Lecture 20: Dynamics of Open Quantum Systems II

In the previous lecture we generalized the Unitary evolution of a state for a close quantum system to an completely positive map on the density operator:

Closed

$$\hat{\rho}(t) = \hat{U}(t)\,\hat{\rho}(0)\,\hat{U}^{\dagger}(t)$$

Open

$$\hat{\rho}(t) = \mathcal{Q}_t[\hat{\rho}(0)] = \sum_{\mu} \hat{M}_{\mu}(t)\,\hat{\rho}(0)\,\hat{M}_{\mu}^{\dagger}(t)$$

Kraus decomposition

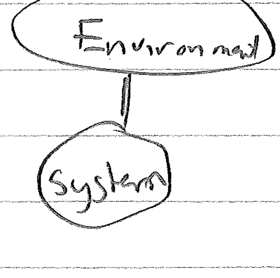
We now seek the analog of the differential Schrödinger equation

Closed

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}(t), \hat{\rho}(t)] \Rightarrow \hat{U}(t) = T\left[e^{-i\int_0^t \hat{H}(t')dt'}\right]$$

$$\text{Open} \longrightarrow \frac{\partial \hat{\rho}}{\partial t} = ?$$

To solve this problem, we begin with the Schrodinger equation for the closed bipartite system + environment



$$\hat{H}_{SE} = \hat{H}_S + \hat{H}_E + \hat{H}_{SE}^{int}$$

Total Hamiltonian

$$\frac{\partial \hat{\rho}_{SE}}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_{SE}, \hat{\rho}_{SE}\right]$$

$$\Rightarrow \frac{\partial \hat{\rho}_S}{\partial t} = Tr_E\left(\frac{\partial \hat{\rho}_{SE}}{\partial t}\right) = -\frac{i}{\hbar} Tr_E\left(\left[\hat{H}_{SE}, \hat{\rho}_{SE}\right]\right)$$

Let us formally integrate this equation for a short time Δt

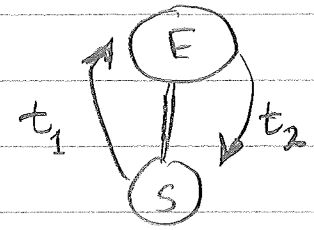
$$\hat{\rho}_S(t+\Delta t) = \hat{\rho}_S(t) - \frac{i}{\hbar}\int_t^{t+\Delta t} Tr_E\left[\hat{H}_{SE}(t'), \hat{\rho}_{SE}(t')\right] dt'$$

$$\Rightarrow \frac{d\hat{\rho}_S}{dt} = \lim_{\Delta t \to 0} \frac{\hat{\rho}_S(t+\Delta t) - \hat{\rho}_S(t)}{\Delta t} = -\frac{i}{\hbar}\int_t^{t+\Delta t} Tr_E\left[\hat{H}_{SE}(t'), \hat{\rho}_{SE}(t')\right]\frac{dt'}{\Delta t}$$

We see from this equation that the right hand side doesn't depend of $Tr_E(\hat{\rho}_{SE}(t)) = \hat{\rho}_S(t)$, but instead depends on $\hat{\rho}_S$ at all times between $t \to t+\Delta t$

The equation for $\frac{d\hat{\rho}_S}{\partial t} = \int_0^t K(t-t')\,\hat{\rho}_S(t')\,dt'$,

i.e., an integro-differential equation, with a "memory kerneal" $K(t-t')$ that depends not only on time t, but all earlier times. We can understand this since excitations transfered to the environment from the system a one time can return to system from the environment at a later time.



Excitation flows from $S \to E$ at t_1 and then from $E \to E$ at $t_2 > t_1$

There is a special case of interest – irreversibility. As discussed in an earlier lecture, if E is true an environment then it is complex, (as in complicated), composed of many, many degrees of freedom. Excitations can flow from $S \to E$, but for all intents and purposes, never come back. Moreover, for a sufficiently larger environment, like a "thermal reservoir", the correlation between the system and the enviroment is quickly lost and the environment equilibrates to its steady value.

