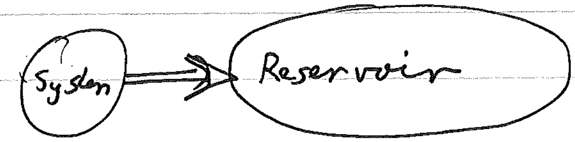
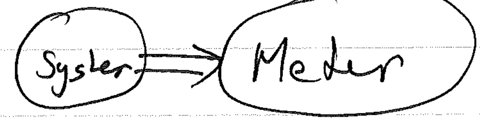
Lecture 30: Open Quantum Systems and Continuous Measurement

There is an intimate relationships between the study of open quantum systems and the theory of quantum measurement. An "open quantum system" is a system ~~coon~~ coupled to a reservoir. The reservoir is considered to be one with an effectively infinite number of degrees of freedom so that energy and information flow irreversibly from the "system" to the "bath"

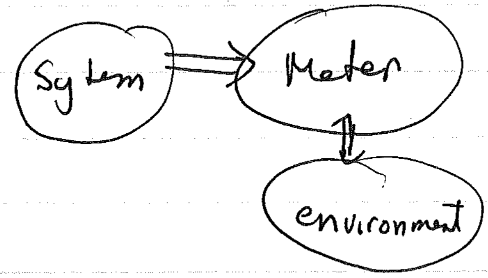


In a measurement the "system" is coupled to a "meter". The states of the meter are correlated to states of system in such a way the ~~a~~ an observation of the meter gives information about the state of the system



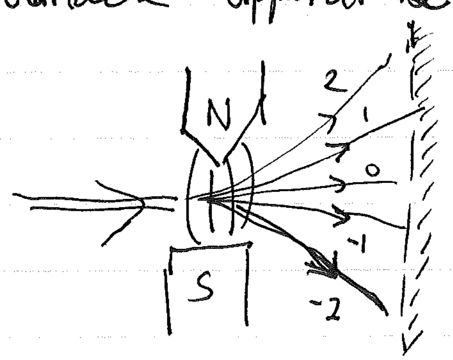
A true measurement is also considered to be irreversible; Once the meter is "macroscopic" enough to be observed, coherence between the different alternatives is to be lost. ~~The meter's degrees of freedom are~~ How this happens returns us again the system + reservoir paradigm. Indeed, in most circumstances, the measurement arises from an "amplification" of a quantum event.

The paradigm of "decoherence" explaining the lack a quantum coherence in a system-meter interaction, with the "Schrödinger Cat" the most famous paradox, was studied and analyzed in detail by Zurek.



In quantum optics, there was considerable work on models of decoherence via ~~amplif~~ amplification. We will not have time to discuss that here. For detail see the text "Quantum Optics" by Walls + Milburn

<u>Von Neumann Model of Measurement</u>
Putting aside the deep issues at the foundations of quantum mechanics, let us recall the basic von Neumann model for a projective measurement. Consider the Stern-Garlach apparatus to measure a ~~spin~~ $\hat{J}_z$



In the example above $J=2$ with 5 possible eigenvalues of J_z

In this apparatus, the atom's motion acts a "meter" for its internal spin component along the direction of the magnetic field gradient. Of course the real measurement occurs when one "observes" the atom's position, which is correlated to its momentum after the gradient field. One can do this, in principle, by having the atoms ~~fluoresce~~ fluoresce as the pass a "sheet of light" (shown as a dashed screen).

To model ~~the~~ the basic measurement, consider a Hamiltonian for the spin-motion coupling

$$\hat{H}_{int} = \hbar\lambda \hat{z} \hat{J}_z$$

We assume an atom enters the S-G apparatus with its motion uncorrelated from its spin. For simplicity we take both to be in a pure state.

$$|\Psi\rangle_{total} = |\phi\rangle_{spin} \otimes |\psi_0\rangle_{motion}$$

$$= \sum_{m_J} c_{m_J} |m_J\rangle_{spin} \otimes |\psi_0\rangle_{motion}$$

After the interaction

$$e^{-i\hat{H}_{int}t_1/\hbar} |\Psi\rangle = \sum_{m_J} c_{m_J} |m_J\rangle \otimes \underbrace{e^{-i m_J t_1 \lambda \hat{z}} |\psi_0\rangle_{motion}}_{|\psi_{m_J}(t_1)\rangle_{motion}}$$

