Physics 566: Quantum Optics
Quantization of the Electromagnetic Field

Maxwell's Equations and Gauge invariance

In lecture we learned how to quantize a one dimensional scalar field corresponding to vibrations on an elastic rod. The procedure involved decomposing the field into its normal modes of oscillation and then quantizing each mode as a simple harmonic oscillator. We now want to apply this procedure to the electromagnetic field. One might think of this a quantization of the vibrational modes of the "ether" in the pre-relativity description of electrodynamics.

The dynamics of this field are determined by Maxwell's equations, which in the absence of charges (and in CGS units) read

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}.$$

In contrast to our discussion of the quantization of the scalar field, this problem is complicated by two factors. First we are dealing with a three dimensional field, and second it is a vector field. A vector field in three dimensions should be describable in terms of three scalar fields. However, at this point we have described our field in terms of six fields $\{E_x, E_y, E_z, B_x, B_y, B_z\}$. We can get a description of the dynamics with fewer fields by using the vector a scalar potentials, which transforms like the four vector in Minkowski space, $A^{\mu} = (\Phi, \mathbf{A})$. The physical fields which exert forces on the charges are the electric an magnetic fields which are defined in terms of the four-potential as

$$\mathbf{E} = -\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}, \quad \mathbf{B} = \nabla \times \mathbf{A}.$$

The equation of motion for Φ and $\mathbf{A}$ follow from Maxwell's equations

$$\nabla^2\Phi = -\frac{1}{c}\frac{\partial}{\partial t}\nabla\cdot\mathbf{A}$$
$$\nabla^2\mathbf{A} - \frac{1}{c}\frac{\partial^2}{\partial t^2}\mathbf{A} = \nabla\left(\nabla\cdot\mathbf{A} + \frac{1}{c}\frac{\partial\Phi}{\partial t}\right)$$

At this point the four components of A^{μ} describe our field. This set is actually over complete. One of the fundamental properties of Maxwell's equation is *gauge invariance*.

That is, the physical fields **E** and **B** are unchanged if we redefine our four potential by making the transformation $A^{\mu} \to A^{\mu} + \partial^{\mu}\Lambda$, or in terms of the components

$$\phi \to \phi + \frac{1}{c}\frac{\partial \Lambda}{\partial t}, \quad \mathbf{A} \to \mathbf{A} + \nabla\Lambda$$

where Λ is an arbitrary scalar field.

Maxwell's Equations in the Coloumb Gauge

The choice of gauge is completely arbitrary. For quantization of the field the convenient choice is the **Coulomb gauge** (a name we will understand soon) which is defined by choosing the vector potential such that

$$\nabla \cdot \mathbf{A} = 0 .$$

This is known as the "transversality" condition. If the vector potential we start with does not satisfy this condition it is always possible to make a gauge transformation so that **A** is transverse by choosing a gauge function that satisfies $\nabla^2\Lambda = 0$. The equation for the scalar potential is the Laplace's equation,

$$\nabla^2\phi = 0 .$$

Thus, ϕ is no longer a dynamical variable. It is just depends on the geometry of the problem, and is constant in time. The equation of motion for $\mathbf{A}_{\perp}$ (where $\perp$ indicates that **A** satisfies the transversality condition) is the wave equation,

$$\nabla^2\mathbf{A}_{\perp} - \frac{1}{c}\frac{\partial^2}{\partial t^2}\mathbf{A}_{\perp} = 0 .$$

The only remaining dynamical field which we must quantize is $\mathbf{A}_{\perp}$. The dynamical physical fields are $\mathbf{E}_{\perp} = -\frac{1}{c}\frac{\partial \mathbf{A}_{\perp}}{\partial t}$ and $\mathbf{B}_{\perp} = \nabla \times \mathbf{A}_{\perp}$.

