Lecture #23: Quantum Trajectories (II)

In the last lecture we developed the following interpretation of the evolution of the density operator governed by a Lindblad master equation. The environment is performing a continuous measurement of the state. If the state started pure and we had complete access to the measurement record, then the state would stay pure, but evolve stochastically in a manner conditioned on the random measurement outcomes. This stochastic dense evolution is known as a quantum trajectory. However, we don't have access to the measurement record by the very nature of an "environment". Tracing over the environment is thus equivalent to averaging over all possible quantum trajectories weighted by the probability of such a record occurring. This provides an algorithm for calculating $\hat{\rho}(t)$:

1. For $\hat{\rho}(0) = |\Psi(0)\rangle \langle \Psi(0)|$, calculate the stochastic evolution of $|\Psi(t)\rangle$ as dictated by the possible quantum jumps the system can make and their relative probabilities.

$$\hat{\rho}(t) = \langle \langle \text{Average over } \text{N trajectories} \rangle$$

Note: If $\rho(0) = \sum P_k |\Psi_k\rangle \langle \Psi_k|$, we perform the simulation for each $|\Psi_k\rangle$ and average...
As a concrete example, we look at a two-level atom coupled to the vacuum evolving as

\[ \frac{d\rho}{dt} = -\frac{i}{\hbar} \left[ \hat{A}_{\text{eff}}, \rho \right] + \Gamma_{-} \rho \frac{\sigma_{-}}{2} \]

where

\[ \hat{A}_{\text{eff}} = \hat{H} - \frac{\hbar^2}{2} \hat{L} = \hat{H} - i\frac{\hbar}{2} \Gamma \sigma_{+} \sigma_{-} \]

\[ = \hat{H} - i\frac{\hbar}{2} \lambda \text{Heck} \]

where \( \hat{L} = \sqrt{\Gamma} \hat{\sigma}_{-} \) is the lone Lindblad "jump" operator.

The QMCWF algorithm for stochastic evolution of a quantum trajectory can be expressed as

(i) Suppose \( \hat{\psi}(0) = |\psi(0)\rangle \langle \psi(0)| \)

(ii) Choose a very small time step \( \Delta t \ll \min \left( \frac{1}{\Gamma}, \frac{\Delta E}{E_{\gamma}} \right) \)

(iii) At any time \( t \), calculate the probability of a jump \( \delta t \)

\[ dp(t) = \langle \psi(t) \hat{L}^{2} \psi(t) \rangle dt = \int dt \text{K}_{\text{E}} |\hat{\psi}(t)|^{2} \]

(iv) Simulate the probability with a random number generator

(a) Pick a random number \( \epsilon \), \( 0 < \epsilon < 1 \)

(b) If \( \epsilon < dp(t) \) \( \Rightarrow \) Jump \( |\psi(t)\rangle \rightarrow \hat{\sigma}_{-} |\psi(t)\rangle \)

(c) Else \( |\psi(t+\Delta t)\rangle = e^{-i\hat{A}_{\text{eff}} \Delta t} |\psi(t)\rangle \)

(v) Renormalize \( |\psi(t+\Delta t)\rangle = \frac{|\psi(t+\Delta t)\rangle}{\| |\psi(t+\Delta t)\|} \)

(vi) Go to (iii), repeat until final time \( t_f \)

The example above includes the case of the two-level atom driven by a laser near resonance and decaying via spontaneous emission (damped Rabi oscillations). Here the effective Hamiltonian

\[
H_{\text{eff}} = -\hbar (\Delta + i \frac{\Gamma}{2}) \sigma_+ \sigma_- + \frac{\hbar}{2} \Omega \left( \sigma_+ + \sigma_- \right)
\]

(in the RWA)

A sample quantum trajectory might look like

\[
\text{Pe} \quad 1 \quad \cdots \quad \cdots \\
\text{Jump} \quad \text{Jump} \quad \text{Jump} \quad \text{Jump} \quad \text{Jump} \quad t
\]

\[
\langle \mathcal{E} \rangle_1 \quad \frac{1}{2}
\]

Once we average over many trajectories we see dephasing and thus damped Rabi oscillation.

Here the "smooth" curve is the exact (Corey) solution to the master equation and the "noisy" curve is the actual average over many trajectories. The "noise" is due to the fact that we only average over a finite number of trajectories, whereas the correspondence of QMEWF simulation and the exact master equation is for infinite number of trajectories. In fact this is
a numerical experiment, equivalent to a real
experiment with a finite number of atoms. The 
fluorescence one would measure in a true experiment 
would exhibit finite signal-to-noise due to 
counting statistics.

To determine the efficacy of the QMCWF simulation, 
we must do a proper error analysis (see Madiman notes).