At this point, the system and meter are correlated, but the different meter states $\{|\Psi_{m_J}\rangle\}$ are not distinguishable by a position measurement,

$$|\langle \psi_{m_J'} | \psi_{m_J} \rangle|^2 = 1 \quad \forall m_J, m_J'$$

Intuitively, the different branches of the wave function receive a momentum impulse $p_{m_J} = \hbar \lambda t_1 m_J$.

The different beams separate after propagating for a sufficient time it free space

$$e^{-i\hat{H}_0(t)/\hbar} |\Psi_{m_J}(t_1)\rangle = e^{-i\frac{\hat{p}^2 t}{2m\hbar}} e^{-ip_{m_J}\hat{z}/\hbar} |\psi_0\rangle$$

$$= |\tilde{\psi}_{m_J}(t)\rangle$$

To good approximation (ignoring the spread of wavepacket, dispersion)

$$|\langle z | \tilde{\psi}_{m_J}(t) \rangle| = |\psi_0(z - \frac{p_{m_J}}{m} t)|$$

That is, the wave packet is displaced along z by an amount depending on the momentum impulse, an the free propagation time.

For times t s.t. $p_{m_J} t/m \gg (\Delta z)_{\psi_0}$

$$|\langle \tilde{\psi}_{m_J}(t) | \tilde{\psi}_{m_J'}(t) \rangle|^2 = \delta_{m_J m_J'}$$

When the different beams are completely distinguishable (orthogonal) the system (spin) and meter (position) are maximally entangled. At that point, measurement of the meter (atom position) determines the eigenvalue of $\hat{J}_z$, m_J.

- $$|\Psi(t)\rangle_{total} = \sum_{m_J} c_{m_J} |m_J\rangle \otimes |\tilde{\Psi}_{m_J}(t)\rangle$$

- Finding motional state $|\tilde{\Psi}_{m_J}\rangle$

$\Rightarrow$ Spin in state $|m_J\rangle$ with probability $|c_{m_J}|^2$

- Post measurement state $= |m_J\rangle = \dfrac{\hat{P}_{m_J}|\phi\rangle_{spin}}{\| \hat{P}_{m_J}|\phi\rangle_{spin} \|}$

This is a "strong" von Neuman projective measurement, Strong in the sense that the post-measurement state of the atom's spin is strongly perturbed from its initial state by a projection operator.

~~From the theory of completely positive maps, the Kraus operator~~

This types of measure is known as "quantum nondemotion" (QND) in that if we started in an eigenstate of $\hat{J}_z$, we remain there after the measurement. This is because $[\hat{J}_z, \hat{H}_{int}] = 0$.

From the perspective of the theory of completely positive maps, the Kraus operators become projection operators when the beams are fully resolvable. For shorter times

$\langle z | \hat{U} | \Psi_0 \rangle \neq$ Projection on spin

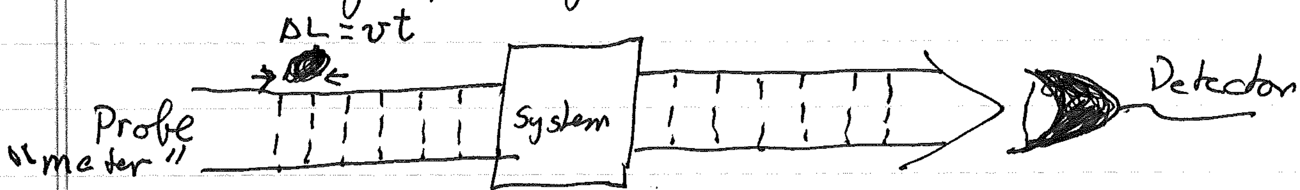
The measurement will not give complete information about the eigenvalue m_J, and will not lead to strong backaction on the spin state. These "weak measurements" are POVMs, they exemplify the information gain / disturbance tradeoff.

