Physics 566: Quantum Optics

Lecture 4: The Optical Bloch Equations (I)

For the problem of Rabi flopping, we were able to solve for the time-evolution operator exactly. This is an exception rather than a rule. In most cases we must resort to approximate solutions. As a warm up, let's familiarize ourselves with some of the basics for the case of Rabi flopping.

Three representations of the state vector

- Probability amplitudes:

In the rotating frame $|\tilde{\Psi}\rangle = \tilde{c}_+(t)\,|+_z\rangle + \tilde{c}_-(t)\,|-_z\rangle$

$$\hat{\tilde{H}} = -\frac{\hbar}{2}\left(\Delta\hat{\sigma}_z + \Omega\hat{\sigma}_x\right)$$

$$\Rightarrow \frac{d}{dt}\begin{bmatrix}\tilde{c}_+\\ \tilde{c}_-\end{bmatrix} = \frac{i}{2}\begin{bmatrix}\Delta & \Omega\\ \Omega & -\Delta\end{bmatrix}\begin{bmatrix}\tilde{c}_+\\ \tilde{c}_-\end{bmatrix} \Leftrightarrow \begin{aligned}\dot{\tilde{c}}_+ &= i\frac{\Delta}{2}\tilde{c}_+ + i\frac{\Omega}{2}\tilde{c}_-\\ \dot{\tilde{c}}_- &= -i\frac{\Delta}{2}\tilde{c}_- + i\frac{\Omega}{2}\tilde{c}_+\end{aligned}$$

On resonance: $\dot{\tilde{c}}_+ = i\frac{\Omega}{2}\tilde{c}_- \qquad \dot{\tilde{c}}_- = i\frac{\Omega}{2}\tilde{c}_+$

$$\Rightarrow \ddot{\tilde{c}}_+ = -\frac{\Omega^2}{4}\tilde{c}_+ \quad \text{(SHO diff' eq)}$$

$$\tilde{c}_+(t) = \tilde{c}_+(0)\cos\left(\frac{\Omega t}{2}\right) + \frac{2}{\Omega}\dot{\tilde{c}}_+(0)\sin\left(\frac{\Omega}{2}t\right)$$

$$= \tilde{c}_+(0)\cos\left(\frac{\Omega t}{2}\right) + i\,\tilde{c}_-(0)\sin\left(\frac{\Omega t}{2}\right)$$

General case ($\Delta \neq 0$ arbitrary phase $\phi \neq 0$)

$$H = -\frac{\hbar \vec{\Omega}_{eff}}{2} \cdot \vec{\sigma}$$

where $\vec{\Omega}_{eff} = \Omega(\cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y) + \Delta \vec{e}_z$

$$|\vec{\Omega}_{eff}| = \sqrt{\Omega^2 + \Delta^2}$$

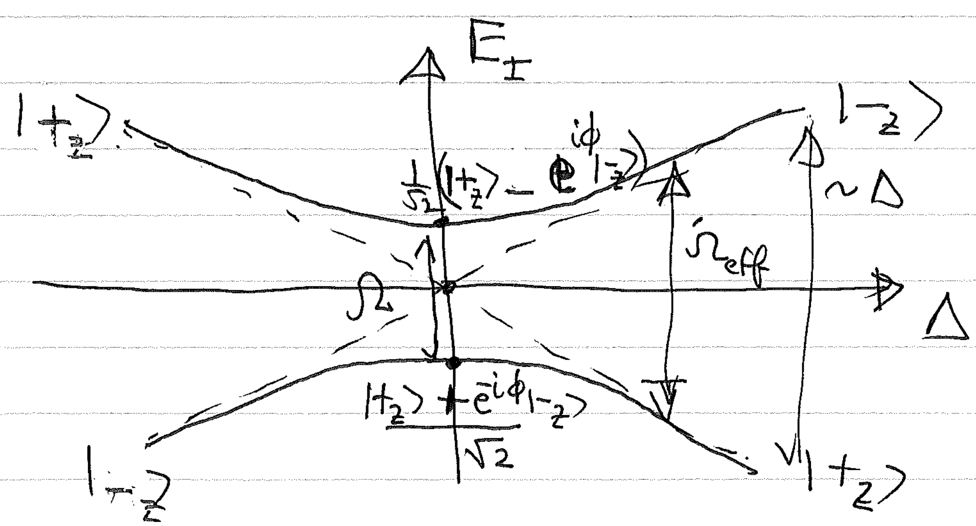
$$\vec{e}_n \equiv \frac{\vec{\Omega}_{eff}}{\Omega_{eff}} = \underbrace{\frac{\Omega}{\Omega_{eff}}}_{\equiv \sin\theta} (\cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y) + \underbrace{\frac{\Delta}{\Omega_{eff}}}_{\equiv \cos\theta} \vec{e}_z$$

mixing angle $\theta = \tan^{-1}\left(\frac{\Omega}{\Delta}\right)$

$\Rightarrow$ Eigenvalues $E_\pm = \mp \frac{\hbar \Omega_{eff}}{2} = \mp \sqrt{\Omega^2 + \Delta^2}\, \frac{\hbar}{2}$