The name Coulomb gauge becomes clear when we include the sources: charge density ρ, and current density **J**. In the Coulomb gauge, the equations are

$$\nabla^2 \Phi = 4\pi\rho$$
$$\nabla^2 \mathbf{A}_\perp - \frac{1}{c}\frac{\partial^2}{\partial t^2}\mathbf{A}_\perp = -\frac{4\pi}{c}\mathbf{J}_\perp$$

The first equation is Poisson's equation, now defined for the *instantaneous* charge density (which may be time varying), whose solution is

$$\Phi(\mathbf{x}, t) = \int d^3\mathbf{x}' \frac{\rho(\mathbf{x}', t)}{|\mathbf{x} - \mathbf{x}'|}.$$

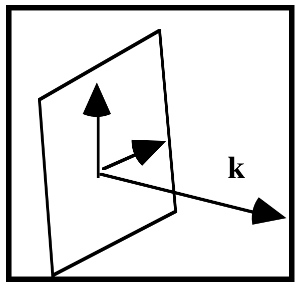
This is just the instantaneous Coulomb potential, and hence the name of the gauge. Note there is **no retarded time** in this integral. That is, the potential at time t and a point $\mathbf{x}$ is instantaneously related to the charge density at a distant point $\mathbf{x}'$ the same time. One might worry if the causal laws of special relativity are violated by this fact. Of course this is not the case for the original Maxwell equation were Lorentz invariant, and thus satisfy all the causal laws we know and love. Remember that the **physical** fields are $\mathbf{E}$ and $\mathbf{B}$. These always satisfy causal equations of motion, no matter what gauge we choose. The Coulomb gauge is a bit bizarre in that Φ seems to be related to the instantaneous values of the source. However, this effect is canceled by the fact that the vector potential is not driven by the physical current, but only by the "transverse" part of $\mathbf{J}$, which itself has an instantaneous part. In fact the transversality condition is itself dependent on the choice of reference frame (i.e. it is not manifestly covariant). However, we again emphasize that no laws of relativity are violated.

<u>Normal Modes in the Coulomb Gauge</u>

In order to quantize the field by the procedure we developed in lecture notes on "quantum field theory", we must introduce a normal mode expansion for the field. Our dynamical variable is the transverse delta function $\mathbf{A}_\perp$, which satisfies the wave equation. To arrive at a discrete set of modes, we will introduce an artificial "quantization volume" in the same way that we took a finite length of rod in our study of the vibrational modes for a scalar field. We will take periodic boundary conditions for complete generality to allow for propagating solutions. The normal modes are then plane waves with a given polarization

$$\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x}) = \vec{\varepsilon}_{\mathbf{k},\lambda}\frac{e^{i\mathbf{k}\cdot\mathbf{x}}}{\sqrt{V}},$$

where $\mathbf{k} = \frac{n_x 2\pi}{L}\mathbf{e}_x + \frac{n_y 2\pi}{L}\mathbf{e}_y + \frac{n_z 2\pi}{L}\mathbf{e}_z$, for periodic boundary conditions in a volume V=L^3, for some integers $\{n_x, n_y, n_z\}$. The vectors $\vec{\epsilon}_{\mathbf{k},\lambda}$ are the unit polarization vectors perpendicular to wave vector **k**, $\mathbf{k}\cdot\vec{\epsilon}_{\mathbf{k},\lambda} = 0$. This insures the transversality condition. The parameter λ (not to be confused with the wave length) is an index with value 1 or 2 representing any two basis vectors in the plane perpendicular to **k:**



The polarization vectors, $\vec{\epsilon}_{\mathbf{k},\lambda}$, may be complex if they represent a general elliptic polarization (for example right and left circularly polarized fields).

The normal modes are orthogonal

$$\int_V d^3x\, \mathbf{u}^*_{\mathbf{k},\lambda}(\mathbf{x})\cdot\mathbf{u}_{\mathbf{k}',\lambda'}(\mathbf{x}) = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}.$$

The completeness relation is a bit more complicated than for our scalar field. Consider the sum over all modes of the tensor $u^{(i)^*}_{\mathbf{k},\lambda}(\mathbf{x}')\cdot u^{(j)}_{\mathbf{k},\lambda}(\mathbf{x})$,

$$\sum_{\mathbf{k},\lambda} u^{(i)^*}_{\mathbf{k},\lambda}(\mathbf{x}')\cdot u^{(j)}_{\mathbf{k},\lambda}(\mathbf{x}) = \sum_{\mathbf{k}} \frac{e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}}{V}\sum_{\lambda}\epsilon^{(i)^*}_{\mathbf{k},\lambda}\epsilon^{(j)}_{\mathbf{k},\lambda}.$$