Continuous Weak Measurement

The discussion of measurement above assumed the measurement process to be instantaneous. In reality, all measurements occur over a finite time. Physically, a probe is detected an integrated over some time before enough information is gathered to register a choice of outcome. In the Stern-Gerlach example, fluorescense from the atoms must be integrated for suffient time to resolve the position and thus the spin state $|m_J\rangle$.

The main goal of this lecture, thus, is to describe the ~~theor~~ basic theory of continuous measurement.

Continuous Quantum Measurement can be understood in the following paradigm



A probe such as a laser beam, electron current, or other traveling beam is passed over the system to be measured where they interact. Downstream, the probe is detected in some way, extracting information that is correlated to the observable we seek to measure of the system. We break up the traveling probe into packets of length $\Delta L = v\Delta t$ where v is the speed at which the probe travels and Δt is the duration the detector must integrate in order to register the signal of interest.

We can thus think about this continuous measurement paradigm as a series of "weak measurements". Each probe packet is a meter which becomes weakly entangled with the system and thus there is little back-action is a single detection. Integrating over a long time, the information gain becomes greater and greater, eventually approaching a projective measurement in the asymptotic limit. In this way, we will see the von Neumann projection arising from a dynamical evolution (dynamical "collapse of the wave function").

We seek, thus, a stochastic differential equation that describes the dynamical measurement. We are guided by the theory of quantum trajectories discussed in the previous lectures. Though not a "reservoir" in the traditional sense, the probe carries away information in an irreversible way. Without detection of the probe, we expect decoherence of the system's density matrix. With perfect detection of the probe, we expect stochastic evolution of the system's pure state.

Continuous Measurement SSE

The key step in deriving the Stochastic Schrödinger Equation for continuous measurement to write the approach completely positive map, that is the Kraus operators.

We take a model interaction Hamiltonian for the continuous measurement of an observable $\hat{O}$ through coupling to observable $\hat{A}_i$ in the i^{th} packet of the probe.

$$\hat{H}_{int} = \sum_i \frac{\hbar \chi}{\Delta t} \hat{O} \otimes \hat{A}_i$$

Here we have assume $\hat{O}$ and $\hat{A}_i$ have been ~~scaled~~ scaled by characteristic units so they are dimensionless, and χ is a dimensionless coupling constant.

In this form, the Unitary Coupling matrix takes a Markov form, under the assumption that the ~~oper~~ observable $\hat{A}_i$ interacts with the system only for a time $(i-1)\Delta t \leq t < i\Delta t$.

$$\Rightarrow \quad U(t) = \prod_{i=1}^{t/\Delta t} U_i(\Delta t)$$

$$U_i(\Delta t) = e^{-i\chi \hat{O} \otimes \hat{A}_i}$$

Note: $\hat{O}$ is an "QND" variable

Then, the Kraus operator for a time step Δt given a fiducial state of the apparatus (probe packet) $|\alpha\rangle_i$ and measureme result $|m\rangle_i$:

$$\hat{M}(\Delta t) = {}_i\langle m| U_i(\Delta t) |\alpha\rangle_i = {}_i\langle m| e^{-i\chi \hat{O} \otimes \hat{A}_i} |\alpha\rangle_i$$

The assumption is that the measurement result are Gaussian distributed.

$$\hat{M}(\Delta t) \propto e^{-\frac{\kappa \Delta t}{4}(\hat{O} - O_m)^2}$$

So that the POVM element

$$\hat{E}(\Delta t) = \hat{M}^\dagger(\Delta t) \hat{M}(\Delta t) \propto e^{-\frac{\kappa \Delta t}{2}(\hat{O} - O_m)^2}$$

where O_m is the value of $\hat{O}$ that is correlated to m

The parameter κ is known as the "measurement strength". It corresponds to the rate at which we gain information about $\hat{O}$ through observation of the probe meter as measure by the rate at which the variance in the POVM element decreases (Note: My definition differs by a factor of 8 compared to other definitions - more below).

Given the Gaussian form of the POVM element, the probability of deducing value O_m of the system observable, given an outcome m of the probe, when the system is in state $|\phi\rangle$ is:

$$P_{O_m} = \langle\phi|\hat{E}(\Delta t)|\phi\rangle \propto e^{-\frac{\kappa\Delta t}{2}(O_m - \langle\hat{O}\rangle)^2}$$

That is, the probability is Gaussian distributed about mean $\langle\hat{O}\rangle$ with variance $\frac{1}{\kappa\Delta t}$.

