# Lecture #23: Quantum Trajectories (II)

In the last lecture we developed the following interpretation of the evolution of the density operator governed by a Lindblad master equation. The environment is performing a continuous measurement of the state. If the state started pure and we had complete access to the measurement record, then the state would stay pure, but evolve stochastically in a manner conditioned on the random measurement outcomes. This stochastic time evolution is known as a quantum trajectory. However, we don't have access to the measurement record by the very nature of an "environment". Tracing over the environment is thus equivalent to averaging over all possible quantum trajectories weighted by the probability of such a record occuring. This provides an algorithm for calculating $\hat{\rho}(t)$

- For $\hat{\rho}(0) = |\psi(0)\rangle\langle\psi(0)|$, calculate the stochastic evolution of $|\psi(t)\rangle$ as dictated by the possible quantum jumps the system can make and there relative probabilities
- $\hat{\rho}(t) = \ll |\psi(t)\rangle\langle\psi(t)| \gg$ ← average over $N \to \infty$ trajectories

Note: If $\rho(0) = \sum P_\alpha |\psi_\alpha\rangle\langle\psi_\alpha|$ (mixed) we perform the simulation for each $|\psi_\alpha\rangle$ and average

As a concrete example, we looked at a two-level atom coupled to the vacuum evolving as

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}] + \Gamma \hat{\sigma}_- \hat{\rho} \hat{\sigma}_+$$

where $\hat{H}_{eff} = \hat{H} - \frac{i\hbar}{2}\hat{L}^\dagger \hat{L} = \hat{H} - i\hbar\frac{\Gamma}{2}\hat{\sigma}_+\hat{\sigma}_-$

$$= \hat{H} - i\hbar\frac{\Gamma}{2}|e\rangle\langle e|$$

where $\hat{L} = \sqrt{\Gamma}\,\hat{\sigma}_-$ is the lone Lindblad "jump" operator

The QMCWF algorithm for stochastic evolution of a quantum trajectory can be expressed as

(i) Suppose $\hat{\rho}(0) = |\psi(0)\rangle\langle\psi(0)|$

(ii) Choose a very small time step $\Delta t \ll \min\left(\frac{1}{\Gamma}, \frac{\hbar}{\Delta E}\right)$ ← energy gap of $\hat{H}$

(iii) At any time $t$, calculate the probability of a jump as $dp(t) = \langle\psi(t)|\hat{L}^\dagger\hat{L}|\psi(t)\rangle dt = \Gamma dt\,|\langle e|\psi(t)\rangle|^2$

(iv) Simulate the probability with a random number generator

(a) Pick a random number $\epsilon$, $0 < \epsilon < 1$

(b) If $\epsilon < dp(t) \Rightarrow$ Jump $|\tilde{\psi}(t)\rangle = \hat{\sigma}_-|\psi(t)\rangle$

(c) Else $|\tilde{\psi}(t+\Delta t)\rangle = e^{-\frac{i}{\hbar}\hat{H}_{eff}\Delta t}|\psi(t)\rangle$

(v) Renormalize $|\psi(t+\Delta t)\rangle = \dfrac{|\tilde{\psi}(t+\Delta t)\rangle}{\| |\tilde{\psi}(t+\Delta t)\rangle \|}$

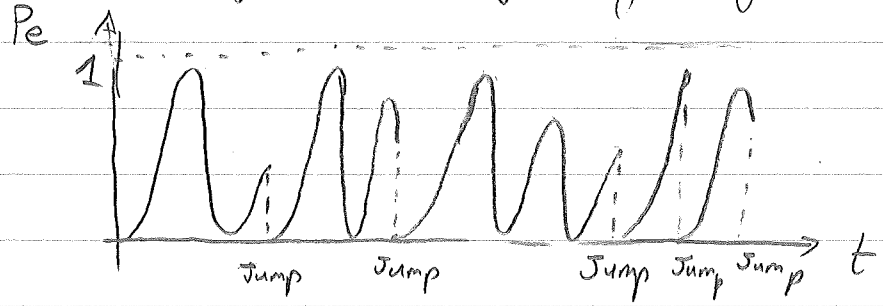
(vi) Goto (iii), Repeat until final time $t_f$

