Chapter 1

Basic Concepts of the Quantum Theory (I): Heisenberg Uncertainty Principle

1.1 Vectors and operators

In a classical system, there exists the direct correspondence between the state of the system and the dynamical variables. Such direct correspondence does not exist in a quantum system. In Dirac formulation of quantum mechanics [1], we can deal with two strange creatures, vectors and operators, in a Hilbert space to describe the state and the dynamical variable, respectively. In order to predict an experimental result, we have to project the operator onto the vector.

1.1.1 State vectors

The state of a quantum object is described by a state vector, $|\varphi\rangle$ (ket vector)(or equivalently by $\langle\varphi|$ (bra vector)), both of which describe the identical physical state. If the state is a linear superposition state, expressed by

$$|\varphi\rangle = \sum_n C_n |\varphi_n\rangle \quad , \tag{1.1}$$

the corresponding bra vector is given by its hermitian adjoint

$$\langle\varphi| = \sum_n C_n^* \langle\varphi_n| = \left(\sum_n C_n |\varphi_n\rangle\right)^+ \quad . \tag{1.2}$$

With each pair of ket vectors $|\psi\rangle$ and $|\varphi\rangle$, we can define a scalar product, $\langle\varphi|\psi\rangle = \langle\psi|\varphi\rangle^*$, which is a c-number. The Schrödinger wavefunction in q-representation, $\psi(q) \equiv \langle q|\psi\rangle$ is just the projected coordinate of a state vector $|\psi\rangle$ as shown in Fig. 1.1. $\langle q|$ is an eigen-bra vector of coordinate. In contrast to a real vector in an ordinary space, those projected coordinates are complex numbers rather than real numbers. The c-number coordinate is the probability amplitude, by which a quantum system is found in a position eigen-state $|q_i\rangle$. An important departure of the quantum theory from the classical theory originates from (1.3) the fact that those probability amplitudes are c-numbers which carry the amplitude and phase information simultaneously.

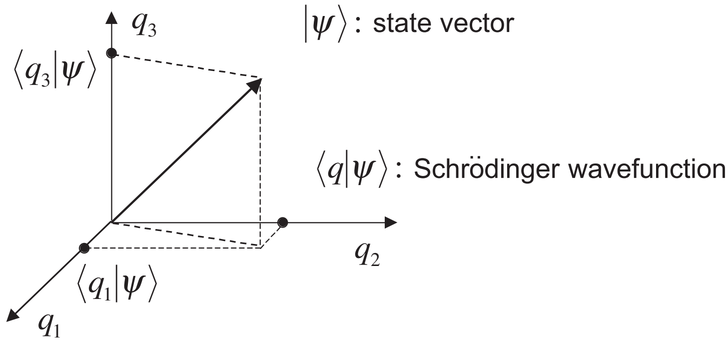

Figure 1.1: The state vector $|\psi\rangle$ and the Schrödinger wavefunction $\psi(q)$.

The Schrödinger wavefunction in p-representation is given by [2]

$$\varphi(p) \equiv \langle p|\psi\rangle = \frac{1}{\sqrt{2\pi\hbar}} \int \psi(q) \exp\left(-\frac{i}{\hbar}pq\right) dq \quad . \tag{1.3}$$

$\varphi(p)$ and $\psi(q)$ are a Fourier transform pair and includes exactly same information of a quantum object.

If we know $\psi(q)$ for all q values, it is said we have a complete information about the system and this situation is called a "pure state". If we do not know $\psi(q)$ completely but have only partial information, that situation is called a "mixed state". A state vector is insufficient to describe such a situation. We need a new mathematical tool (density operator) for describing a mixed state, which will be introduced in Sec.2.1.

1.1.2 Linear operators

If we associate each ket $|a\rangle$ in the space to another ket $|b\rangle$ by an operator $\hat{D}$,

$$|b\rangle = \hat{D}|a\rangle \quad , \tag{1.4}$$

and $\hat{D}$ satisfies the relation

$$\hat{D}(|a_1\rangle + |a_2\rangle) = \hat{D}|a_1\rangle + \hat{D}|a_2\rangle \quad , \tag{1.5}$$

$$\hat{D}(c|a\rangle) = c\hat{D}|a\rangle \quad , \tag{1.6}$$

$\hat{D}$ is called a linear operator.