In our measurement model, we will see that

$$\kappa \propto \chi^2 \frac{dN}{dt}$$

where $\frac{dN}{dt}$ is the mean flux of particles in the probe.

The unnormalized S.S.E. then follow from

$$|\tilde{\psi}(t+\Delta t)\rangle = \hat{M}(\Delta t)|\psi(t)\rangle \propto e^{-\frac{\kappa\Delta t}{4}(\hat{O}-\hat{O}_m)^2}|\psi(t)\rangle$$

As discussed in Lect. 29, the macroscopic nature of the probe being measured implies, by the central limit theorem, a signal plus Gaussian noise. The measurement result is thus a stochastic variable described according to a Weiner process

$$O_m = \langle \hat{O} \rangle + \frac{\Delta W}{\sqrt{\kappa}\,\Delta t}$$

ΔW = Weiner stochastic increment of zero mean and variance Δt

$\Rightarrow$ Variance of $O_m = \dfrac{\langle \Delta W^2 \rangle}{\kappa \Delta t^2} = \dfrac{1}{\kappa \Delta t}$ as expected

thus, the S.S.E. for the unnormalized states is

$$|\tilde{\psi}(t+\Delta t)\rangle = e^{-\frac{\kappa \Delta t}{4}\left(\hat{O} - \langle \hat{O} \rangle - \frac{\Delta W}{\sqrt{\kappa}\,\Delta t}\right)^2} |\psi(t)\rangle$$

$$= \exp\left\{-\frac{\kappa \Delta t}{4}\hat{O}^2 + \frac{\kappa \Delta t}{2}\langle \hat{O} \rangle \hat{O} + \frac{\sqrt{\kappa}}{2} dW \hat{O}\right\} |\psi(t)\rangle$$

We now take the limit $\Delta t \to dt$, $\Delta W \to dW$ as in lect 29 with "Ito rules" of stochastic differentials: $\begin{cases} dW^2 = dt \\ dt^2 = dt\,dW = 0 \end{cases}$

$\Rightarrow$ Unnormalized State:

$$|\tilde{\psi}(t+dt)\rangle = \left[1 - \frac{\kappa}{4} dt\, \hat{O}^2 + \left(\frac{\kappa}{2} dt \langle \hat{O} \rangle + \frac{\sqrt{\kappa}}{2} dW\right)\hat{O} + \frac{1}{2}\left(\frac{\sqrt{\kappa}}{2} dW\, \hat{O}\right)^2\right] |\psi\rangle$$

$$\Rightarrow \boxed{|\tilde{\psi}(t+dt)\rangle = \left[1 - \frac{\kappa}{8}dt\,\hat{O}^2 + \left(\frac{\kappa dt}{2}\langle\hat{O}\rangle + \frac{\sqrt{\kappa}}{2}dW\right)\hat{O}\right]|\psi\rangle}$$

We recognize this S.S.E. as equivalent to that we studied in Lect. 29, corresponding to the quantum trajectory under continuous homodyne measurement. In that case, the Linblad operator was $\hat{L} = \sqrt{\Gamma}\,\hat{a}$, as the unnormalized S.S.E. could be written

$$|\tilde{\psi}(t+dt)\rangle = \left[1 - \frac{dt}{2}\hat{L}^\dagger\hat{L} + \left(\langle\hat{L}^\dagger+\hat{L}\rangle dt + dW\right)\hat{L}\right]|\psi\rangle$$

In the continuous measurement case, we can define the Linblad operator $\hat{L} = \sqrt{\frac{\kappa}{4}}\,\hat{O}$

$$\Rightarrow |\tilde{\psi}(t+dt)\rangle = \left[1 - \frac{\kappa dt}{8}\hat{O}^2 + \left(\frac{\kappa dt}{2}\langle\hat{O}\rangle + \frac{\sqrt{\kappa}}{2}dW\right)\hat{O}\right]|\psi(t)\rangle$$