We thus coarse-grain the evolution of $\hat{\rho}_S(t)$ for a time Δt which satisfy the following inequalities:

$$\tau_c \ll \Delta t \ll t_{decay}$$

Correlation time of system + environment

decay time of energy in system out of equilibrium

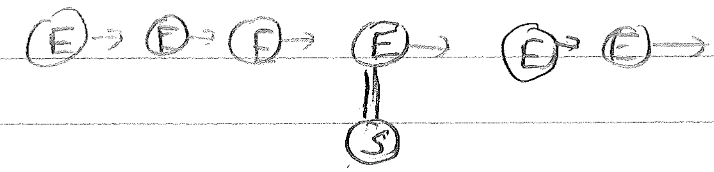
Under this approximation the memory Kernal is local differential operator

$$K(t-t') \Longrightarrow \mathcal{L}_t \, \delta(t-t')$$

This is known as the <u>Markov approximation</u>.

$$\boxed{\frac{d\rho_s}{dt} = \mathcal{L}_t[\rho_s(t)]}$$

We can physically understand the meaning of the Markov approximation as follows. The system interacts locally with the environment, they become entangled, by these correlations are carried off into the uncountably infinite # of degrees of freedom of environment. A "fresh" copy of the environment then interacts with the system



For example, for a two-level atom each copy of E can be though of as a short wave packet of the quantized field that propagate across the atom. The propagation time is extremely short, corresponding to τ_c, and mostly each little interaction does nothing. But, every so often the atom jumps from $|e\rangle \to |g\rangle$ with a rate Γ. The Markov approximation is thus based on the approximation that for all times $> \tau_c$, the system and environment remain <u>factorized</u>

$$\rho_{SE}(t') = \rho_S(t')\,\rho_E(0)$$

Formally, start with Schrödinger equation in the interaction picture

$$\frac{\partial \hat{\rho}_{SE}}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_{SE}(t), \hat{\rho}_{SE}(t)\right]$$

Formally integrate $\Rightarrow$ $\hat{\rho}_{SE}(t) = \hat{\rho}_{SE}(0) - \frac{i}{\hbar}\int_0^t dt' \left[\hat{H}_{SE}(t'), \hat{\rho}_{SE}(t')\right]$

Plug back in

$$\Rightarrow \frac{\partial \hat{\rho}_S}{\partial t} = \mathrm{Tr}_E\left(-\frac{i}{\hbar}\left[\hat{H}_{SE}(t), \hat{\rho}_{SE}(0)\right] - \frac{1}{\hbar^2}\int_0^t dt' \left[\hat{H}_{SE}(t), \left[\hat{H}_{SE}(t'), \hat{\rho}_{SE}(t')\right]\right]\right)$$

$\hat{\rho}_{SE}(0) = \hat{\rho}_S(0)\,\hat{\rho}_E(0)$

and $\mathrm{Tr}_E\left(\hat{H}_{SE}(t)\,\hat{\rho}_E(0)\right) = 0$

(e.g. $\hat{H}_{SE} = \sum_k \left(g_k \hat{a}_k \hat{\sigma}_+ + g_k^* \hat{a}_k^\dagger \sigma_-\right)$)

$$\Rightarrow \boxed{\frac{\partial \hat{\rho}_S}{\partial t} \cong -\frac{1}{\hbar^2}\int_0^t dt' \; \mathrm{tr}_E \left[\hat{H}_{SE}(t), \left[\hat{H}_{SE}(t'), \hat{\rho}_E(0)\,\hat{\rho}_S(t')\right]\right]}$$

"Born Markov"

$$\Rightarrow \frac{\partial \hat{\rho}_S}{\partial t} = -\frac{1}{\hbar^2}\int_0^t dt' \Big(\langle \hat{H}_{SE}(t)\,\hat{H}_{SE}(t')\rangle_E\, \hat{\rho}_S(t')$$

$$+ \hat{\rho}_S(t') \langle \hat{H}_{SE}(t')\,\hat{H}_{SE}(t')\rangle_E$$

$$- \langle \hat{H}_{SE}(t)\,\hat{\rho}_S(t')\,\hat{H}_{SE}(t')\rangle$$

$$- \langle \hat{H}_{SE}(t')\,\hat{\rho}_S(t')\,\hat{H}_{SE}(t)\rangle \Big)$$

The Markov approximation $\Rightarrow$ delta correlated noise

We thus now have a local diff'eqn for $\hat{\rho}_S(t)$

$$\boxed{\frac{d\hat{\rho}_S}{dt} = -\frac{1}{\hbar^2}\int_0^t dt' \, Tr_E\left\{\left[\hat{H}_{SE}(t), \left[\hat{H}_{SE}(t'), \hat{\rho}_S(t)\hat{\rho}_E(0)\right]\right]\right\} = \mathcal{L}_t\left[\hat{\rho}_S(t)\right]}$$

We can now ask, what is the most general form of $\mathcal{L}_t$, consistent with a completely positive map?