The example above includes the case of the two-level atom driven by a laser near resonance and decaying via spontaneous emission (damped Rabi oscillations). Here the effective Hamiltonian

$$\hat{H}_{eff} = -\hbar\left(\Delta + i\frac{\Gamma}{2}\right)|e\rangle\langle e| - \frac{\hbar\Omega}{2}\left(\hat{\sigma}_+ + \hat{\sigma}_-\right)$$

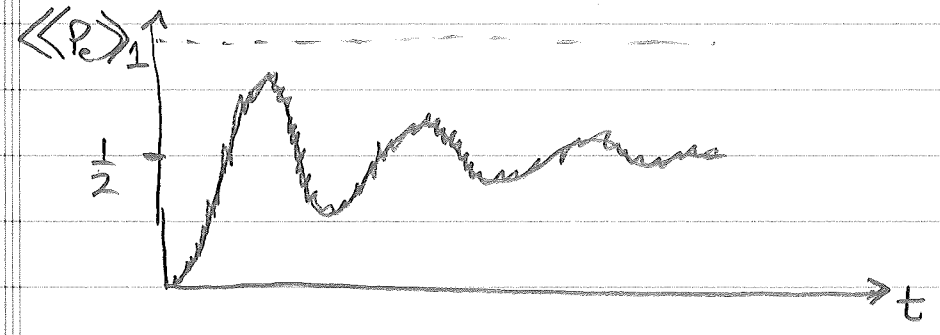
(in the RWA)

A sample quantum trajectory might look like




e.g.
$\Delta = 0$
$\Omega = 10\Gamma$

Once we average over many trajectories we see dephasing and thus damped Rabi oscillation




Here the "smooth" curve is the exact (Torey) solution to the master equation and the "noisy" curve is the actual average over many trajectories. The "noise" is due to the fact that we only average over a finite number of trajectories, whereas the correspondence of QMCWF simulation and the exact master equation is for infinite number of trajectories. In fact this is

a numerical experiment, equivalent to a real experiment with a finite number of atoms. The fluorescence one would measure in a true experiment would exhibit finite signal-to-noise due to counting statistics

To determine the efficacy of the QMCWF simulation, we must do a proper error analysis (see Mølmer notes).

## General Case: Multiple Lindblad Jump Operators

-Master equation:

$$\frac{\partial \hat{\rho}}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_{eff}, \hat{\rho}\right] + \sum_{\mu=1}^{M} \hat{L}_\mu \hat{\rho} \hat{L}_\mu^\dagger$$

$$\hat{H}_{eff} = \hat{H} - i\frac{\hbar}{2}\sum_\mu \hat{L}_\mu^\dagger \hat{L}_\mu$$

e.g. Two-level atom plus thermal reservoir

$$\hat{L}_1 = \sqrt{\Gamma(\bar{n}+1)}\,\hat{\sigma}_- \quad , \quad \hat{L}_2 = \sqrt{\Gamma\bar{n}}\,\hat{\sigma}_+$$

The algorithm is shown schematically:

$|\psi(t)\rangle$ → (1−δp) no-jump: $|\psi(t+dt)\rangle = \dfrac{e^{-\frac{i}{\hbar}\hat{H}_{eff}\delta t}|\psi(t)\rangle}{\| e^{-\frac{i}{\hbar}\hat{H}_{eff}\delta t}|\psi(t)\rangle\|}$

(δp) jump → $\delta p_1$ … $\delta p_M$: $|\psi(t+dt)\rangle = \dfrac{\hat{L}_\mu|\psi(t)\rangle}{\|\hat{L}_\mu|\psi(t)\rangle\|}$