The polarization vectors $\vec{\epsilon}_{\mathbf{k},\lambda}$ span the plane perpendicular to **k**. Therefore

$$\sum_{\lambda}\epsilon^{(i)^*}_{\mathbf{k},\lambda}\epsilon^{(j)}_{\mathbf{k},\lambda} = \delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j.$$

That is, $\delta_{ij} - \hat{\mathbf{k}}_i\hat{\mathbf{k}}_j$ is the tensor that projects any vector into the plane perpendicular to **k**. The completeness relation reads,

$$\sum_{\mathbf{k},\lambda} u^{(i)^*}_{\mathbf{k},\lambda}(\mathbf{x}')\cdot u^{(j)}_{\mathbf{k},\lambda}(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}}\left(\delta_{ij}-\hat{\mathbf{k}}_i\hat{\mathbf{k}}_j\right)e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} = \delta^{ij}_{\perp}(\mathbf{x}-\mathbf{x}').$$

The tensor, $\delta^{ij}_{\perp}(\mathbf{x}-\mathbf{x}')$, is known as the "transverse" delta function, defined by the property that it projects any vector field $\mathbf{V}(\mathbf{x})$ into the transverse component $\mathbf{V}_{\perp}(\mathbf{x})$,

$$\int d^3\mathbf{x}'\, V^i(\mathbf{x}')\,\delta^{ij}_{\perp}(\mathbf{x}-\mathbf{x}') = V^j_{\perp}(\mathbf{x}),\ \text{ with zero divergence, }\ \nabla\cdot\mathbf{V}_{\perp} = 0.$$

Since the vector potential in the Coulomb gauge is transverse, it can be expanded in the complete set of normal mode functions $\mathbf{u}_{\mathbf{k},\lambda}$,

$$\mathbf{A}_{\perp}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}}}\left(\alpha_{\mathbf{k},\lambda}(t)\,\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x}) + \alpha^*_{\mathbf{k},\lambda}(t)\,\mathbf{u}^*_{\mathbf{k},\lambda}(\mathbf{x})\right).$$

Here we have used the normal mode expansion for periodic boundary conditions, given on page 12 of lecture "Intro. to Quantum Field Theory" in terms of the dimensionless complex amplitudes

$$\alpha_{\mathbf{k},\lambda}(t) = \alpha_{\mathbf{k},\lambda}(0)\,e^{-i\omega_{\mathbf{k}}t},\ \text{where } \omega_{\mathbf{k}} = c|\mathbf{k}|.$$

The normalization constant, $\sqrt{\dfrac{2\pi\hbar c^2}{\omega_{\mathbf{k}}}}$, is the appropriate scale factor $q_{0,k}$. Substituting in for the normal modes we obtain,

$$\mathbf{A}_{\perp}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{\omega_{\mathbf{k}}V}}\left(\alpha_{\mathbf{k},\lambda}(0)\,\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \alpha^*_{\mathbf{k},\lambda}(0)\,\vec{\varepsilon}^{\,*}_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right).$$

The transverse component of the electric and magnetic fields can be written using the definition at the bottom of page 2

$$\mathbf{E}_{\perp}(\mathbf{x},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\left(\alpha_{\mathbf{k},\lambda}(0)\,\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} - \alpha^*_{\mathbf{k},\lambda}(0)\,\vec{\varepsilon}^{\,*}_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right)$$

$$\mathbf{B}_{\perp}(\mathbf{x},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_{\mathbf{k}}}}\left(\alpha_{\mathbf{k},\lambda}(0)\,\mathbf{k}\times\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} - \alpha^*_{\mathbf{k},\lambda}(0)\,\mathbf{k}\times\vec{\varepsilon}^{\,*}_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right)$$

The Quantized Electromagnetic Field

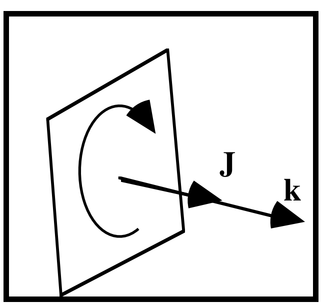
Now that we've expanded the field in terms of its normal modes, we can immediately quantize it according the procedure in lecture. Thus, we associate the complex amplitudes with creation and annihilation operators

$$\alpha^*_{\mathbf{k},\lambda}(0) \rightarrow \hat{a}^\dagger_{\mathbf{k},\lambda}, \quad \alpha_{\mathbf{k},\lambda}(0) \rightarrow \hat{a}_{\mathbf{k},\lambda},$$

