Quantum measurement theory dictates:

- Probability of finding a given outcome

Example: pure state \( |\psi\rangle \)

observable \( \hat{A} \), eigenvectors \( |a\rangle \)

\[
P_a = |\langle a | \psi \rangle|^2 = \langle a | \hat{A} | \psi \rangle
\]

\[
\rightarrow |a\rangle \times |a\rangle
\]

- Post-measurement state

"Collapse of the wave function"

\[
|\psi\rangle_{\text{after}} = \frac{\hat{A}_a |\psi\rangle}{|\langle \psi | \hat{A}_a |\psi\rangle|} = \frac{|a\rangle \times |a\rangle}{\sqrt{|\langle a | a \rangle|^2}}
\]

\[
= e^{i\phi} |a\rangle
\]

irrelevant phase

Quintessential Example: Stern-Gerlach

\[
\hat{\mu} = g \frac{\hat{J}}{\hbar}
\]

gyro-magnetic ratio

\[
F = \hat{\mu} \times \nabla B
\]
\[
\text{Spin-1/2} \quad \hat{S} = \frac{\hbar}{2} \sigma \quad \text{Pauli operator}
\]

Eigenvalues: Spin-up/down along any direction
\[
e_{n} \cdot \hat{\sigma}_{n} |\pm_{n}\rangle = \pm |\pm_{n}\rangle
\]
\[
\hat{\sigma}_{z} |\pm_{z}\rangle = \pm |\pm_{z}\rangle \quad e + e^{c}
\]

Recall
\[
\begin{align*}
|\pm_{x}\rangle &= \frac{1}{\sqrt{2}} (|+_{z}\rangle \pm |-_{z}\rangle) \\
|\pm_{y}\rangle &= \frac{1}{\sqrt{2}} (|+_{z}\rangle \pm i|-_{z}\rangle)
\end{align*}
\]

Short-hand notation

Preparing a pure state
\[
P(|_{+_{x}|}_{+_{z}}) = |\langle+_{x}|+_{z}\rangle|^{2} = |\frac{1}{\sqrt{2}}|^{2} = \frac{1}{2}
\]
\[
P(|_{-_{x}|}_{+_{z}}) = |\langle-_{x}|+_{z}\rangle|^{2} = |\frac{1}{\sqrt{2}}|^{2} = \frac{1}{2}
\]

Conditional probability
- Probability to find \(1_{-x}\rangle\) given preparation in \(1_{+z}\rangle\)
Two experiments

(I) \[ H_z \rightarrow \sigma_z \rightarrow 1_z \rightarrow \]

(II) \[ H_z \rightarrow \sigma_x \rightarrow \sigma_z \rightarrow 1_z \rightarrow \]

Case (I) \[ P(-z|+z) = |\langle -z|+z \rangle|^2 = 0 \]

Case (II) \[ P(-z|+x, +z) = |\langle -z|+x \rangle|^2 |\langle +x|+z \rangle|^2 \]

\[ = \left| \frac{1}{\sqrt{2}} \right|^2 \left| \frac{1}{\sqrt{2}} \right|^2 = \frac{1}{4} \]

Intermediate state is superposition of \[ |+z \rangle \]

(Like cross-polarizer with \(45°\) in middle)

Pure vs. Mixed States

Suppose we have a preparer with two ovens in a black box and a coin
The state prepared for measurement is not a pure preparation of spin-up along some direction. It is a statistical mixture of $|+\rangle$ and $|-\rangle$

State: Ensemble \{ $p_{+z}$, $|+\rangle$ ; $p_{-z}$, $|-\rangle$ \}

Preparer sends $|+\rangle$ with probability $p_{+z}$

Calculate probability of a measurement outcome

\[
\begin{array}{c}
\text{Mixed state} \\
\Phi \\
\text{state} \rightarrow |0\rangle \\
\rightarrow |+\rangle
\end{array}
\]

\[
P(+_{x}) = p(+_{x} |+\rangle \cdot p_{+z} + p(+_{x} |-\rangle \cdot p_{-z}
\]

\[
\text{Probability of measuring } |+_{x}\rangle
\]

\[
= |\langle +_{x} |-\rangle |^2 p_{+z} + |\langle +_{x} |-\rangle |^2 p_{-z}
\]

\[
= \frac{1}{2} (p_{+z} + p_{-z}) = \frac{1}{2}
\]

