Physics 566: Quantum Optics

Lecture 2: Coherence and Density Matrix

Quantum measurement theory dictates:

- Probability of finding a given outcome

Example: pure state $|\psi\rangle$

observable $\hat{A}$, eigenvectors $|a\rangle$

$$P_a = |\langle a|\psi\rangle|^2 = \langle\psi|\underbrace{\hat{\mathbb{P}}_a}_{|a\rangle\langle a|}|\psi\rangle$$

- Post-measurement state

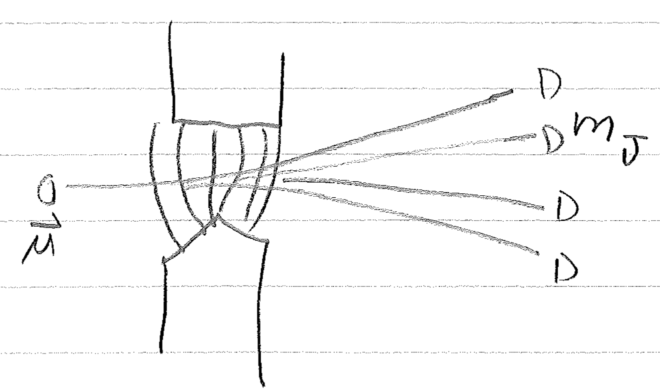
"Collapse of the wave function"

$$|\psi\rangle_{after} = \frac{\hat{\mathbb{P}}_a|\psi\rangle}{\|\hat{\mathbb{P}}_a|\psi\rangle\|} = \frac{|a\rangle\langle a|\psi\rangle}{\sqrt{|\langle a|\psi\rangle|^2}}$$

$$= e^{i\phi}|a\rangle$$

irrelevant Phase

Quintescential Example: Stern-Gerlach



$$\hat{\vec{\mu}} = \gamma\hat{\vec{J}}$$

gyro-magnetic ratio

$$\vec{F} = \vec{\mu}\cdot\vec{\nabla}\vec{B}$$

Spin-1/2 $\hat{\vec{S}} = \frac{\hbar}{2}\hat{\vec{\sigma}}$ ← Pauli operator

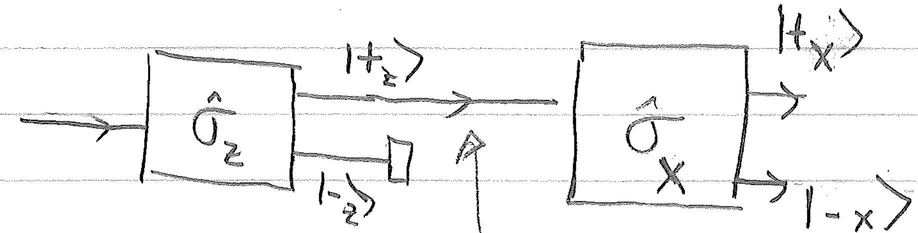
Eigenvectors: Spin-up/down along any direction

$$\vec{e}_n \cdot \hat{\vec{\sigma}}\,|\pm_n\rangle = \pm|\pm_n\rangle$$

$\sigma_z|\pm_z\rangle = \pm|\pm_z\rangle$ etc.

Recall
$$\begin{cases} |\pm_x\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle \pm |-_z\rangle\right) \\ |\pm_y\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle \pm i|-_z\rangle\right) \end{cases}$$

<u>Short-hand notation</u>



preparing a pure state

$$\mathbb{P}(+_x|+_z) = |\langle +_x|+_z\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

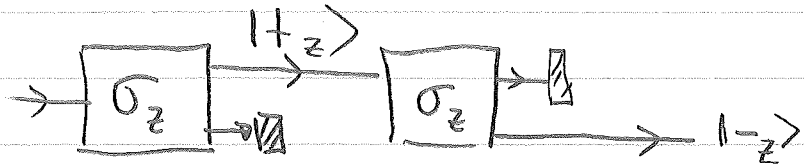
$$\mathbb{P}(-_x|+_z) = |\langle -_x|+_z\rangle|^2 = \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}$$

Conditional probability

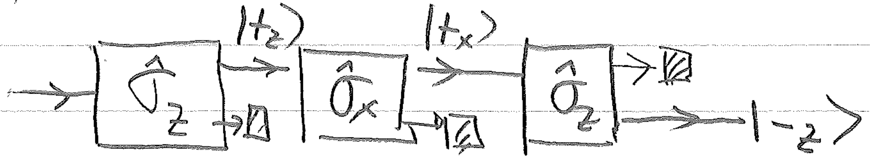
- Probability to find $|-_x\rangle$ given preparation in $|+_z\rangle$