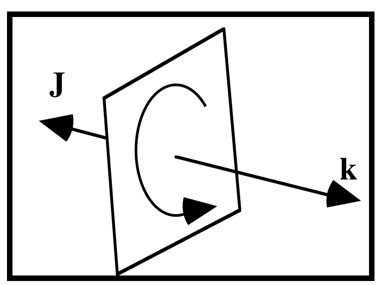
are instate canonical commutation relations

$$[\hat{a}_{\mathbf{k},\lambda}, \hat{a}^\dagger_{\mathbf{k}',\lambda'}] = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\lambda,\lambda'}.$$

We have used the commutator, rather than the anti-commutator, implying that the quanta of the field, otherwise known as **photons**, are bosons. This follows from the fact that we have a vector field with three components and thus must correspond to angular momentum J=1 (2J+1=3 components). Although the photon is a spin 1 particle, the transversality condition requires that the polarization (which describes the angular momentum of the light) must rotate in the plane perpendicular to the direction of propagation. Thus the spin vector can only be parallel, or anti-parallel to the direction of propagation. Photons with the spin along the direction of polarization are circularly polarized with their polarization rotating according to the right hand rule. These photons have **positive helicity**. the counter-rotating polarization describes **negative helicity**.



Positive helicity



Negative helicity

The classical fields now become vector operators,

$$\hat{\mathbf{A}}_{\lambda}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k},\lambda}(0)\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + \hat{a}^{\dagger}_{\mathbf{k},\lambda}(0)\vec{\varepsilon}^{\,*}_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right),$$
$$\hat{\mathbf{E}}_{\lambda}(\mathbf{x},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\left(\hat{a}_{\mathbf{k},\lambda}(0)\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} - \hat{a}^{\dagger}_{\mathbf{k},\lambda}(0)\vec{\varepsilon}^{\,*}_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right),$$
$$\hat{\mathbf{B}}_{\lambda}(\mathbf{x},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_{\mathbf{k}}}}\left(\hat{a}_{\mathbf{k},\lambda}(0)\mathbf{k}\times\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} - \hat{a}^{\dagger}_{\mathbf{k},\lambda}(0)\mathbf{k}\times\vec{\varepsilon}^{\,*}_{\mathbf{k},\lambda}e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}\right).$$

The Hilbert space describing the quantized field is a Fock space. A state with n photons in the mode with wave vector $\mathbf{k}$ and polarization λ is given by acting with the creation operator $\hat{a}^{\dagger}_{\mathbf{k},\lambda}$ n times on the vacuum state, and normalizing.

$$\left|n_{\mathbf{k},\lambda}\right\rangle = \frac{(\hat{a}^{\dagger}_{\mathbf{k},\lambda})^n}{\sqrt{n!}}\left|0\right\rangle.$$

Any state will be a superposition of such states for all modes.

At times one will see the field operator decomposed as

$$\hat{\mathbf{E}}_{\lambda}(\mathbf{x},t) = \hat{\mathbf{E}}^{(+)}_{\lambda}(\mathbf{x},t) + \hat{\mathbf{E}}^{(-)}_{\lambda}(\mathbf{x},t)$$

where $\hat{\mathbf{E}}^{(+)}_{\lambda}(\mathbf{x},t) = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\hat{a}_{\mathbf{k},\lambda}(0)\vec{\varepsilon}_{\mathbf{k},\lambda}e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}$ is known as the **positive frequency component** of the electric field, and $\hat{\mathbf{E}}^{(-)}_{\lambda}(\mathbf{x},t) = \left(\hat{\mathbf{E}}^{(+)}_{\lambda}(\mathbf{x},t)\right)^{\dagger}$ is known as the negative frequency component. The positive frequency component contains only annihilation operators, and is thus responsible for absorption, while the negative frequency components are responsible for emission of photons. From this picture, the "rotating wave" approximation, introduced in lecture for classical fields becomes clear. The dipole interaction Hamiltonian then has then form