$$\rho_S(t+dt) = \hat{\rho}_S(t) + \hat{\mathcal{O}}(dt)$$

$$= \sum_\mu \hat{M}_\mu(dt)\, \hat{\rho}_S(t)\, \hat{M}_\mu^\dagger(dt)$$

The is one Kraus operator of the form

$$M_0(dt) = \hat{\mathbb{1}} + dt\,\hat{G}$$

All other Kraus operator must be of the form

$$M_\mu(dt) = \sqrt{dt}\,\hat{L}_\mu$$

If this trace preserving then

$$\sum_{\mu=1}^N M_\mu^\dagger M_\mu + M_0^\dagger M_0 = \hat{\mathbb{1}}$$

$$\Rightarrow dt\sum_{\mu=1}^N \hat{L}_\mu^\dagger \hat{L}_\mu + \hat{\mathbb{1}} + dt(\hat{G} + \hat{G}^\dagger) = \hat{\mathbb{1}}$$

$$\Rightarrow \hat{G}^\dagger + \hat{G} = -\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu$$

If we write $\hat{G} = \hat{K} - \frac{i}{\hbar}\hat{H}$

$$\Rightarrow \hat{G} = -\frac{i}{\hbar}\hat{H} - \frac{1}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu$$

Thus, the equation of motion for $\hat{\rho}_S(t)$ in the Markov approximation, consistent with a CP map has the general form:

$$\boxed{\begin{aligned}\frac{d\rho_S}{dt} &= \mathcal{L}_t[\rho_S(t)] \\ &= -\frac{i}{\hbar}[\hat{H},\hat{\rho}] + \sum_\mu -\frac{1}{2}\left(\hat{L}^\dagger_\mu \hat{L}_\mu \hat{\rho} + \hat{\rho}\hat{L}^\dagger_\mu \hat{L}_\mu\right) + \hat{L}_\mu \hat{\rho} \hat{L}^\dagger_\mu\end{aligned}}$$

This important form of the Master equation is known as the "Lindblad form", and the operators $\{\hat{L}_\mu\}$ are known as Lindblad operators

For a time-independent set of $\hat{H}$ and $\hat{L}_\mu$ the formal superoperator solution $\rho_S(t) = e^{\mathcal{L}t}[\rho_S(0)]$

<u>Two Important Examples</u>

- Two level atom damped in the E-M vacuum,
  - One Lindblad operator $\hat{L} = \sqrt{\Gamma}\,\hat{\sigma}_-$ $\quad \Gamma =$ Einstein-A

$$\Rightarrow \frac{d\rho_A}{dt} = -\frac{i}{\hbar}[\hat{H}_A,\hat{\rho}] - \frac{\Gamma}{2}\left(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho}_A + \hat{\rho}_A\hat{\sigma}_+\hat{\sigma}_-\right) + \Gamma\hat{\sigma}_-\rho_A\hat{\sigma}_+$$

- Damped Simple Harmonic oscillator @ zero temperature
  - One Lindblad operator $\hat{L} = \sqrt{\Gamma}\,\hat{a}^\dagger$ $\quad \Gamma =$ decay

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H},\hat{\rho}] - \frac{\Gamma}{2}(\hat{a}^\dagger\hat{a}\hat{\rho} + \hat{\rho}\hat{a}^\dagger\hat{a}) + \Gamma\hat{a}\hat{\rho}a^\dagger$$

To get a basic understanding of the Lindblad operators, consider that the relaxation terms in the Master equation donot Couple populations (diagonal matrix elements) and coherences (off-diagonal matrix elements). Thus we can look at rate equations for the populations, ignoring the off-diagonal elements of the density operator.

Consider the basis of energy eigenstates $\{|g\rangle\}$

Let $\hat{\rho} = \sum_g P_g |g\rangle\langle g|$ (ignoring off-diagonal)

$$\Rightarrow \dot{P}_g = \langle g| \frac{d\hat{\rho}}{dt} |g\rangle$$

$$= -\gamma_g P_g + \sum_{g'} \gamma_{g\leftarrow g'} P_{g'}$$

where $\gamma_{g\leftarrow g'} \equiv \sum_\mu |\langle g| \hat{L}_\mu |g'\rangle|^2$

$$\text{and } \gamma_g = \sum_\mu \langle g| \hat{L}_\mu^\dagger \hat{L}_\mu |g\rangle$$

$$= \sum_{g'} \left(\sum_\mu |\langle g'| \hat{L}_\mu |g\rangle|^2 \right)$$

$$= \sum_{g'} \gamma_{g'\leftarrow g}$$

Thus the terms $-\frac{1}{2}\sum_\mu (\hat{L}_\mu^\dagger \hat{L}_\mu \hat{\rho} + \hat{\rho}\hat{L}_\mu^\dagger \hat{L}_\mu)$ lead to <u>decay</u> out of level j, γ_j whereas the terms $\sum_\mu \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$ lead to <u>feeding</u> of population into level j from all other levels j', $\gamma_{j \leftarrow j'}$.

These two different terms will be seen to have different physical signifance in terms of "quantum trajectories" to be studied soon.