Eigenvectors: $\begin{cases} |+_n\rangle = \cos\frac{\theta}{2} |+_z\rangle + e^{-i\phi} \sin\frac{\theta}{2} |-_z\rangle \\ |-_n\rangle = \sin\frac{\theta}{2} |+_z\rangle - e^{+i\phi} \cos\frac{\theta}{2} |-_z\rangle \end{cases}$

$\equiv$ $\boxed{\text{Dressed States}}$



Avoided crossing

• <u>Bloch vector dynamics</u>

$$\vec{Q} = \langle \vec{\sigma} \rangle = (\tilde{u}, \tilde{v}, \tilde{w}) \quad \text{in rotating frame}$$

Heisenberg equations of motion $\Rightarrow \dfrac{d\vec{\tilde{Q}}}{dt} = \vec{\tilde{Q}} \times \vec{\Omega}_{eff}$

("Gyroscope")

$$\frac{d}{dt}\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{w} \end{bmatrix} = \begin{bmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & +\Omega \\ 0 & -\Omega & 0 \end{bmatrix} \Leftrightarrow \begin{cases} \dot{\tilde{u}} = \Delta \tilde{v} \\ \dot{\tilde{v}} = -\Delta \tilde{u} + \Omega \tilde{w} \\ \dot{\tilde{w}} = -\Omega \tilde{v} \end{cases}$$

"Bloch equations"

• <u>Density matrix</u>: $\dfrac{d\tilde{\rho}}{dt} = -i[H, \tilde{\rho}]$ "Master equation"

$$\tilde{\rho} = |\tilde{\psi}\rangle\langle\tilde{\psi}| \Rightarrow \tilde{\rho}_{ij} = \langle i|\tilde{\psi}\rangle\langle\tilde{\psi}|j\rangle$$
$$= \tilde{c}_i \tilde{c}_j^*$$

$$\Rightarrow \frac{d}{dt}\tilde{\rho}_{+-} = \dot{\tilde{c}}_+ \dot{\tilde{c}}_-^* + \dot{\tilde{c}}_+ \tilde{c}_-^* = i\Delta \tilde{c}_+ \tilde{c}_-^* - i\frac{\Omega}{2}(|\tilde{c}_+|^2 - |\tilde{c}_-|^2)$$

$$\Rightarrow \boxed{\dot{\tilde{\rho}}_{+-} = i\Delta \tilde{\rho}_{+-} - \frac{i}{2}\Omega(\tilde{\rho}_{++} - \tilde{\rho}_{--})}$$

$$\frac{d}{dt}\tilde{\rho}_{++} = \dot{\tilde{c}}_+ \tilde{c}_+^* + \tilde{c}_+ \dot{\tilde{c}}_+^* = i\frac{\Omega}{2}(\tilde{c}_- \tilde{c}_+^* - \tilde{c}_+ \tilde{c}_-^*)$$

$$\Rightarrow \boxed{\dot{\tilde{\rho}}_{++} = i\frac{\Omega}{2}(\tilde{\rho}_{-+} - \tilde{\rho}_{+-})}$$

Unitary evolution $\Rightarrow$ $Tr(\hat{\rho}) = 1 \quad \forall t$

$$\Rightarrow \quad \tilde{\rho}_{++} + \tilde{\rho}_{--} = 1$$

$$\Rightarrow \quad \dot{\tilde{\rho}}_{--} = -\dot{\tilde{\rho}}_{++} = -i\frac{\Omega}{2}\left(\tilde{\rho}_{-+} - \tilde{\rho}_{+-}\right)$$

Note: In Rabi flopping population differences drive coherences and coherences drive

All of these different representations are inter-related.