$$\hat{H}_{\text{int}} = -\hat{\mathbf{d}}\cdot\left(\hat{\mathbf{E}}^{(+)}_{\lambda}(\mathbf{x},t) + \hat{\mathbf{E}}^{(-)}_{\lambda}(\mathbf{x},t)\right) = -\langle e|\hat{\mathbf{d}}|g\rangle\left(|e\rangle\langle g| + |g\rangle\langle e|\right)\cdot\left(\hat{\mathbf{E}}^{(+)}_{\lambda}(\mathbf{x},t) + \hat{\mathbf{E}}^{(-)}_{\lambda}(\mathbf{x},t)\right)$$
$$\approx -\langle e|\hat{\mathbf{d}}|g\rangle\left(|e\rangle\langle g|\hat{\mathbf{E}}^{(+)}_{\lambda}(\mathbf{x},t) + |g\rangle\langle e|\hat{\mathbf{E}}^{(-)}_{\lambda}(\mathbf{x},t)\right).$$

That is, the annihilation of the photon is always accompanied by the transition of the atom from the ground to excited state, and the creation of a photon is accompanied by the transition from the excited to ground state.

The Hamiltonian of the field is given by the integral of the energy density over the quantization volume

$$\hat{H} = \frac{1}{8\pi}\int_V d^3\mathbf{x}\left[\left|\hat{\mathbf{E}}_\perp\right|^2 + \left|\hat{\mathbf{B}}_\perp\right|^2\right].$$

If we write the field operators in terms of their normal mode expansions,

$$\hat{\mathbf{E}}_\perp(\mathbf{x},t) = i\sum_l \sqrt{2\pi\hbar\omega_l}\left(\hat{a}_l e^{-i\omega_l t}\mathbf{u}_l(\mathbf{x}) - \hat{a}_l^\dagger e^{i\omega_l t}\mathbf{u}_l^*(\mathbf{x})\right),$$
$$\hat{\mathbf{B}}_\perp(\mathbf{x},t) = i\sum_l \sqrt{\frac{2\pi\hbar c^2}{\omega_l}}\left(\hat{a}_l e^{-i\omega_l t}\mathbf{k}_l \times \mathbf{u}_l(\mathbf{x}) - \hat{a}_l^\dagger e^{i\omega_l t}\mathbf{k}_l \times \mathbf{u}_l^*(\mathbf{x})\right),$$

where l is a composite index for the mode $(\mathbf{k},\lambda)$, and $\mathbf{u}_l(\mathbf{x})$ is the normal mode vector function, then,

$$\int_V d^3\mathbf{x}\left|\hat{\mathbf{E}}_\perp\right|^2 = \sum_{l,m} 2\pi\hbar\sqrt{\omega_l\omega_m}\,\Big(-\hat{a}_l\hat{a}_m e^{-i(\omega_l t+\omega_m t)}\int d^3\mathbf{x}\,\mathbf{u}_l(\mathbf{x})\cdot\mathbf{u}_m(\mathbf{x})$$
$$+\,\hat{a}_l^\dagger\hat{a}_m e^{i(\omega_l t-\omega_m t)}\int d^3\mathbf{x}\,\mathbf{u}_l^*(\mathbf{x})\cdot\mathbf{u}_m(\mathbf{x})$$
$$+\,H.c\Big)$$

where H.c. stands for the Hermitian conjugate of the terms in parentheses. Using the completeness relations on the normal mode functions we have

$$\int d^3\mathbf{x}\,\mathbf{u}_l^*(\mathbf{x})\cdot\mathbf{u}_m(\mathbf{x}) = \delta_{lm}, \quad \int d^3\mathbf{x}\,\mathbf{u}_l(\mathbf{x})\cdot\mathbf{u}_m(\mathbf{x}) = \delta_{-lm}$$

so that,

$$\int_V d^3\mathbf{x}\left|\hat{\mathbf{E}}_\perp\right|^2 = \sum_l 2\pi\hbar\omega_l\left(\hat{a}_l^\dagger\hat{a}_l + \hat{a}_l\hat{a}_l^\dagger - \hat{a}_l\hat{a}_{-l}e^{-2i\omega_l t} - \hat{a}_l^\dagger\hat{a}_{-l}^\dagger e^{2i\omega_l t}\right).$$