Contrast: Pure state preparation $|-\rangle$

\[
P(+_{x} |-\rangle) = |\langle +_{x} |-\rangle |^2 = 0
\]

\[
= |\langle +_{x} |\left( \sum_{m_{z} = \pm} |m_{z}\rangle \langle m_{z}| \right) |-\rangle |^2
\]
\[ P(\pm_x \pm_z) = |\langle \pm_x \pm_z | \pm_z \mp_x \rangle |^2 + |\langle \pm_x \mp_z | \pm_z \mp_x \rangle |^2 \]

\[ = \left| \langle \pm_x \pm_z | \pm_z \mp_x \rangle \right|^2 + \left| \langle \pm_x \mp_z | \pm_z \mp_x \rangle \right|^2 \]

\[ + \langle \pm_x \pm_z \rangle \langle \pm_z \mp_x \rangle \langle \mp_x \pm_z \rangle \langle \mp_x \mp_z \rangle + \text{c.c.} \]

\[ = \frac{1}{2} \left( P(\pm_x \pm_z) P(\pm_z \mp_x) + P(\pm_x \mp_z) P(\mp_z \pm_x) \right) \]

\[ + \text{INTERFERENCE} \]

\[ \left( \frac{1}{\sqrt{2}} \right) \left( -\frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \right) + \text{c.c.} \]

\[ = \frac{1}{2} - \frac{1}{2} = 0 \]

- Mixed state \( \{ \frac{1}{2}, \pm_z \rangle, \frac{1}{2}, \pm_z \rangle \} \)
  
  50/50

is a statistical mixture of \( \pm_z \rangle \) and \( \pm_z \rangle \)

- \( \pm_x \rangle = \frac{1}{\sqrt{2}} \pm_z \rangle + \frac{1}{\sqrt{2}} \mp_z \rangle \)
  
  is a coherent superposition of \( \pm_z \rangle \)
Formal description: Density operator
Consider ensemble \( \{ \psi_{t_z}, |t_z\rangle \}; t_z = \pm 1 \)

Proof to find \( |t_n\rangle \) (spin-up in arbitrary direction)

\[
P(t_n) = P_{t_z} |\langle t_n | t_z \rangle|^2 + P_{-t_z} |\langle t_n |-t_z \rangle|^2
\]

\[
= P_{t_z} |\langle t_n | t_z \rangle|^2 + P_{-t_z} |\langle t_n |-t_z \rangle|^2
\]

\[
= \langle t_n | (P_{t_z} |t_z\rangle \langle t_z | + P_{-t_z} |-t_z\rangle \langle -t_z |) |t_n\rangle
\]

State depending on the preparation

\[
\Rightarrow P(t_n) = \hat{\rho} \quad \text{Density Operator}
\]

More generally (not just spin-\(1/2\))

Given an ensemble which is prepared as a statistical mixture of states \( \{ |\Psi_i\rangle \} \)

with probability \( \{ p_i \} \), the state is described by a density operator:

\[
\hat{\rho} = \sum_i p_i |\Psi_i\rangle \langle \Psi_i |
\]
Properties of the density operator

- **Hermitean**: \( \hat{\rho}^\dagger = \hat{\rho} \)
- **Positive**: \( \langle \phi | \hat{\rho} | \phi \rangle \geq 0 \quad \forall |\phi\rangle \)
  (also has positive eigenvalues)
- **Eigenvector decomposition**: \( \hat{\rho} = \sum_{i=1}^d \lambda_i |e_i \rangle \langle e_i| \)
  (An ensemble, decomposition
  eigenvectors \( |e_i \rangle \),
  with probabilities \( \lambda_i \))
- **Normalization**: Eigenvectors orthonormal
  \[ \sum_{i=1}^d \lambda_i = \text{Tr} (\hat{\rho}) = 1 \]
  (trace)
- **Pure vs. mixed state**
  For a pure state, \( |\psi\rangle \) s.t.
  \( \hat{\rho} = |\psi\rangle \langle \psi| \)
  \( \Rightarrow \) One eigenvector with eigenvalue \( 1 \)
  For general mixed state \( \hat{\rho} = \sum_{i=1}^d \lambda_i |e_i \rangle \langle e_i| \)
  \( \Rightarrow \) \( \beta^2 = \sum_{i=1}^d \lambda_i^2 |e_i \rangle \langle e_i| \)
  \( \Rightarrow \) \( \text{Tr}(\beta^2) < 1 \quad \text{mixed} \)
  Maximal mixed \( \lambda_i = \frac{1}{d} \quad \forall i \)
Note: Density operator is the most general way to represent a quantum state. It appears in different contexts:
- Random preparation
- "Open quantum" system
  e.g. Thermodynamic (canonical ensemble)
  \[ \rho = \frac{1}{Z} e^{-\hat{H}/k_B T} \]
  \[ Z = \text{Tr}(e^{-\hat{H}/k_B T}) \]

Using \( \rho \) to make predictions

- Probability of finding 1a) given \( \rho \)
  \[ P_a = \langle a | \rho | a \rangle \quad (\text{Diagonal matrix element in basis } |\alpha\rangle) \]

- Expectation values
  \[ \langle \hat{A} \rangle = \sum_a a P_a = \sum_a a \langle a | \rho | a \rangle \]
  \[ = \sum_a \text{Tr}(1_a X_a \rho) \]
  \[ = \text{Tr}(\sum_a 1_a X_a \rho) \]

\[ \Rightarrow \langle \hat{A} \rangle = \text{Tr}(\hat{A} \rho) \]