E.g. Bloch vector vs. Density Operator

- $$u = \langle \sigma_x \rangle = \langle \sigma_+ + \sigma_- \rangle = \rho_{+-} + \rho_{-+}$$
$$= 2\mathrm{Re}(\rho_{-+}) \leftarrow \text{Coherence}$$

- $$v = \langle \sigma_y \rangle = \left\langle \frac{\sigma_+ - \sigma_-}{i} \right\rangle = \frac{\rho_{-+} - \rho_{+-}}{i} = +2\mathrm{Im}(\rho_{-+})$$

- $$w = \langle \sigma_z \rangle = \rho_{++} - \rho_{--} \quad \text{(Population difference)}$$

Check

$$\frac{d}{dt}\rho_{+-} = \frac{1}{2}(\dot{u} - i\dot{v})$$
$$= \frac{1}{2}(\Delta v + i\Delta u + i\Omega w)$$
$$= i\Delta\left(\frac{u - iv}{2}\right) + i\frac{\Omega}{2}w$$
$$= i\Delta\rho_{+-} + i\frac{\Omega}{2}(\rho_{++} - \rho_{--}) \quad \checkmark$$

Atomic resonance as Pseudo spin

Dipole approximation $\lambda_L \gg a_0 \leftarrow$ Bohr radius

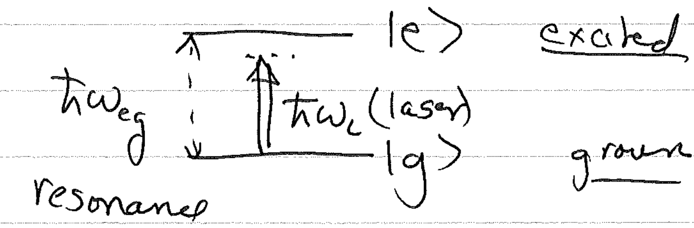
$\Rightarrow$ Atom-Laser interaction $\hat{H}_{AL} = -\hat{\vec{d}} \cdot \vec{E}_L(\vec{X}_{cm}, t)$

dipole operator — center of mass of atom

$\hat{\vec{d}} = -e\hat{\vec{X}}_{rel} \leftarrow$ relative coordinate of electron w.r.t. core

<u>Two-level atom:</u>
- Well defined laser polarization
- Near resonance to only <u>one</u> excited state

$\approx$



Atomic Hamiltonian $\hat{H}_A = E_g|g\rangle\langle g| + E_e|e\rangle\langle e|$

take zero of energy half-way between $|e\rangle$ and $|g\rangle$

$E_g = -\frac{\hbar\omega_{eg}}{2}, \quad E_e = +\frac{\hbar\omega_{eg}}{2}$

$$\Rightarrow \hat{H}_A = +\frac{\hbar\omega_{eg}}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right)$$

Interaction: (restricted to 2-levels)

$$\hat{H}_{AL} = -\hat{\vec{d}} \cdot \vec{E}_L(\vec{x}_A, t)$$

Monochromatic polarized plane wave:

$$\vec{E}_L(\vec{x}, t) = \vec{\epsilon}_L E_0 \cos(\omega_L t + \phi(\vec{x}))$$

$$\phi(\vec{x}) = -\vec{k}_L \cdot \vec{x} + \phi_0$$

$$\Rightarrow \hat{H}_{AL} = -(\hat{\vec{d}} \cdot \vec{\epsilon}_L)\, E_0 \cos(\omega_L t + \phi(\vec{x}))$$

Aside:

$$\begin{aligned}\hat{\vec{d}} \cdot \vec{\epsilon}_L = {} & \langle e|\hat{\vec{d}} \cdot \vec{\epsilon}_L|g\rangle |e\rangle\langle g| \\ & + \langle g|\hat{\vec{d}} \cdot \vec{\epsilon}_L|e\rangle |g\rangle\langle e| \\ & + \underbrace{\langle g|\hat{\vec{d}} \cdot \vec{\epsilon}_L|g\rangle}_{0} |g\rangle\langle g| \\ & + \underbrace{\langle e|\hat{\vec{d}} \cdot \vec{\epsilon}_L|e\rangle}_{0} |e\rangle\langle e|\end{aligned}$$

Diagonal matrix elements vanish by the parity selection rule.