**General Case: Multiple Lindblad Jump Operators**

Master equation:

\[
\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar} \left[ \hat{H}_{\text{eff}}, \hat{\rho} \right] + \sum_{\mu=1}^{M} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger
\]

\[
\hat{H}_{\text{eff}} = \hat{H} - i\frac{\hbar}{2} \sum_{\mu} \hat{L}_\mu^\dagger \hat{L}_\mu
\]

e.g. Two-level atom plus thermal reservoir

\[
\hat{L}_1 = \sqrt{\Gamma (\hbar + 1)} \hat{\sigma}_- , \quad \hat{L}_2 = \sqrt{\Gamma \hbar} \hat{\sigma}_+
\]

The algorithm is shown schematically:

\[
\begin{align*}
|\psi(t)\rangle & \quad \xrightarrow{\text{no-jump}} \quad 12\psi(t+dt) = e^{\frac{-i}{\hbar} \hat{H}_{\text{eff}} dt} |\psi(t)\rangle \\
12\psi(t+dt) & \quad \xrightarrow{\text{jump}} \quad S_p |\psi(t)\rangle \\
S_{pM} \quad 12\psi(t+dt) & \quad = \frac{\hat{L}_\mu \langle 12\psi(t) | \mu \rangle}{\| e^{\frac{-i}{\hbar} \hat{H}_{\text{eff}} dt} |\psi(t)\rangle \|}
\end{align*}
\]

First we pick a random \( \xi \) \( 0 \leq \xi \leq 1 \). Then we calculate

\[S_M = \langle 12\psi(t) | \sum \hat{L}_\mu \rangle, \quad \text{If } S_M > \xi \Rightarrow \text{Jump. If Jump choose another random } \eta, \mu^{th} \text{ jump } \frac{\hat{L}_\mu}{S_M} \eta \]
Further considerations

The procedure outlined, while formally correct, is not necessarily very efficient, only accurate to order $\Delta t$ in the true time step. For very small $\Delta t$, the probability of a jump $\sigma_p = \sum \Delta t / \Delta t < 1$.

Most of the time a jump does not occur and we "waste" "random" numbers in our algorithm.

A more efficient procedure is to use a "waiting time" or "delay function". For the simple case studies in Lectures 2, decaying atom in the absence of a driving field, we found the probability that no jump occurs for a duration $\tau$

$$P(\tau) = |\langle \Psi(0) | e^{i \frac{\text{H}_{\text{eff}} \tau}{\hbar}} | \Psi(0) \rangle|^2$$

More generally, we have $P(\tau)$ satisfying

$$P(\tau + d\tau) = S_{0} (\tau, d\tau) \ P(\tau)$$

where

$$1_{\Psi(0)} = \frac{e^{i \frac{\text{H}_{\text{eff}} \tau}{\hbar}} | \Psi(0) \rangle}{\sqrt{\langle \Psi(0) | e^{i \frac{\text{H}_{\text{eff}} \tau}{\hbar}} | \Psi(0) \rangle}}$$

Solution: $P(\tau) = |\langle \Psi(0) | e^{-i \frac{\text{H}_{\text{eff}} \tau}{\hbar}} | \Psi(0) \rangle|^2$

Thus, an efficient MCWF simulation can be generated by picking a random number and evolving $|\Psi(t)\rangle$ deterministically by $e^{-i \text{H}_{\text{eff}} \Delta t}$ until $|\Psi(t)\rangle |^2 < \epsilon$. Then jump, renormalize, and repeat.
Example: Multilevel atom with Zeeman degeneracy

Consider an atom with good quantum number $J = \text{total angular momentum}$, $M_J = \text{projection along a "quantization axis"}$ $z$.

\[ J = 2 \quad \begin{array}{cccc} -2 & -1 & 0 & 1 & 2 \end{array} \quad \text{excited manifold } |e, J, M_J \rangle \\
J = 1 \quad \begin{array}{cccc} -1 & 0 & 1 \end{array} \quad \text{ground manifold } |g, J, M_g \rangle \\

States $|e, J, M_J \rangle$ and $|g, J, M_g \rangle$ are connected by electric dipole transitions according to selection rules: $M_e - M_g = \Delta J = 0, \pm 1$. These transitions are driven by photons of polarization $\hat{e^\perp}$ where

\[ \hat{e^0} = \hat{e^z}, \quad \hat{e^\perp} = \frac{1}{\sqrt{2}} \left( \hat{e^x} \pm \hat{e^y} \right) \quad \text{w.r.t. } z = \text{axis} \]

The dipole matrix elements are determined by the Wigner-Eckart theorem

\[
\langle e; J, M_J | \hat{D}_q | g; J, M_g \rangle = \langle J, M_J | J, M_g, \Delta J = \pm 1, \pm 1 | \hat{e^\perp} \rangle \\
\text{Clebsch-Gordan coefficient for addition of angular momentum}
\]