Two experiments

(I)



(II)



Case (I) $P(-_z | +_z) = |\langle -_z | +_z \rangle|^2 = 0$

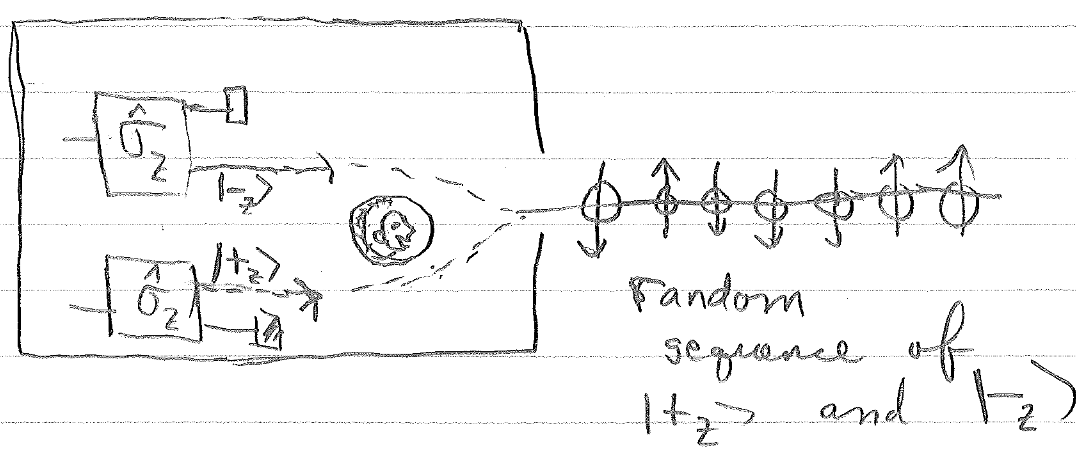
Case (II) $P(-_z | +_x ; +_z) = |\langle -_z | +_x \rangle|^2 \, |\langle +_x | +_z \rangle|^2$

$$= \left|\frac{1}{\sqrt{2}}\right|^2 \left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{4}$$

Intermediate state is superposition of $|\pm_z\rangle$
(Like cross-polarizer with 45° in middle)

Pure vs. Mixed States

Suppose we have a preparer with two ovens in a black box and a coin

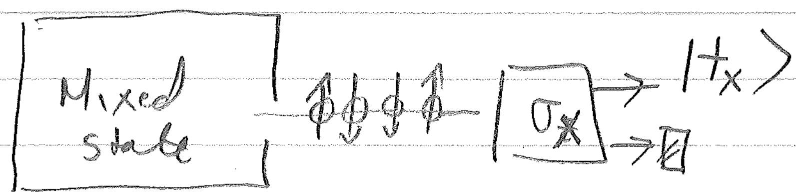


The state prepared for measurement is not a <u>pure</u> preparation of spin-up along some direction. It is a <u>statistical mixture</u> of $|+_z\rangle$ and $|-_z\rangle$

State : Ensemble $\{ p_{+_z}, |+_z\rangle \, ; \, p_{-_z}, |-_z\rangle \}$

Preparer sends $|\pm_z\rangle$ with probability $p_{\pm_z}$

Calculate probability of a measurement outcome



$$P(+_x) = P(+_x | +_z)\, p_{+_z} + P(+_x | -_z)\, p_{-_z}$$

Probability of measuring $|+_x\rangle$

$$= |\langle +_x | -_z \rangle|^2 p_{+_z} + |\langle +_x | -_z \rangle|^2 p_{-_z}$$

$$= \frac{1}{2}\left(p_{+_z} + p_{-_z}\right) = \frac{1}{2}$$

Contrast : Pure state preparation $|-_x\rangle$

$$P(+_x | -_x) = |\langle +_x | -_x \rangle|^2 = 0$$

$$= \left| \langle +_x | \left(\sum_{m_z = \pm} |m_z\rangle\langle m_z| \right) |-_x\rangle \right|^2$$

$$\Rightarrow P(+_x|-_x) = |\langle +_x|+_z\rangle\langle +_z|-_x\rangle + \langle +_x|-_z\rangle\langle -_z|-_x\rangle|^2$$

$$= |\langle +_x|+_z\rangle|^2 \, |\langle +_z|-_x\rangle|^2 + |\langle +_x|-_z\rangle|^2 \, |\langle -_z|-_x\rangle|^2$$

$$+ \langle +_x|+_z\rangle\langle +_z|-_x\rangle\langle -_x|-_z\rangle\langle -_z|+_x\rangle + \text{c.c.}$$

$$= \overbrace{P(+_x|+_z)\,P(+_z|-_x) + P(+_x|-_z)\,P(-_z|-_x)}^{=1/2}$$

$$+ \underbrace{\text{INTERFERENCE}}_{\left(\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) + \text{c.c}}$$

$$= \frac{1}{2} - \frac{1}{2} = 0 \quad \checkmark$$

- Mixed state $\left\{ \frac{1}{2}, |+_z\rangle, \ \frac{1}{2}, |-_z\rangle \right\}$ is a 50/50 statistical mixture of $|+_z\rangle$ and $|-_z\rangle$