Similarly for the magnetic field we have,

$$\int_V d^3\mathbf{x}\left|\hat{\mathbf{B}}_\perp\right|^2 = \sum_{l,m}\frac{2\pi\hbar c^2}{\sqrt{\omega_l\omega_m}}\Big(-\hat{a}_l\,\hat{a}_m e^{-i(\omega_l t+\omega_m t)}\int d^3\mathbf{x}(\mathbf{k}_l\times\mathbf{u}_l)\cdot(\mathbf{k}_m\times\mathbf{u}_m)$$
$$+\,\hat{a}_l^\dagger\hat{a}_m e^{i(\omega_l t-\omega_m t)}\int d^3\mathbf{x}(\mathbf{k}_l\times\mathbf{u}_l^*)\cdot(\mathbf{k}_m\times\mathbf{u}_m)$$
$$+\,H.c\Big)$$

Using vector identities, and the fact that $\mathbf{k}\cdot\mathbf{u}_l = 0$

$$(\mathbf{k}_l\times\mathbf{u}_l)\cdot(\mathbf{k}_m\times\mathbf{u}_m) = (\mathbf{k}_l\cdot\mathbf{k}_m)(\mathbf{u}_l\cdot\mathbf{u}_m)\,.$$

Therefore, the orthogonality relation yields,

$$\int d^3\mathbf{x}\,(\mathbf{k}_l\times\mathbf{u}_l)\cdot(\mathbf{k}_m\times\mathbf{u}_m) = -k_l^{\,2}\,\delta_{-l,m}\,,$$
$$\int d^3\mathbf{x}\,(\mathbf{k}_l\times\mathbf{u}_l^*)\cdot(\mathbf{k}_m\times\mathbf{u}_m) = k_l^{\,2}\,\delta_{l,m}\,.$$

The integral of the norm of the magnetic field operator is thus,

$$\int_V d^3\mathbf{x}\left|\hat{\mathbf{B}}_\perp\right|^2 = \sum_l 2\pi\hbar\omega_l(\hat{a}_l^\dagger\hat{a}_l + \hat{a}_l\hat{a}_l^\dagger + \hat{a}_l\hat{a}_{-l}e^{-2i\omega_l t} + \hat{a}_l^\dagger\hat{a}_{-l}^\dagger e^{2i\omega_l t})$$

Finally we arrive at the expression for the Hamiltonian in terms of the creation and annihilation operators,

$$\hat{H} = \frac{1}{2}\sum_{\mathbf{k},\lambda}\hbar\omega_{\mathbf{k}}(\hat{a}_{\mathbf{k},\lambda}^\dagger\hat{a}_{\mathbf{k},\lambda} + \hat{a}_{\mathbf{k},\lambda}\hat{a}_{\mathbf{k},\lambda}^\dagger) = \sum_{\mathbf{k},\lambda}\hbar\omega_{\mathbf{k}}(\hat{a}_{\mathbf{k},\lambda}^\dagger\hat{a}_{\mathbf{k},\lambda} + \frac{1}{2})\,,$$

where we have reinstated the full indexing of the modes, and used the commutator to move all the creation operators to the right. We recognize this Hamiltonian as the sum of simple harmonic oscillator Hamiltonians for the modes.

Vacuum Fluctuations

In lecture we discussed that even in a state with no quanta, i.e. the vacuum $|0\rangle$, the quantum uncertainties lead to fluctuations in the field. The average value of the electric field operator vanishes in the vacuum,

$$\left\langle \hat{\mathbf{E}} \right\rangle_{vac} = \langle 0|\hat{\mathbf{E}}|0\rangle = \langle 0|\hat{\mathbf{E}}^{(+)}|0\rangle + \langle 0|\hat{\mathbf{E}}^{(-)}|0\rangle = 0,$$

since $\hat{\mathbf{E}}^{(+)}|0\rangle = i\sum_{\mathbf{k},\lambda}\sqrt{2\pi\hbar\omega_{\mathbf{k}}}\,\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x})e^{-i\omega_{\mathbf{k}}t}\hat{a}_{\mathbf{k},\lambda}|0\rangle = 0$, and $\langle 0|\hat{\mathbf{E}}^{(-)} = \left(\hat{\mathbf{E}}^{(+)}|0\rangle\right)^{\dagger} = 0$.