A linear operator which associates a ket vector $|\psi\rangle$ to another ket vector $|\varphi\rangle$ is also called a projection operator, and is expressed by

$$\hat{A} \equiv |\varphi\rangle\langle\psi| \quad . \tag{1.7}$$

A linear operator $\hat{A}^+$ which associates a bra vector $\langle\psi|$ to another bra vector $\langle\varphi|$ is called an adjoint operator to $\hat{A}$:

$$
\begin{array}{lcl}
lcl|\psi\rangle & \xrightarrow{\hat{A}=|\varphi\rangle\langle\psi|} & |\varphi\rangle = \hat{A}|\psi\rangle \\
\langle\psi| & \xrightarrow{\hat{A}^{+}=|\psi\rangle\langle\varphi|} & \langle\varphi| = \langle\psi|\hat{A}^{+} \quad .
\end{array} \tag{1.8}
$$

If $\hat{A} = \hat{A}^{+}$, the projection operator $\hat{A}$ is called an Hermitian operator (or self-adjoint operator). If

$$
\hat{A}|a\rangle = a|a\rangle \quad , \tag{1.9}
$$

is satisfied, we have the following relations:

$$
\begin{array}{rcl}
\langle a|\hat{A}|a\rangle & = & a\langle a|a\rangle \\
\langle a|\hat{A}|a\rangle^{*} & = & \langle a|\hat{A}^{+}|a\rangle = a^{*}\langle a|a\rangle \\
\text{But} \quad \hat{A} & = & \hat{A}^{+} \rightarrow a = a^{*} \quad .
\end{array} \tag{1.10}
$$

The eigenvalue of an Hermitian operator is a real number.

If we measure a dynamical variable of a physical system, such as position, momentum, angular momentum, energy, etc., the obtained values are always "real numbers". Since the eigenvalues of Hermitian operators are "real numbers", we can let an Hermitian operator represent a dynamical variable. An Hermitian operator is in this sense called an observable.

1.1.3 Probability interpretation

If an observable $\hat{A}$ is measured for a quantum object in a state $|\psi\rangle$, a measurement result is one of the eigenvalues of $\hat{A}$. Which specific eigenvalue a_i is obtained for a single measurement is totally unknown. However, if we repeat the preparation of a quantum object in the same state and the measurement of the same observable, the probability of obtaining a specific result a_i is equal to $|\langle a_i|\psi\rangle|^2$, which is the square of the Schrödinger wavefunction. This correspondence between the Schrödinger wavefunction and the "ensemble" measurement is only connection between the quantum theory and experimental result.

The sum of the probabilities for all possible measurement results is unity :

$$
\sum_q |\langle q|\psi\rangle|^2 = \sum_q \langle\psi|q\rangle\langle q|\psi\rangle = 1 \quad , \tag{1.11}
$$

$$
\sum_q |q\rangle\langle q| = \hat{I} \quad . \tag{1.12}
$$

This relation(1.12) is called "completeness". The eigen–states of an Hermitian operator (observable) form a complete set. If an Hermitian operator has continuous eigenvalue rather than discrete eigenvalues, the completeness relation is replaced by $\int |q\rangle\langle q|dq = \hat{I}$.

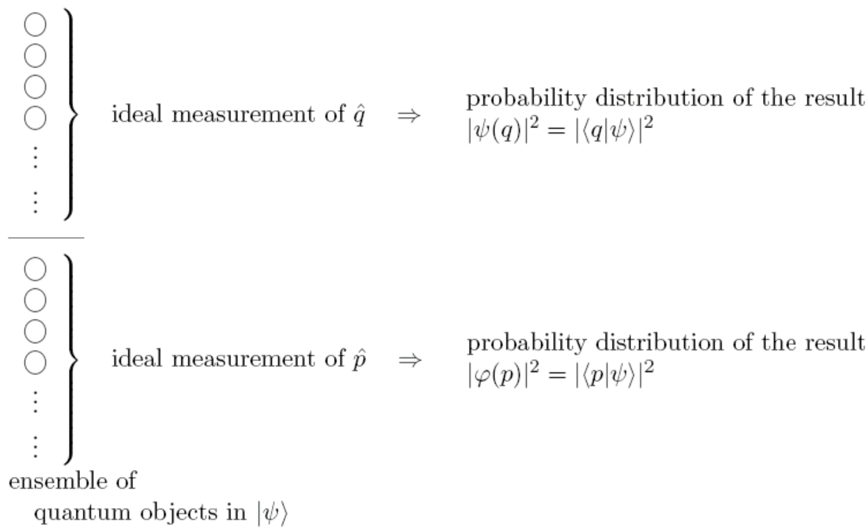

Figure 1.2: Connection between the squared Schrödinger wavefunction and the probability of measurement results.