- $|+_x\rangle = \frac{1}{\sqrt{2}}|+_z\rangle + \frac{1}{\sqrt{2}}|-_z\rangle$ is a coherent superposition of $|\pm_z\rangle$

Formal description: Density operator

Consider ensemble $\{ p_{+z}, |+_z\rangle \,;\, p_{-z} |-_z\rangle \}$

Prob to find $|+_n\rangle$ (spin-up in arbitrary direction)

$$P(+_n) = p_{+z} |\langle +_n | +_z \rangle|^2 + p_{-z} |\langle +_n | -_z \rangle|^2$$

$$= p_{+z} \langle +_n | +_z \rangle \langle +_z | +_n \rangle + p_{-z} \langle +_n | -_z \rangle \langle -_z | +_n \rangle$$

$$= \langle +_n | \underbrace{\left(p_{+z} |+_z\rangle\langle +_z| + p_{-z} |-_z\rangle\langle -_z| \right)}_{} |+_n\rangle$$

State depending on the preparation

$\equiv \hat{\rho}$ Density Operator

$\Rightarrow P(+_n) =$

More generally (not just spin-½)

Given an ensemble which is prepared as a statistical mixture of states $\{|\psi_i\rangle\}$ with probabilitys $\{p_i\}$, the the state is described by a density operator:

$$\boxed{\hat{\rho} = \sum_i p_i |\psi_i\rangle\langle\psi_i|}$$

Properties of the density operator

- Hermitian: $\hat{\rho}^{\dagger} = \hat{\rho}$
- Positive: $\langle \phi | \hat{\rho} | \phi \rangle \geq 0 \quad \forall |\phi\rangle$

  (also has positive eigenvalues)
- Eigenvector decomposition: $\hat{\rho} = \sum_{i=1}^{d} \lambda_i |e_i\rangle\langle e_i|$

  (An ensemble decomposition
  eigenvectors $\{|e_i\rangle\}$
  with probabilities λ_i
- Normalization: Eigenvectors orthornormal

  $$\Rightarrow \sum_{i=1}^{d} \lambda_i = \underbrace{\mathrm{Tr}}_{\text{trace}}(\hat{\rho}) = 1$$

- Pure vs. mixed state

For a pure state, $\exists |\psi\rangle$ s.t.

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$\Rightarrow$ One eigenvector with eigenvalue 1

For general mixed state $\hat{\rho} = \sum_{i=1}^{d} \lambda_i |e_i\rangle\langle e_i|$

$$\Rightarrow \hat{\rho}^2 = \sum_{i=1}^{d} \lambda_i^2 |e_i\rangle\langle e_i|$$

$\Rightarrow \boxed{\mathrm{Tr}(\hat{\rho}^2) < 1}$ mixed

Maximal mixed $\lambda_i = \frac{1}{d} \quad \forall i$

Note: Density operator is the most general way to represent a quantum state, It appears in different contexts

- Random preparation
- "Open quantum" system

e.g. Thermodynamics (canonical ensemble)

$$\hat{\rho} = \frac{1}{Z} e^{-\hat{H}/k_B T} \qquad Z = \mathrm{Tr}(e^{-\hat{H}/k_B T})$$

Using $\hat{\rho}$ to make predictions

- Probability of finding $|a\rangle$ given $\hat{\rho}$
  - $P_a = \langle a|\hat{\rho}|a\rangle$ (Diagonal matrix element in basis $\{|a\rangle\}$)
- Expectation values

$$\langle \hat{A} \rangle = \sum_a a P_a = \sum_a a \langle a|\hat{\rho}|a\rangle$$

$$= \sum_a a \,\mathrm{Tr}(|a\rangle\langle a| \hat{\rho})$$

$$= \mathrm{Tr}\left(\sum_a a|a\rangle\langle a| \hat{\rho}\right)$$

$$\Rightarrow \boxed{\langle \hat{A} \rangle = \mathrm{Tr}(\hat{A}\rho)}$$

Aside $\mathrm{Tr}(|\psi\rangle\langle\phi| \hat{M}) = \langle\phi|M|\psi\rangle$