First we pick a random # $0<\epsilon<1$. Then we calculate $\delta p_\mu = \langle\psi|\hat{L}_\mu^\dagger\hat{L}_\mu|\psi\rangle$. If $\delta p = \Sigma\delta p_\mu > \epsilon \Rightarrow$ Jump. If Jump choose another random # $\eta$. $\mu^{th}$ jump $\frac{\delta p_\mu}{\delta p} \geq \eta$

## Further considerations

The procedure outlined, while formally correct is not necessarily very efficient, only accurate to order $\delta t$ in the true time step. For very small $\delta t$, the probability of a jump $\delta p = \sum_m \hat{L}_m^\dagger \hat{L}_m \, \delta t \ll 1$.

$\Rightarrow$ Most of the time a jump does not occur and we "waste" "random" numbers in our algorithm.

A more efficient procedure is to use a "waiting time" or "delay function". For the simple case studies in Lect #21, decaying atom in the absence of a driving field we found the probability that no jump occurs for a duration $\tau$

$$P(\tau) = |c_g|^2 + e^{-\Gamma\tau}|c_e|^2 = \| e^{-iH_{eff}\tau/\hbar}|\psi(0)\rangle\|^2$$

$$|\psi(0)\rangle = c_g|g\rangle + c_e|e\rangle$$

More generally, we have $P(\tau)$ satisfying

$$P(\tau+d\tau) = \delta p_0(\tau, d\tau)\, P(\tau)$$

$$= \langle\psi(\tau)| e^{\frac{i}{\hbar}(H_{eff} - H_{eff}^\dagger)d\tau} |\psi(\tau)\rangle \, P(\tau)$$

where $|\psi(\tau)\rangle = \dfrac{e^{-\frac{i}{\hbar}H_{eff}\tau}|\psi(0)\rangle}{\sqrt{\langle\psi(0)| e^{\frac{i}{\hbar}(H_{eff}^\dagger - H_{eff})\tau}|\psi(0)\rangle}}$

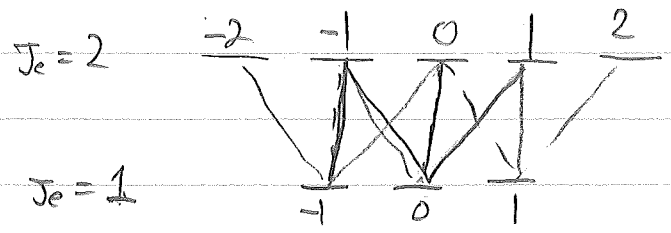
Solution: $P(\tau) = \| e^{-iH_{eff}\tau}|\psi(0)\rangle\|^2$

Thus, an efficient MCWF simulation can be generated picking a random number and evolving $|\psi\rangle$ deterministically by $e^{-iH_{eff}t/\hbar}$ until $\|\tilde{\psi}(\tau)\|^2 \equiv \langle\psi(0)| e^{+\frac{i}{\hbar}(H_{eff}^\dagger - H_{eff})\tau}|\psi(0)\rangle < \epsilon$.

Then jump, renormalize, and repeat.

## Example: Multilevel atom with Zeeman degeneracy

Consider an atom with good quantum number $J$ = total angular momentum, $M_J$ = projection along a "quantization axis" $z$.




excited manifold $|e, J_e, M_e\rangle$

ground manifold $|g, J_g, M_g\rangle$

States $|e, J_e, m_e\rangle$ and $|g, J_g, m_g\rangle$ are connected by electric dipole transitions according to selection rules: $M_e - M_g \equiv q = 0, \pm 1$. These transitions are driven by photons of polarization $\vec{e}_q$ where

$$\vec{e}_0 = \vec{e}_z, \quad \vec{e}_{\pm 1} = \mp \frac{(\vec{e}_x \pm i\vec{e}_y)}{\sqrt{2}} \quad \text{w.r.t } z\text{-axis}.$$