1.1.4 Single quantum system

The standard quantum theory describes a following ensemble measurement:

1. Preparation of an ensemble of identical systems

2. Noise-free measurement of a specific observable

The uncertainty (probability distribution) of the measurement results is attributed to the characteristics of the initial state.

However, a following experimental situation is often encountered and becomes more and more important recently:

1. Prepare one and only one quantum system

2. This is single quantum system couples to an unknown force (information source).

3. To extract the information of the unknown force, a second quantum system (called probe) couples to the quantum system and the observable of the probe is measured.

4. Repeat the process 2 and process 3 to monitor a time dependent unknown force, as shown in Fig. 1.3.

In order to analyze the above situation, we must know not only the influence of the unknown force on the quantum system but also the influence of the coupling of the quantum probe and the measurement of the probe observable on the quantum system. We must go beyond the standard probability interpretation of the Schrödinger wavefunction. Dirac formulation of quantum mechanics is particularly useful for this goal, as will be discussed in Chapter 2.

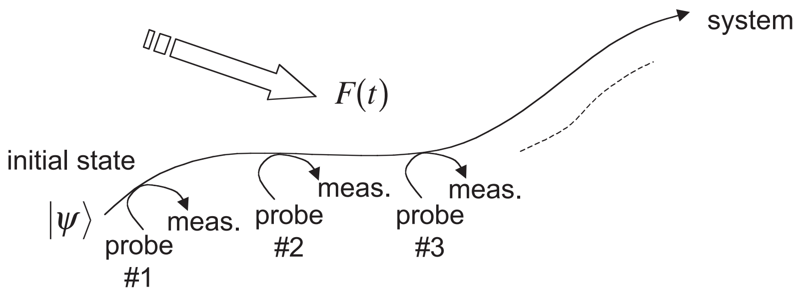

Figure 1.3: A continuous monitoring unknown force $F(t)$ by a single quantum system.

1.2 Heisenberg uncertainty principle

Today, the Heisenberg uncertainty principle is considered as the property of a "measured" quantum system. In fact, it is usually formulated in the context of the probability interpretation for the two ideal quantum measurements of conjugate observables such as $\hat{q}$ and $\hat{p}$ as shown in Fig. 1.2. However, when it was originally discussed by Heisenberg [3], it was clearly the statement about the measurement error and back action, which are traced back on the property of a "measuring" quantum probe. The goal of this section is to demonstrate that there are two types of uncertainty relations, one for the quantum system and the other for the quantum probe and that these two uncertainty relations are independent but intimately related. Even though it is referred to as "principle", it is a consequence of the fundamental assumption (postulate) of the quantum theory: commutation relation, as we will see in this section.

1.2.1 von Neumann's Doppler speed meter

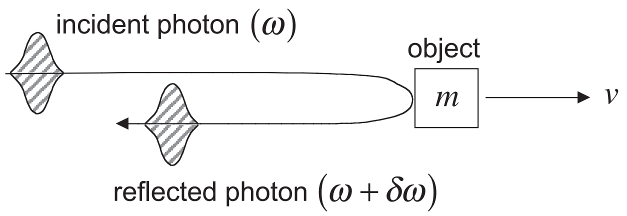

Figure 1.4: von Neumann's Doppler speed meter.

Let us consider the measurement of the momentum p by using the Dopper shift of a single photon wavepacket in an experimental setup shown in Fig. 1.4. The single photon wave packet acquires the Doppler shift: $\delta\omega = -\frac{2v}{c}\omega$ upon the reflection from an object. We assume a Fourier transform limited photon wavepacket : $\Delta\omega \cdot \Delta\tau \sim 1$. The measurement error of the velocity (momentum) is then given by

$$\Delta v_{\text{meas.error}} \simeq \frac{c}{2}\frac{\Delta\omega}{\omega} = \frac{c}{2\omega\Delta\tau} \quad , \tag{1.13}$$

$$\Delta p_{\text{meas.error}} \simeq \frac{mc}{2\omega\Delta\tau} \quad . \tag{1.14}$$

The object acquires a photon recoil in its momentum, $2\hbar k = \frac{2\hbar\omega}{c}$, upon reflection of the photon and changes its velocity by $\frac{2\hbar\omega}{cm}$. An exact time the photon recoil is transferred to the object is, however, uncertain due to the finite pulse duration $\Delta\tau$ of a single photon wave packet, which results in the uncertainty in the center position of the object after the collision between the object and the photon. This is the back action noise of the measurement. Back action noise of the position is given by

$$\Delta x_{\text{back action}} \simeq \frac{2\hbar\omega}{cm} \times \frac{\Delta\tau}{2} = \frac{\hbar\omega\Delta\tau}{cm} \quad , \tag{1.15}$$

and thus we have

$$\Delta p_{\text{meas.error}} \cdot \quad \Delta x_{\text{back action}} \simeq \frac{\hbar}{2} \quad . \tag{1.16}$$

This example illustrates the main elements of quantum measurements:

1. Measurement error $\Delta p_{\text{meas.error}}$ is governed by the uncertainty of the readout observable of a quantum probe $\Delta\omega$.