Proof:

$$\mathrm{Tr}(|\phi\rangle\langle\psi|\hat{M}) = \sum_{i=1}^{d} \langle i|\phi\rangle\langle\psi|\hat{M}|i\rangle$$

$$= \sum_i \langle\psi|\hat{M}|i\rangle\langle i|\phi\rangle = \langle\psi|\hat{M}|\phi\rangle \checkmark$$

Diagonal vs. off-diagonal matrix elements of $\hat{\rho}$

Consider two states:

(i) $|\psi\rangle \equiv |+_x\rangle = \frac{1}{\sqrt{2}}\left(|+_z\rangle + |-_z\rangle\right)$

Pure $\quad \rho^{(i)} \equiv |\psi\rangle\langle\psi| = \frac{1}{2}|+_z\rangle\langle+_z| + \frac{1}{2}|-_z\rangle\langle-_z| + \frac{1}{2}|+_z\rangle\langle-_z| + \frac{1}{2}|-_z\rangle\langle+_z|$

(ii) A mixed state:

$$\hat{\rho}^{(ii)} \equiv \frac{1}{2}|+_z\rangle\langle+_z| + \frac{1}{2}|-_z\rangle\langle-_z|$$

Both states predict the same probability for measuring $|\pm_z\rangle$

$$P^{(i)}_{\pm_z} = \langle\pm_z|\hat{\rho}^{(i)}|\pm_z\rangle = \frac{1}{2}$$

$$P^{(ii)}_{\pm_z} = \langle\pm_z|\hat{\rho}^{(ii)}|\pm_z\rangle = \frac{1}{2}$$

However, they lead to different probabilities for finding $|\pm x\rangle$

$$P^{(i)}_{+_x} = \langle+_x|\hat{\rho}^{(i)}|+_x\rangle = 1$$

$$P^{(i)}_{-_x} = \langle-_x|\hat{\rho}^{(i)}|-_x\rangle = 0$$

$$P^{(ii)}_{+_x} = P^{(ii)}_{-_x} = \frac{1}{2}$$

What distinguishes these two states are the off-diagonal matrix elements in the basis $\{|+_z\rangle, |-_z\rangle\}$

$$\hat{\rho} \doteq \begin{bmatrix} \langle +_z|\hat{\rho}|+_z\rangle & \langle +_z|\hat{\rho}|-_z\rangle \\ \langle -_z|\hat{\rho}|+_z\rangle & \langle -_z|\hat{\rho}|-_z\rangle \end{bmatrix}$$

$$\hat{\rho}^{(i)} \doteq \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad \hat{\rho}^{(ii)} \doteq \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

The off-diagonal elements w.r.t. a given basis represents the capacity for the quantum system to exhibit <u>interference</u> in that basis, i.e. the degree to which there is <u>coherence</u> between those states

- In state $\hat{\rho}^{(i)}$ there is perfect coherence between $|+_z\rangle$ and $|-_z\rangle$. Interference is seen between probability amplitudes c_{+_z} and c_{-_z}
- In state $\hat{\rho}^{(ii)}$ there is no coherence between $|+_z\rangle$ and $|-_z\rangle$ since $\rho_{+-} = \rho_{-+} = 0$. In fact, for this example $\hat{\rho}^{(ii)} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$ Completely mixed

$\Rightarrow$ No coherence in any basis

Note, for <u>arbitrary pure state</u>

$$|\psi\rangle = c_{+_z}|+_z\rangle + c_{-_z}|-_z\rangle$$

$$\hat{\rho} = |\psi\rangle\langle\psi|$$

$$\Rightarrow \hat{\rho} \doteq \begin{bmatrix} |c_{+_z}|^2 & c_{+_z} c_{-_z}^* \\ c_{-_z} c_{+_z}^* & |c_{-_z}|^2 \end{bmatrix}$$

- Diagonal elements: "Population" = probability

$$P_m = |c_m|^2$$

- Off diagonal: "Coherence"

$$c_{-_z} c_{+_z}^* = |c_{-_z}||c_{+_z}| \, e^{i(\phi_- - \phi_+)}$$

phase

<u>Statistical Mixture</u>

average over ensemble

$$\hat{\rho} \doteq \begin{bmatrix} \overline{|c_{+_z}|^2} & \overline{c_{+_z} c_{-_z}^*} \\ \overline{c_{-_z} c_{+_z}^*} & \overline{|c_{-_z}|^2} \end{bmatrix}$$

typically washes out ("phase coherence")