However, the fluctuation of the field, given by the statistical variance, does not vanish

$$\begin{aligned}\left\langle \Delta\hat{\mathbf{E}}^2 \right\rangle_{vac} &= \langle 0|\hat{\mathbf{E}}^2|0\rangle - \underbrace{\langle 0|\hat{\mathbf{E}}|0\rangle^2}_{0} \\ &= \langle 0|\hat{\mathbf{E}}^{(+)}\cdot\hat{\mathbf{E}}^{(+)}|0\rangle + \langle 0|\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{E}}^{(-)}|0\rangle + \langle 0|\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{E}}^{(+)}|0\rangle + \langle 0|\hat{\mathbf{E}}^{(+)}\cdot\hat{\mathbf{E}}^{(-)}|0\rangle\end{aligned}.$$

From the relation above $\langle 0|\hat{\mathbf{E}}^{(+)}\cdot\hat{\mathbf{E}}^{(+)}|0\rangle = \langle 0|\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{E}}^{(-)}|0\rangle = \langle 0|\hat{\mathbf{E}}^{(-)}\cdot\hat{\mathbf{E}}^{(+)}|0\rangle = 0$. Thus,

$$\left\langle \Delta\hat{\mathbf{E}}^2 \right\rangle_{vac} = \langle 0|\hat{\mathbf{E}}^{(+)}\cdot\hat{\mathbf{E}}^{(-)}|0\rangle = \sum_{l,m} 2\pi\hbar\sqrt{\omega_l\omega_m}\,\mathbf{u}_l^*(\mathbf{x})\cdot\mathbf{u}_m(\mathbf{x})e^{-i(\omega_l-\omega_m)t}\langle 0|\hat{a}_m\hat{a}_l^{\dagger}|0\rangle,$$

$$\langle 0|\hat{a}_m\hat{a}_l^{\dagger}|0\rangle = \underbrace{\langle 0|\hat{a}_l^{\dagger}\hat{a}_m|0\rangle}_{0} + \underbrace{\langle 0|[\hat{a}_m,\hat{a}_l^{\dagger}]|0\rangle}_{\delta_{lm}} = \delta_{lm}$$

$$\left\langle \Delta\hat{\mathbf{E}}^2 \right\rangle_{vac} = \langle 0|\hat{\mathbf{E}}^{(+)}\cdot\hat{\mathbf{E}}^{(-)}|0\rangle = \sum_{\mathbf{k},\lambda} 2\pi\hbar\omega_{\mathbf{k}}|\mathbf{u}_{\mathbf{k},\lambda}(\mathbf{x})|^2 = \sum_{\mathbf{k},\lambda}\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}.$$

Therefore, the vacuum fluctuation of the electric field is on the order of $\hbar\omega_{\mathbf{k}}/V$ per mode.

The expectation value of the Hamiltonian in the vacuum is,

$$\left\langle \hat{H} \right\rangle_{vac} = \langle 0|\hat{H}|0\rangle = \sum_{\mathbf{k},\lambda}\hbar\omega_{\mathbf{k}}\left(\langle 0|\hat{a}_{\mathbf{k},\lambda}^{\dagger}\hat{a}_{\mathbf{k},\lambda}|0\rangle + \frac{1}{2}\right) = \sum_{\mathbf{k},\lambda}\frac{\hbar\omega_{\mathbf{k}}}{2} = \infty\ !$$

That is, each mode possesses a zero point energy $\frac{\hbar\omega_{\mathbf{k}}}{2}$, and for an infinite number of modes in the vacuum the total vacuum energy goes to infinity! Such divergences plague the mathematics of quantum field theory. A proper treatment requires the sophisticated theory of renormalization. For our purposes we will use the physical argument that the true energy of the system must be measured with respect to the ground state energy, i.e. the energy of the vacuum. In other words, we will set the vacuum energy to zero.

This is not to say the vacuum energy has no observable effects. In the next lecture we will show that the vacuum fluctuations in the electromagnetic field are responsible for the decay of an atom initially in the excited state to its ground sate. If we were able in some way to experimentally effect the environment of modes which interact with the atom, then we may be able to manipulate the vacuum fluctuations which interact with the atom and thereby modify the spontaneous emission rate with respect to the free space value. This may be done by placing the atom in a highly reflecting cavity which alters the spectrum of modes that can be supported inside. Such experiments have been performed. This a research field known as "cavity QED", which again and again demonstrates the power of quantum mechanics to describe the interactions of electromagnetic fields and atoms.