For the atom with Zeeman degeneracy, we thus have multiple jump operators corresponding to emission of $q = 0, \pm 1, -1$ photons

\[
\hat{L}_q = \sqrt{q} \hat{D}_q = \sqrt{q} \sum_{M_J, M_g} \langle e; J, M_J | \hat{D}_q | g; J, M_g \rangle \quad |e; J, M_J \rangle \langle g; J, M_g | q, \Delta J = 0, \pm 1, \pm 1 \rangle
\]
Note \( \langle J_g M_g | \hat{L}_q | J_e M_e \rangle \) = \( \Gamma_q = \Gamma_{J_e M_e J_g M_g} \)

= Spontaneous emission rate from \( | J_e M_e \rangle \to | J_g M_g \rangle \) through the emission of a photon with polarization \( q \)

\[ \sum_q \Gamma_q = \Gamma \sum_q \langle J_e M_e | \hat{J}_q M_g \hat{J}_q | J_e M_e \rangle \]

= \( \Gamma \)

\( \Rightarrow \) Master Equation

\[ \frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \Gamma \sum_q \hat{D}_q \hat{\rho} \hat{D}^+_q \]

\( \hat{H}_{\text{eff}} = \hat{H} - \frac{i \hbar \Gamma}{2} \sum_q \hat{D}^+_q \hat{D}_q = \hat{H} - \frac{i \hbar \Gamma}{2} \sum_q | J_e M_e \rangle \langle J_e M_e | \)

= \( \hat{H} - \frac{i \hbar \Gamma}{2} \hat{P}_e \) is projector onto excited manifold

**Monte-Carlo Simulation**

- With probability \( dp = \Gamma dt \langle \hat{\psi} | \hat{P}_e | \psi \rangle \) a jump occurs

- With probability \( dp_q = \Gamma dt \langle \hat{\psi} | \hat{D}_q^+ \hat{D}_q | \psi \rangle \)

\[ \Gamma dt \sum_q \langle J_e M_e | \hat{D}_q^+ \hat{D}_q | J_e M_e \rangle \]

= \( \Gamma dt \sum_q \langle J_e M_e | 1_q J_g M_g = J_e - q \rangle \langle J_e M_e | 1_{J_e M_e} \psi \rangle \)

A photon of polarization \( q \) is emitted
Example: Coherent Population Trapping (CPT)

Consider an atom with $J_g = 1$ and $J_e = 1$

The Wigner-–Eckart coefficients are shown.
Note the $|J_g=1, M_g=0\rangle \Rightarrow |J_e=1, M_e=0\rangle$ has a zero C-G coefficient.

Suppose the atom is driven by linearly polarized light. Let us choose the quantization axis along $\vec{k} = \hat{z}$ and polarization $\vec{\varepsilon}_y = \frac{\varepsilon_+ + \varepsilon_-}{i \sqrt{2}}$. There is only $\sigma^+ \text{ light}$ (no $\sigma^- \text{ light}$).

The state $|2 \text{ dark}\rangle = (|g, -1\rangle + |g, +1\rangle) \sqrt{\frac{1}{2}}$

is a dark state; it is uncoupled from the base due to destructive interference.

This is a "Lambda" system of the sort studied in P.S. #3.
If an atom starts in an arbitrary initial state, it will, through absorption and spontaneous emission, eventually land in the dark state. This process is known as laser spectroscopy as "coherent population trapping" (CPT).

That is, in steady state $\rho_{ss} \to |\Psi_{dark}\rangle \langle \Psi_{dark}|$

We can see this in a quantum trajectory in the following way. The master equation is

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar} [\hat{H}_{\text{eff}}, \hat{\rho}] + \Gamma \sum_q D_q^+ \rho D_q$$

$$\hat{H}_{\text{eff}} = \hat{H} - i \frac{\hbar}{2} \hat{P} e$$

In resonance in RWA: $\hat{H} = \hbar \frac{\Omega}{2} (D_y + D_y^\dagger)$

where $D_y^\dagger = \sum_q D_q^\dagger (\hat{e}_q \cdot \hat{e}_y)$

Suppose at $t=0$ $|\Psi(0)\rangle = |g, m=-1\rangle$. Let us plot $|\langle \Psi_{dark} | \Psi(t) \rangle|^2 = N_{dark}(t)$ for a $g$ trajectory.
As expected, in steady state the population is trapped in the dark state. However, it reached this state through a long series of null measurements.

Because $|1\rangle \rightarrow |1\rangle$ is forbidden, jumps put the atom into $|1g\rangle = |\pm 1\rangle$

However two important states:

$|12_Dark\rangle = |1\rangle + |1-1\rangle \over \sqrt{2}$

$|12_Bright\rangle = |1\rangle - |1-1\rangle \over \sqrt{2}$

In the absence of a detector:

$|\Psi(t)\rangle = C_D(t) |12_Dark\rangle + C_B(t) |12_Bright(t)\rangle$

As time increases $C_B(t)$ decays, $C_D(t)$ rises and asymptotes.

Thus, in this unravelling evolution to the dark state is seen as a contentious violation of the state. After averaging over many trajectories:

$N_{dark}(t) \equiv \langle \Psi_{dark}(t) | \Psi_{dark} \rangle$