To normalize:

$$\|\tilde{\psi}(t+dt)\|^2 = 1 + \cancel{\frac{\kappa dt}{4}\langle\hat{O}^2\rangle} + \kappa dt\,\langle\hat{O}\rangle^2 + \sqrt{\kappa}\,\langle\hat{O}\rangle dW + \cancel{\frac{\kappa}{4}\langle\hat{O}^2\rangle dW^2}$$

$$\Rightarrow \frac{1}{\|\tilde{\psi}(t+dt)\|} = \left[\|\tilde{\psi}(t+dt)\|^2\right]^{-1/2} = 1 - \frac{\kappa dt}{2}\langle\hat{O}^2\rangle - \frac{\sqrt{\kappa}}{2}\langle\hat{O}\rangle dW + \frac{3}{8}\left(\sqrt{\kappa}\,\langle\hat{O}\rangle dW\right)^2$$

$$\Rightarrow \frac{1}{\|\tilde{\psi}(t+dt)\|} = 1 - \frac{\kappa dt}{8}\langle\hat{O}^2\rangle - \frac{\sqrt{\kappa}}{2}\langle\hat{O}\rangle dW$$

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Thus, the normalized S.S.E. for continuous measurement:

$$|\psi(t+dt)\rangle = \frac{|\tilde{\psi}(t+dt)\rangle}{\| \tilde{\psi}(t+dt) \|}$$

$$= \Big[1 - \frac{\kappa}{8}dt(\hat{O}^2 + \langle\hat{O}^2\rangle) + \frac{\sqrt{\kappa}}{2}(\hat{O} - \langle\hat{O}\rangle)dW - \frac{\kappa}{4}\hat{O}\langle\hat{O}\rangle dW^2\Big]|\psi(t)\rangle$$

$$\Rightarrow \boxed{|\psi(t+dt)\rangle = \Big[1 - \frac{\kappa}{8}dt(\hat{O} - \langle\hat{O}\rangle)^2 + \frac{\sqrt{\kappa}}{2}dW(\hat{O} - \langle\hat{O}\rangle)\Big]|\psi(t)\rangle}$$

This S.S.E. leads to "dynamical collapse" of the wave function. That is, given an arbitrary state, in steady state $|\psi\rangle$ converges on an eigenstate of $\hat{O}$. Which eigenstate it converges to is random, with probability of reach $|o\rangle$ given by the Born rule $|\langle o|\psi(t_0)\rangle|^2$.

We will show this in detail below. As a first step, note that is

$$\hat{O}|\psi\rangle = \langle\hat{O}\rangle|\psi\rangle,$$

the state does not change. This is true for an eigenstate of $\hat{O}$. The change in $|\psi\rangle$ depends of

$$(\hat{O} - \langle\hat{O}\rangle)|\psi\rangle$$

S.M.E. in Continuous measurement

We write the Stochastic Master equation for the conditioned state

$$\hat{\rho}_c(t+dt) = |\psi(t+dt)\rangle\langle\psi(t+dt)|$$

$$= \hat{\rho}_c(t) - \frac{\kappa}{8}dt\{(\hat{O}-\langle\hat{O}\rangle)^2, \hat{\rho}_c\}$$

$$+ \frac{\sqrt{\kappa}}{2}dW\{\hat{O}-\langle\hat{O}\rangle, \hat{\rho}_c\} + \frac{\kappa}{4}dW^2(\hat{O}-\langle\hat{O}\rangle)\hat{\rho}_c(\hat{O}-\langle\hat{O}\rangle)$$

$$= \hat{\rho}_c(t) - \frac{\kappa}{8}dt\left(\hat{O}^2\hat{\rho}_c + \hat{\rho}_c\hat{O}^2 - 2\hat{O}\hat{\rho}_c\hat{O}\right)$$

$$+ \frac{\sqrt{\kappa}}{2}dW\left(\hat{O}\hat{\rho}_c + \hat{\rho}_c\hat{O} - 2\langle\hat{O}\rangle\hat{\rho}_c\right)$$

$$\Rightarrow \boxed{d\rho_c = -\frac{\kappa}{8}dt\,[\hat{O},[\hat{O},\hat{\rho}_c]] + \frac{\sqrt{\kappa}}{2}dW\left(\hat{O}\hat{\rho}_c + \hat{\rho}_c\hat{O} - 2\langle\hat{O}\rangle\hat{\rho}_c\right)}$$

