1), \quad \langle \hat{f}_2 \rangle = \int dX_1 dX_2 W(X_1, X_2) f_2(X_2)
\]

and thus show, for example, the quantum uncertainties \(\Delta X_1\) and \(\Delta X_2\) are the respective rms widths of the Wigner function.

(c) Generalize these results to the case of the rotated quadratures,

\[
\hat{X}_1(\theta) = \cos \theta \hat{X}_1 + \sin \theta \hat{X}_2
\]
\[
\hat{X}_2(\theta) = \cos \theta \hat{X}_2 - \sin \theta \hat{X}_1
\]

Epilog: Given the marginals, we can invert to find the joint distribution. This process is well known in medical imaging – tomography. It turns out one can perform the same procedure here to use the marginals to obtain the Wigner function. This process is known as “quantum tomography”, see e.g. M. G. Raymer et al., Phys. Rev. Lett. 72, 1137 (1994).
**Problem 3:** A “Schrödinger cat” state. (10 Points)

Consider a superposition state of two “macroscopically” distinguishable coherent states,

\[ |\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle), |\alpha_1 - \alpha_2| >> 1, \]

where \( N = \left[ 2\left(1 + \exp\{-|\alpha_1 - \alpha_2|^2\}\right)\right]^{-1/2} \) is normalization.

This state is often referred to as a “Schrodinger cat”, and is very nonclassical.

(a) Calculate the Wigner function, for the simpler case \( |\psi\rangle = N(|\alpha\rangle + |\alpha\rangle) \), with \( \alpha \) real, and plot it for different values of \( |\alpha_1 - \alpha_2| = 2\alpha \). Comment please.

(b) Calculate the marginals in \( X_1 \) and \( X_2 \) and show they are what you expect.

**Problem 4:** Thermal Light (15 points)

Consider a single mode field in thermal equilibrium at temperature \( T \), Boltzmann factor \( \beta = 1/k_B T \). The state of the field is described by the “canonical ensemble”,

\[ \hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}, \quad \hat{H} = \hbar \omega \hat{a}^\dagger \hat{a} \] the Hamiltonian and \( Z = Tr\left(e^{-\beta \hat{H}}\right) \) is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- \( \langle n \rangle = \frac{1}{e^{\beta \omega} - 1} \) (the Planck spectrum)
- \( P_n = \frac{\langle n \rangle^n}{\langle 1 + \langle n \rangle \rangle^{n+1}} \) (the Bose-Einstein distribution). Plot a histogram for various \( \langle n \rangle \).
- \( \Delta n^2 = \langle n \rangle + \langle n \rangle^2 \). How does this compare to a coherent state?
- \( \langle \hat{a} \rangle = 0 \Rightarrow \langle \hat{E} \rangle = 0 \). How does this compare to a coherent state?

(b) Find the \( P, Q, \) and \( W \) distributions for this field, and show they are Gaussian functions. For example, you should find \( P(\alpha) = \frac{1}{\pi \langle n \rangle} \exp\left(-\frac{|\alpha|^2}{\langle n \rangle}\right) \). Show that these three distributions give the proper functions in the limit, \( \langle n \rangle \rightarrow 0 \), i.e. the vacuum.

(c) Calculate \( \Delta n^2, (\Delta X_1(\theta))^2, (\Delta X_2(\theta))^2 \) using an appropriate quasi-probability distribution. Interpret \( \Delta n^2 \) as having a “particle” and a “wave” component.