2. Back action noise $\Delta x_{\text{back action}}$ is determined by the uncertainty of the conjugate observable of a quantum probe $\Delta\tau$.

3. Uncertainty relation $\Delta p_{\text{meas.error}}\Delta x_{\text{back action}} \simeq \frac{\hbar}{2}$ originates from the minimum uncertainty relation of a quantum probe $\Delta\omega\Delta\tau \sim 1$.

4. Irreversible process, i.e. death of photon and birth of photoelectron, occurs in the quantum probe. After the death of a photon, the back action noise imposed on the measured system becomes permanent. We even do not know what the back action noise was. In this way, the measured system jump into a new state, which is an essence of the collapse of the wavefunction (state reduction).

5. Initial uncertainty of the quantum system $(\Delta p_{\text{initial}}, \Delta x_{\text{initial}})$ introduces an independent source of the uncertainty for the measurement result, which stems from the fact that the measured quantum system has its own uncertainty on the measured observable and conjugate observable. Even if the measurement error is zero, the measurement result is unpredictable due to this type of uncertainty. This is often referred to as lack of causality in quantum measurements.

1.2.2 Commutation relation, the minimum uncertainty wavepacket and squeezing

The most profound and fundamental postulate of the quantum theory probably a commutation relation. A certain pair of observables do not commute :

$$[\hat{q}, \hat{p}] = \hat{q}\hat{p} - \hat{p}\hat{q} = i\hbar \quad . \tag{1.17}$$

Classically, the position q and the momentum p commute. Therefore, the classical theory and the quantum theory depart with each other due to this postulate. In general, the commutation relation is expressed by

$$[\hat{A}, \hat{B}] = i\hat{C} \quad . \tag{1.18}$$

where $\hat{C}$ is an observable or real number. Let us introduce the fluctuating operators:

$$\begin{aligned} \hat{\alpha} &= \hat{A} - \langle \hat{A} \rangle \\ \hat{\beta} &= \hat{B} - \langle \hat{B} \rangle \quad . \end{aligned} \tag{1.19}$$

Then, we have

$$[\hat{\alpha}, \hat{\beta}] = i\hat{C} \quad . \tag{1.20}$$

We assume linear operators $\hat{\alpha}$ and $\hat{\beta}$ project the system state $|\psi\rangle$ onto new states

$$\hat{\alpha}|\psi\rangle = \langle \varphi \rangle \quad , \tag{1.21}$$

$$\hat{\beta}|\psi\rangle = \langle \chi \rangle \quad , \tag{1.22}$$

We now introduce Schwartz inequality: $\langle \varphi|\varphi\rangle\langle\chi|\chi\rangle \geq |\langle\varphi|\chi\rangle|^2$ to obtain

$$\langle \hat{\alpha}^2 \rangle \langle \hat{\beta}^2 \rangle \geq |\langle \hat{\alpha}\hat{\beta} \rangle|^2 \tag{1.23}$$

If we use

$$\hat{\alpha}\hat{\beta} = \frac{1}{2}(\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}) + \frac{1}{2}i\hat{C} \quad , \tag{1.24}$$

in (1.23), we can derive the Heisenberg uncertainty relation:

$$\begin{aligned} \langle \Delta \hat{A}^2 \rangle \langle \Delta \hat{B}^2 \rangle = \langle \hat{\alpha}^2 \rangle \langle \hat{\beta}^2 \rangle &\geq \frac{1}{4} |\langle \hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha} \rangle + i \langle \hat{C} \rangle|^2 \\ &\geq \frac{1}{4} |\langle \hat{C} \rangle|^2 \quad . \end{aligned} \tag{1.25}$$

For the equality to be held, we need

1. $\hat{\alpha}|\psi\rangle = C_1\hat{\beta}|\psi\rangle \quad \Rightarrow$ two linear operators $\hat{\alpha}$ and $\hat{\beta}$ project the system state $|\psi\rangle$ onto the identical state, except for the c-number C_1.
   $\Downarrow$
   The mathematical definition of the minimum uncertainty state $|\psi\rangle$.