The first term (proportional to dt) is the Linblad master equation with Linblend operator $\hat{L} = \frac{\sqrt{\kappa}}{2}\hat{O}$

$$d\hat{\rho} = -\frac{dt}{2}\{\hat{\rho}, \hat{L}^\dagger\hat{L}\} + dt\,\hat{L}\hat{\rho}\hat{L}^\dagger$$

$$= -\frac{dt\,\kappa}{8}\{\hat{\rho}, \hat{O}^2\} + \frac{\kappa\,dt}{4}\hat{O}\hat{\rho}\hat{O}$$

$$= -\frac{\kappa\,dt}{8}[\hat{O},[\hat{O},\hat{\rho}]]$$

That is, the ~~[illegible]~~ unconditioned density operator for the system, in the absense of measurement of the probe, decoheres as the probe carries away information.

To understand this decoherence, look at the matrix elements in the eigenstates of $\hat{\sigma}$.
We easily see that the <u>unconditioned</u> density matrix evolves as

$$\boxed{\frac{d}{dt}\langle \sigma' | \hat{\rho} | \sigma \rangle = -\frac{\kappa}{8}(\sigma - \sigma')^2 \; \langle \sigma' | \rho | \sigma \rangle}$$

That is, the populations ~~[illegible]~~ $\langle \sigma | \rho | \sigma \rangle$ remain unchanged, and the coherences in this basis decay at rate $\frac{\kappa}{8}(\sigma - \sigma')^2$ (Note: the factor of 8 is often absorbed into the definition of κ; see e.g. Jacobs + Steck quant-ph/0611067)

Thus, given an intial state $|\psi\rangle = \sum_\sigma c_\sigma |\sigma\rangle$

is steady state $\hat{\rho} \Rightarrow \sum_\sigma |c_\sigma|^2 |\sigma\rangle\langle\sigma|$.

The density matrix decoheres into a statistical mixture of eigenstates of $\hat{\sigma}$, with probability $|c_\sigma|^2$. The basis $\{|\sigma\rangle\}$ is known as the "pointer" basis, corresponding to the eigenstates of the Linblad operator.

To understand the dynamical "collapse" of the wave function, consider that the differential Kraus operator under our Markov approximation is

$$\hat{M}_m(dt) = \frac{e^{-\frac{\kappa dt}{4}(\hat{\theta}-\theta_m)^2}}{(2\pi/\kappa dt)^{1/4}}$$

Thus after a finite time we can easily integrate to

$$\hat{M}_m(t) = \frac{e^{-\frac{\kappa t}{2}(\hat{\theta}-\theta_m)^2}}{(2\pi/\kappa t)^{1/2}}$$

That is, the Kraus operator which (stochastically) maps the state at time $t=0$ ~~to~~ time t is itself a Gaussian filter of width (variance) $1/\kappa t$. The longer the integration time, the narrower the filter:

$$|\psi(t)\rangle_m = \frac{\hat{M}_m(t)\,|\psi(0)\rangle}{\|\hat{M}_m(t)\,|\psi(0)\rangle\|}$$

In the limit $t \to \infty$ (i.e. $t \gg 1/\kappa$)

$$\hat{M}_m \to \delta(\hat{\theta}-\theta_m) = |\theta_m\rangle\langle\theta_m|$$

A projection <u>operator</u>!

The particular value O_m the filter converges to is random, distributed according to $|C_m|^2$, by the rule $P_m = \langle \psi(0) | \hat{M}_m^\dagger \hat{M}_m | \psi(0) \rangle$.

Note that backaction is only important when $\kappa t \ll \langle \Delta \hat{O}^2 \rangle$. that is, the measurement uncertainty is less than the uncertainty in the value of $\hat{O}$ in the initial state preparation. Before that time, the measurement is <u>weak</u>.

For long times, the continuous measurement approaches a projective measurement, strongly perturbing the state conditional on the the measurement history.