The dipole matrix elements are determined by the Wigner-Eckart theorem

$$\langle e; J_e, m_e | \hat{d}_q | g; J_g m_g \rangle = \underbrace{\langle J_e m_e | J_g m_g; 1q \rangle}_{\text{Clebsch-Gordan coefficient}} \underbrace{\langle e J_e \| d \| e J_g \rangle}_{\text{reduced matrix element}}$$

for addition of angular momentum

For the atom with Zeeman degeneracy we thus have multiple jump operators corresponding to emission of $q = 0, +1, -1$ photons

$$\hat{L}_q = \sqrt{\Gamma}\, \hat{D}_q = \sqrt{\Gamma} \sum_{m_e, m_g} \langle J_e \| d \| J_g M_g 1q \rangle \, |J_e M_e\rangle\langle J_g M_g|$$

- Note $|\langle J_g M_g | \hat{\Sigma}_q | J_e M_e \rangle|^2 = \Gamma_q = \Gamma |\langle J_e M_e | J_g M_g 1_q \rangle|^2$

= Spontaneous emission rate from $|J_e M_e\rangle \to |J_g M_g\rangle$ through the emission of a photon with polarization $q$

- $\sum_q \Gamma_q = \Gamma \underbrace{\sum_q \langle J_e M_e | J_g M_g 1_q \rangle \langle J_g M_g 1_q | J_e M_e \rangle}_{=1}$

$= \Gamma$

$\Rightarrow$ Master Equation

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}]' + \Gamma \sum_q \hat{D}_q \hat{\rho} \hat{D}_q^\dagger$$

$$\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma}{2} \sum_q \hat{D}_q^\dagger \hat{D}_q = \hat{H} - i\frac{\hbar\Gamma}{2} \sum_e |J_e M_e\rangle\langle J_e M_e|$$

$$= \hat{H} - i\frac{\hbar\Gamma}{2} \hat{P}_e \quad \leftarrow \text{projector onto excited manifold}$$

Monte-Carlo Simulation

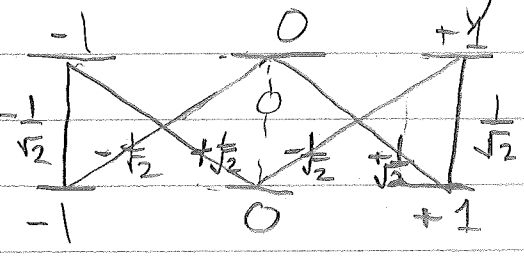
- With probability $dp = \Gamma dt \langle\psi|P_e|\psi\rangle$ a jump occurs
- With probability $dp_q = \Gamma dt \langle\psi| D_q^\dagger D_q |\psi\rangle$

$$= \Gamma dt \sum_{M_e} |\langle J M_e | 1_q \; J_g \; M_g = M_e - q\rangle|^2 \langle\psi|J M_e\rangle\langle J M_e|\psi\rangle$$

a photon of polarization $q$ is emitted

## Example: Coherent-Population-Trapping (CPT)

Consider an atom with $J_g = 1$ and $J_e = 1$

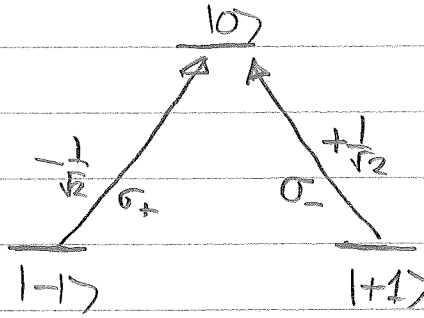



The Clebsch-Gordan coefficients are shown. Note the $|J_g=1, M_g=0\rangle \Rightarrow |J_e=1, M_e=0\rangle$ has a zero C-G coefficient

Suppose the atom is driven by linearly polarized light. Let us choose the quantization axis along $\vec{k} = \vec{z}$ and polarization $\vec{e}_y = \frac{\vec{e}_+ + \vec{e}_-}{i\sqrt{2}}$. There is only $\sigma_\pm$ light (no $\vec{e}_0$ : $\pi$-light)

The state $|\psi_{dark}\rangle = \frac{|g,-1\rangle + |g,+1\rangle}{\sqrt{2}}$

is a dark state; it is uncoupled from the base due to destructive interference




This is a "Lambda system" system of the sort studied in P.S. #3

If an atom starts in an arbitrary initial state, it will, through absorption and spontaneous emission, eventually land in the dark state. This process is known is laser spectroscopy as "coherent population trapping" (CPT).

