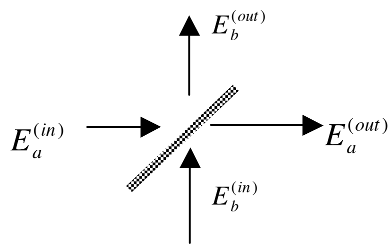
Physics 566, Quantum Optics
Problem Set #7
Due: Monday Oct 27, 2008

Problem1: The beam splitter and other linear transformations (25 points)
We're all familiar with classical linear optics. This problem explores the quantum description.

Consider a symmetric beam splitter



The pair $\left(E_a^{(out)}, E_b^{(out)}\right)$ is related to $\left(E_a^{(in)}, E_b^{(in)}\right)$ through a unitary "scattering matrix"

$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

(a) **Show** that $|t|^2 + |r|^2 = 1$, $Arg(t) = Arg(r) \pm \dfrac{\pi}{2}$, so that a possible transformation is,

$$E_a^{(out)} = \sqrt{T}\, E_a^{(in)} + i\sqrt{1-T}\, E_b^{(in)}, \quad E_b^{(out)} = \sqrt{T}\, E_b^{(in)} + i\sqrt{1-T}\, E_a^{(in)}, \text{ where } T = |t|^2.$$

Classically, if we inject a field only into one input port, leaving the other empty, the field in that mode will become attenuated, e.g., $E_a^{(out)} = \sqrt{T}\, E_a^{(in)} < E_a^{(in)}$.

(b) Consider now the quantized theory for these two modes, $E_a \Rightarrow \hat{a}$, $E_b \Rightarrow \hat{b}$. Suppose again that a field is injected only into the "a-port". Show that

$\hat{a}^{(out)} = \sqrt{T}\hat{a}^{(in)}$ is inconsistent with the quantum uncertainty.

(c) In order to preserve the proper commutation relations we cannot ignore *vacuum fluctuations* entering the unused port. **Show that** if the "in" and "out" creation operators are related by the scattering matrix,

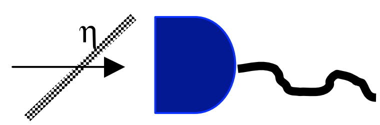
$$\begin{bmatrix} \hat{a}^{(out)\dagger} \\ \hat{b}^{(out)\dagger} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} \hat{a}^{(in)\dagger} \\ \hat{b}^{(in)\dagger} \end{bmatrix},$$ the commutator is preserved.

(d) Suppose a single photon is injected into the a-port, so that the "in-state" is $\left|\psi^{(in)}\right\rangle = \left|1\right\rangle_a \otimes \left|0\right\rangle_b$. The "out-state" is $\left|\psi^{(out)}\right\rangle = \hat{S}\left|\psi^{(in)}\right\rangle$ where $\hat{S}$ is the "scattering operator", defined so that $\hat{S}\hat{a}^{(in)\dagger}\hat{S}^\dagger = \hat{a}^{(out)\dagger}$ and $\hat{S}\hat{b}^{(in)\dagger}\hat{S}^\dagger = \hat{b}^{(out)\dagger}$.

Show that $\left|\psi^{(out)}\right\rangle = t\left|1\right\rangle_a \otimes \left|0\right\rangle_b + r\left|0\right\rangle_a \otimes \left|1\right\rangle_b$.

(e) Suppose a coherent state is injected into the a-port $\left|\psi^{(in)}\right\rangle = \left|\alpha\right\rangle_a \otimes \left|0\right\rangle_b$. Which is the output, $\left|\psi^{(out)}\right\rangle = \left|t\alpha\right\rangle_a \otimes \left|r\alpha\right\rangle_b$ or $\left|\psi^{(out)}\right\rangle = r\left|\alpha\right\rangle_a \otimes \left|0\right\rangle_b + t\left|0\right\rangle_a \otimes \left|\alpha\right\rangle_b$? Explain the difference between these.

(f) We can model a photon counter with a finite quantum efficiency η as perfect detector preceded by a beam splitter of with transmission coefficient η.



Show that the photon counting statistics, i.e. the probability to detect *m* photons is

$$P_m(\eta) = \sum_{n=m}^{\infty} p_n \binom{n}{m} \eta^m (1-\eta)^{n-m},$$ where p_n is the distribution before the beam-splitter.

Explain the meaning of this expression.

(g) A general linear optical system consisting, e.g., of beam-splitters, phase shifters, mirrors, etalons, etc. can be described by a unitary transformation on the modes

$$E_k^{(out)} = \sum_{k'} u_{kk'} E_{k'}^{(in)} .$$

In the quantum description the mode operators transform by the scattering transformation

$$\hat{a}_k^{(out)} = \hat{S}\hat{a}_k^{(in)}\hat{S}^\dagger = \sum_{k'} u_{kk'}\hat{a}_{k'}^{(in)},$$ where $u_{kk'}$ is a unitary matrix.

Show that if we start with a multimode coherent state $\left|\psi^{(in)}\right\rangle = \left|\left\{\alpha_k^{(in)}\right\}\right\rangle$, the output state is ALSO a coherent state, $\left|\psi^{(out)}\right\rangle = \left|\left\{\alpha_k^{(out)}\right\}\right\rangle$, with $\alpha_k^{(out)} = \sum_{k'} u_{kk'} \alpha_{k'}^{(in)}$.

(i) The previous part highlights how linear transformations are essentially classical. This was true for exactly one photon inputs or coherent states. However, this is not true for more general inputs. Suppose we send one photon into *both ports*, of a 50-50 beam-splitter T=1/2, $\left|\psi^{(in)}\right\rangle = |1\rangle_a \otimes |1\rangle_b$. **Show that** the output state is,

$$\left|\psi^{(out)}\right\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_a |0\rangle_b + |0\rangle_a |2\rangle_b\right).$$

This says that the two photons "bunch", both going to port-a or to port-b, but never one in port-a and one in port-b. This is an effect of Bose-Einstein quantum statistics. **Explain** in terms of destructive interference between indistinguishable processes.