Let $d_{eg} = \langle e|\hat{\vec{d}} \cdot \vec{\epsilon}_L|g\rangle$ (can be chosen real by phase convention)

Thus the total Hamiltonian, restricted to the two levels of the atom is

$$\boxed{\begin{aligned} \hat{H} &= \hat{H}_A + \hat{H}_{AL} \\ \hat{H}_A &= \frac{\hbar\omega_{eg}}{2}\left(|e\rangle\langle e| - |g\rangle\langle g|\right) \\ \hat{H}_{AL} &= -d_{eg}E_0\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\cos(\omega_L t + \phi(\vec{x}_A)) \end{aligned}}$$

As for any two-level system, the two level atom interacting with $\vec{E}$-field $\equiv$ spin mag resonance

Let $|-_z\rangle \equiv |g\rangle \qquad |+_z\rangle \equiv |e\rangle$

$$\Rightarrow \boxed{\begin{aligned} &\hat{H}_A = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z, \quad \hat{H}_{AL} = -\hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-)\cos(\omega_L t + \phi) \\ &\text{where } \Omega = \frac{d_{eg}E_0}{\hbar} = \text{Rabi frequency} \end{aligned}}$$

For $|\omega_L - \omega_{eg}| = |\Delta| \ll \omega_{eg}$ and $\Omega \ll \omega_{eg}$ we can make the rotating wave approx.

$\Rightarrow$ In RWA we neglect the counter-rotating terms

$$\hat{H}_{AL} = -\hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-)\,\frac{e^{-i(\omega_L t+\phi)} + e^{i(\omega_L t+\phi)}}{2}$$

$$\underset{\text{RWA}}{\approx} -\frac{\hbar\Omega}{2}\left(\hat{\sigma}_+ e^{-i\phi}e^{-i\omega_L t} + \hat{\sigma}_- e^{i\phi}e^{i\omega_L t}\right)$$

Go the the Rotating frame

$$\boxed{\begin{aligned}\tilde{H} &= -\frac{\hbar\Delta}{2}\hat{\sigma}_z - \frac{\hbar\Omega}{2}\left(\hat{\sigma}_+ e^{i\phi_L(\vec{x}_A)} + \hat{\sigma}_- e^{-i\phi(\vec{x}_A)}\right) \\ &= -\frac{\hbar\vec{\Omega}_{eff}}{2}\cdot\hat{\vec{\sigma}} \\ \vec{\Omega}_{eff} &= \Omega(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y) + \Delta\,\vec{e}_z\end{aligned}}$$

Suppose at $t=0$, atom is in its ground state (take $\phi=0$)

$$\Rightarrow \boxed{\begin{aligned}|\tilde{\psi}(t)\rangle &= \left(\cos\left(\frac{\Omega_{eff}t}{2}\right) - i\frac{\Delta}{\Omega_{eff}}\sin\left(\frac{\Omega_{eff}t}{2}\right)\right)|g\rangle \\ &\quad + i\frac{\Omega}{\Omega_{eff}}\sin\left(\frac{\Omega_{eff}t}{2}\right)|e\rangle \qquad \underline{\text{Rabi flopping}}\end{aligned}}$$

In the

$$\tilde{d}(t) = D^\dagger d\, D = d_{eg}\left(e^{+i\omega_L t}|e\rangle\langle g| + e^{-i\omega_L t}|g\rangle\langle e|\right)$$

$$\begin{aligned}\Rightarrow \langle\tilde{d}(t)\rangle &= d_{eg}\, 2\,\mathrm{Re}\left(e^{-i\omega_L t}\tilde{c}_e\tilde{c}_g^*\right) \\ &= d_{eg}\, 2\,\mathrm{Re}\left(e^{-i\omega_L t}\tilde{\rho}_{eg}\right) \\ &= d_{eg}\,\mathrm{Re}\left[e^{-i\omega_L t}(\tilde{u}-i\tilde{v})\right] \\ &= d_{eg}\left[\underset{\Uparrow}{\tilde{u}(t)}\cos\omega_L t - \underset{\Uparrow}{\tilde{v}(t)}\sin\omega_L t\right]\end{aligned}$$

"in phase" "in quadrature"

$$\boxed{\begin{aligned}\tilde{u}(t) &= 2\,\mathrm{Re}(\tilde{c}_g\,\tilde{c}_e^{*}) = \frac{\Omega\Delta}{\Omega_{eff}^2}\left(1-\sin(\tilde{\Omega}t)\right) \\ \tilde{v}(t) &= -\frac{\Omega}{\tilde{\Omega}}\sin(\tilde{\Omega}t)\end{aligned}}$$

- On resonance $\tilde{v} = -\sin(\tilde{\Omega}t)$, oscillates between absorption and emission
- Off resonance $\tilde{u}$ dominates
  $\Rightarrow$ Index of refraction ("reactive part")

Oscillating dipole with <u>amplitude modulation</u>

Frequency of modulation $\Rightarrow \Omega$

$\Rightarrow$ "Rabi sidebands" on emission spectrum

$\Rightarrow$ "Mollow triplet"