Problem 1: Boson Algebra (10 Points)
This problem is to give you some practice manipulating the boson algebra. A great
source is the classic “Quantum Statistical Properties of Radiation”, by W. H. Louisell,

(a) Gaussian integrals in phase-space are used all the time. Show that
\[
\int \frac{d^2 \beta}{\pi} e^{-A|\beta|^2} e^{\alpha \beta^* - \beta^* \alpha} = \frac{1}{A} e^{-|\alpha|^2 / A}.
\]

(b) Prove the completeness integral for coherent states
\[
\int \frac{d^2 \alpha}{\pi} |\alpha\rangle \langle \alpha| = 1 \quad \text{(Hint: Expand in number states)}.
\]

(c) The “quadrature” operators in optics are the analogs of \(Q\) and \(P\), \(\hat{a} = \hat{X}_1 + i\hat{X}_2\). Show
\[
\hat{U}^\dagger(\theta) \hat{X}_1 \hat{U}(\theta) = \cos \theta \hat{X}_1 + \sin \theta \hat{X}_2, \quad \text{where} \quad \hat{U}(\theta) = e^{-i \theta \hat{a}^\dagger \hat{a}}.
\]
Interpret in phase space.

(d) Prove the group property of the displacement operator
\[
\hat{D}(\alpha) \hat{D}(\beta) = \hat{D}(\alpha + \beta) \exp\{i \text{Im}(\alpha \beta^*)\}.
\]

(e) Show that the displacement operators has the following matrix elements

Vacuum: \(\langle 0 | \hat{D}(\alpha) | 0 \rangle = e^{-|\alpha|^2 / 2}\)

Coherent states: \(\langle \alpha_1 | \hat{D}(\alpha) | \alpha_2 \rangle = e^{-|\alpha_1 + \alpha_2 - \alpha_1|^2 / 2} e^{i \text{Im}(\alpha_2 \alpha^* - \alpha_1 \alpha^* - \alpha_1 \alpha^*_2)}\)

Fock states: \(\langle n | \hat{D}(\alpha) | n \rangle = e^{-|\alpha|^2 / 2} L_n(|\alpha|^2), \) where \(L_n\) is the Laguerre polynomial of order \(n\).
Problem 2: The Wigner Function

The Wigner function for a single mode of the field described by a state $\hat{\rho}$ can be understood as the expectation value of a Hermitian operator,

$$\hat{W}(\alpha) = \frac{1}{\pi} \int d^2 \beta \hat{D}(\beta) \exp(\alpha \beta^* - \alpha^* \beta),$$

so that, $W(\alpha) = \langle \hat{W}(\alpha) \rangle = \frac{1}{\pi} \int d^2 \beta \text{Tr}(\hat{\rho} \hat{D}(\beta)) \exp(\alpha \beta^* - \alpha^* \beta)$.

(a) Show that $\hat{W}(\alpha) = \hat{D}(\alpha) \hat{W}(0) \hat{D}^\dagger(\alpha)$.

Consider, $\hat{W}(0) = \frac{1}{\pi^2} \int d^2 \beta \hat{D}(\beta) = \frac{1}{\pi^2} \int dX_0 dP_0 \hat{D}(X_0, P_0)$, with $\hat{D}(X_0, P_0) = \exp[-2i(X_0 \hat{P} - P_0 \hat{X})]$.

To determine $\hat{W}(0)$, consider the position representation.

(b) Show that the matrix element of the displacement operator in the position representation is

$$\langle X' | \hat{D}(X_0, P_0) | X \rangle = \exp[iP_0(X_0 + 2X)] \delta(X' - X - X_0).$$

(c) Use this to show that $\hat{W}(0) = \frac{2}{\pi} \hat{\Pi}$, where $\hat{\Pi}$ is the parity operator, familiar in wave mechanics.

(d) Show that in the number-state basis, $\hat{W}(0) = \frac{2}{\pi} \sum_{n=0} (-1)^n \langle n | n \rangle$, and thus the Wigner function at the origin is $W(0) = \frac{2}{\pi} \langle \hat{\Pi} \rangle = \frac{2}{\pi} \sum_{n=0} (-1)^n p_n$, where $p_n$ is the probability of finding $n$ photons in the mode.

The above result shows that one can measure the Wigner function of a mode by counting the number of photons in many copies, collecting statistics, and determining the $p_n$.

(e) To get the Wigner function at any other point $\alpha$, one need first displace the state away from the origin as in part (a), and measure the mean value of parity through photon counting on the displaced state.

Show: $W(\alpha) = \frac{2}{\pi} \langle \hat{D}(\alpha) \hat{\Pi} \hat{D}^\dagger(\alpha) \rangle = \frac{2}{\pi} \sum_{n=0} (-1)^n p_{n\alpha}$, where $p_{n\alpha} = \langle n, \alpha | \hat{\rho} | n, \alpha \rangle$, $| n, \alpha \rangle \equiv \hat{D}(\alpha) | n \rangle$. 
**Problem 3:** A “Schrödinger cat” state.

Consider a superposition state of two “macroscopically” distinguishable coherent states, 
\[ |\psi\rangle = N(|\alpha_1\rangle + |\alpha_2\rangle), \quad |\alpha_1 - \alpha_2| >> 1, \]
where \( N = \left[ 2 \left( 1 + \exp\{-|\alpha_1 - \alpha_2|^2\} \right) \right]^{-1/2} \) is normalization.

This state is often referred to as a “Schrodinger cat”, and is very nonclassical.

(a) Calculate the Wigner function, for the simpler case \( |\psi\rangle = N(|\alpha\rangle + |-\alpha\rangle) \), with \( \alpha \) real, and plot it for different values of \( |\alpha_1 - \alpha_2| = 2\alpha \). Comment please.

(b) Calculate the marginals in \( X_1 \) and \( X_2 \) and show they are what you expect.

**Problem 4:** Thermal Light

Consider a single mode field in thermal equilibrium at temperature \( T \), Boltzmann factor \( \beta = 1/k_B T \). The state of the field is described by the “canonical ensemble”,
\[ \hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}, \quad \hat{H} = \hbar \alpha \hat{a}^\dagger \hat{a} \] is the Hamiltonian and \( Z = Tr \left( e^{-\beta \hat{H}} \right) \) is the partition function.

(a) Remind yourself of the basic properties by deriving the following:
- \( \langle n \rangle = \frac{1}{e^{\beta \omega} - 1} \) (the Planck spectrum)
- \( P_n = \frac{\langle n \rangle^n}{(1 + \langle n \rangle)^{n+1}} \) (the Bose-Einstein distribution). Plot a histogram for various \( \langle n \rangle \).
- \( \Delta n^2 = \langle n \rangle + \langle n \rangle^2 \). How does this compare to a coherent state?
- \( \langle \hat{a} \rangle = 0 \Rightarrow \langle \hat{E} \rangle = 0 \). How does this compare to a coherent state?

(b) Find the \( P, Q, \) and \( W \) distributions for this field, and show they are *Gaussian* functions. For example, you should find \( P(\alpha) = \frac{1}{\pi \langle n \rangle} \exp\left( -\frac{|\alpha|^2}{\langle n \rangle} \right) \). Show that these three distributions give the proper functions in the limit, \( \langle n \rangle \to 0 \), i.e. the vacuum.

(c) Calculate \( \Delta n^2, \langle \Delta X_1(\theta) \rangle^2, \langle \Delta X_2(\theta) \rangle^2 \) using an appropriate quasi-probability distribution. Interpret \( \Delta n^2 \) as having a “particle” and a “wave” component.