Aside: \[ \text{Tr}(\| \psi \rangle \langle \phi | \hat{M} | \psi \rangle) = \langle \phi | M | \psi \rangle \]
Proof:
\[ \text{Tr}(\Phi X \Phi^\dagger M) = \sum_{i=1}^{d} \langle i | \Phi X \Phi^\dagger M | i \rangle, \]
\[ = \sum_{i} \langle 2 | M | i \rangle \langle i | \Phi \rangle = \langle 2 | \Phi^\dagger M | \Phi \rangle. \]

**Diagonal vs. off-diagonal matrix elements of \( \hat{M} \)**

Consider two states
\[ (i) \quad | 2 \Phi \rangle = \frac{1}{\sqrt{2}} (| +z \rangle + | -z \rangle) \]

Pure
\[ \rho^{(i)} = | 2 \Phi \rangle \langle 2 | = \frac{1}{2} | +z \rangle \langle +z | + \frac{1}{2} | -z \rangle \langle -z | \]

\[ + \frac{1}{2} | +z \rangle \langle -z | + \frac{1}{2} | -z \rangle \langle +z | \]

A mixed state
\[ \rho^{(ii)} = \frac{1}{2} | +z \rangle \langle +z | + \frac{1}{2} | -z \rangle \langle -z | \]

Both states predict the same probability for measuring \( | \pm z \rangle \)
\[ P_{\pm z}^{(i)} = \langle \pm z | \rho^{(i)} | \pm z \rangle = \frac{1}{2} \]

\[ P_{\pm z}^{(ii)} = \langle \pm z | \rho^{(ii)} | \pm z \rangle = \frac{1}{2} \]

However, they lead to different probabilities for finding \( | \pm x \rangle \)
\[ P_{+x}^{(i)} = \langle +x | \rho^{(i)} | +x \rangle = 1 \]
\[ P_{-x}^{(i)} = \langle -x | \rho^{(i)} | -x \rangle = 0 \]
\[ P_{+x}^{(ii)} = P_{-x}^{(ii)} = \frac{1}{2} \]
What distinguishes these two states are the off-diagonal matrix elements in the basis \( \{ |\pm 2\rangle \} \)

\[
\hat{\rho} = \begin{pmatrix}
\langle t_2 | \hat{\rho} | t_2 \rangle & \langle t_2 | \hat{\rho} | -t_2 \rangle \\
\langle -t_2 | \hat{\rho} | t_2 \rangle & \langle -t_2 | \hat{\rho} | -t_2 \rangle
\end{pmatrix}
\]

\( \hat{\rho}^{(i)} = \begin{pmatrix}
\frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2}
\end{pmatrix} \quad \hat{\rho}^{(ii)} = \begin{pmatrix}
\frac{1}{2} & 0 \\
0 & \frac{1}{2}
\end{pmatrix} \)

The off-diagonal elements w.r.t. a given basis represents the capacity for the quantum system to exhibit interference in that basis, i.e. the degree to which there is coherence between those states.

* In state \( \hat{\rho}^{(i)} \) there is perfect coherence between \( |t_2\rangle \) and \( |-t_2\rangle \). Interference is seen between probability amplitudes \( C_{t_2} \) and \( C_{-t_2} \).
* In state \( \hat{\rho}^{(ii)} \) there is no coherence between \( |t_2\rangle \) and \( |-t_2\rangle \) since \( P_{t_2} = P_{-t_2} = 0 \). In fact, for this example \( \hat{\rho}^{(ii)} = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \)
  
  \( \Rightarrow \) Completely mixed
  
  \( \Rightarrow \) No coherence in any basis.
Note, for arbitrary pure state

\[ |\psi\rangle = c_{+z}|+z\rangle + c_{-z}|-z\rangle \]

\[ \rho = |\psi\rangle \langle \psi | \]

\[ \rho = \begin{bmatrix} |c_{+z}|^2 & c_{+z}^* c_{-z} \\ c_{-z} c_{+z}^* & |c_{-z}|^2 \end{bmatrix} \]

- **Diagonal elements:** "Population" = probability
  \[ P_m = |c_m|^2 \]

- **Off diagonal:** "Coherence"
  \[ c_{-z}^* c_{+z}^* = |c_{-z}| |c_{+z}| e^{i(\phi_+ - \phi_-)} \]

**Statistical Mixture**

average over ensemble

\[ \rho = \langle \begin{bmatrix} \frac{1}{|c_{+z}|^2} & \frac{c_{+z}}{c_{-z}}^* \\ \frac{c_{-z}}{c_{+z}} & \frac{|c_{-z}|^2}{|c_{+z}|^2} \end{bmatrix} \rangle \]

Typically washes out "phase coherence"