That is, in steady state $\hat{\rho}_{ss} \to |\psi_{dark}\rangle\langle\psi_{dark}|$

We can see this in a quantum trajectory in the following way. The master equation is

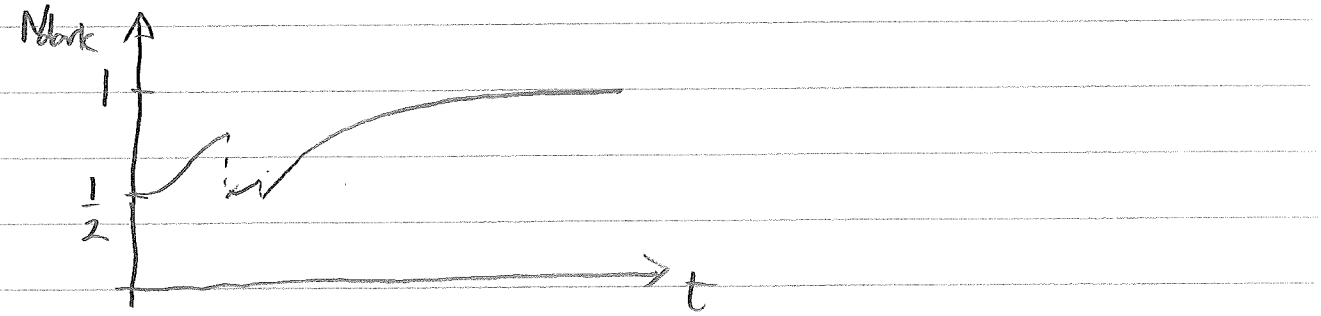
$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}_{eff}, \hat{\rho}] + \Gamma \sum_q \hat{D}_q \hat{\rho} \hat{D}_q^\dagger$$

$$\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma}{2}\hat{P}_e$$

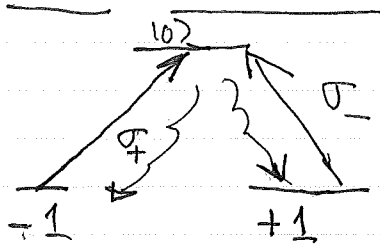
On resonance in RWA: $\hat{H} = \frac{\hbar\Omega}{2}(D_y + D_y^\dagger)$

where $D_y^\dagger = \sum_q D_q^\dagger (\vec{e}_q \cdot \vec{e}_y)^*$ Laser polarization

Suppose at $t=0$ $|\psi(0)\rangle = |g, m=-1\rangle$. Let us plot $|\langle\psi_{dark}|\psi(t)\rangle|^2 = N_{dark}(t)$ for a q. traject.



As expected, in steady state the population is trapped in the dark state. However, it reached this state through a long series a null measurements




Because $|0\rangle \to |0\rangle$ is forbidden, jumps put the atom into $|m_g\rangle = |\pm 1\rangle$

However two importants states

$$|\psi_{Dark}\rangle = \frac{|1\rangle + |-1\rangle}{\sqrt{2}}$$

$$|\psi_{Bright}\rangle = \frac{|1\rangle - |-1\rangle}{\sqrt{2}}$$

In the absence of a detector

$$|\psi(t)\rangle = C_D(t)\,|\psi_{Dark}\rangle + C_B(t)\,|\psi_{Bright}(t)\rangle$$

As time increases $C_B(t)$ decays
$C_D(t)$ rises and asymptotes.

Thus, in this unravelling, evolution to the dark state is seen as a continuous rotation of the state. After averaging over many trajectories