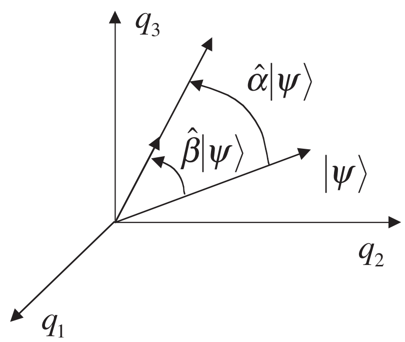

Figure 1.5: Projection property of a minimum uncertainty state $|\psi\rangle$.

2. $\langle\psi|\hat{\alpha}\hat{\beta} + \hat{\beta}\hat{\alpha}|\psi\rangle = 0 \Rightarrow (C_1 + C_1^*)\langle\hat{\beta}^2\rangle = 0$

If $|\psi\rangle$ is not an eigenstate of $\hat{B}$, $\langle\hat{\beta}^2\rangle \neq 0$. Then, C_1 must be a pure imaginary, so that we can write $C_1 = -ie^{-2r}$ without loss of generality, where r is a real number.

Now, the minimum uncertainty state is constructed by the relation:

$$\hat{\alpha}|\psi\rangle = -ie^{-2r}\hat{\beta}|\psi\rangle \quad , \tag{1.26}$$

or

$$(e^r\hat{A} + ie^{-r}\hat{B})|\psi\rangle = (e^r\langle\hat{A}\rangle + ie^{-r}\langle\hat{B}\rangle)|\psi\rangle \quad . \tag{1.27}$$

The minimum uncertainty state $|\psi\rangle$ is an eigenstate of the non- Hermitian operator $e^r\hat{A} + ie^{-r}\hat{B}$ with the complex eigenvalue $e^{-r}\langle\hat{A}\rangle + ie^r\langle\hat{B}\rangle$.

We can evaluate the quantum noise of the two observables $\hat{A}$ and $\hat{B}$ by using (1.26):

$$\begin{array}{ll} \hat{\alpha}|\psi\rangle = -ie^{-2r}\hat{\beta}|\psi\rangle & \\ \quad\quad\quad\quad\quad\searrow & \\ & \langle\Delta\hat{A}^2\rangle = e^{-4r}\langle\Delta\hat{B}^2\rangle \\ \quad\quad\quad\quad\quad\nearrow & \\ \langle\psi|\hat{\alpha} = ie^{-2r}\langle\psi|\hat{\beta} & \end{array} \tag{1.28}$$

If we substitute (1.28) into $\langle\Delta\hat{A}^2\rangle\langle\Delta\hat{B}^2\rangle = \frac{1}{4}|\langle\hat{C}\rangle|^2$, we obtain

$$\begin{aligned} \langle\Delta\hat{A}^2\rangle &= \frac{1}{2}|\langle\hat{C}\rangle|e^{-2r} \\ \langle\Delta\hat{B}^2\rangle &= \frac{1}{2}|\langle\hat{C}\rangle|e^{2r} \end{aligned} \tag{1.29}$$

A real parameter r determines the distribution of the quantum noise between the two conjugate observables under the constraint of the minimum uncertainty product. This property is called "squeezing".

1.2.3 Wave-particle duality

Let us consider a position-momentum minimum uncertainty state. The relevant eigenvalue equation is given by

$$(\hat{q} - \langle\hat{q}\rangle)|\psi\rangle = -ie^{-2r}(\hat{p} - \langle\hat{p}\rangle)|\psi\rangle \quad . \tag{1.30}$$

By multiplying a bra-vector $\langle q'|$ from the left and use the operator identity $\langle q'|\hat{p} = \frac{\hbar}{i}\frac{d}{dq'}$, we have

$$\frac{d}{dq'}\psi(q') = \left[-\frac{(q' - \langle\hat{q}\rangle)}{e^{-2r}\hbar} + \frac{i}{h}\langle\hat{p}\rangle\right]\psi(q') \quad , \tag{1.31}$$

A solution of this first-order differential equation has a general form of

$$\psi(q') = C_3 \exp\left[-\frac{(q' - \langle\hat{q}\rangle)^2}{e^{-2r}\hbar} + \frac{i}{h}\rangle\hat{p}q'\right] \quad , \tag{1.32}$$

where the variance in $\hat{q}$ is identified as

$$\langle\Delta\hat{q}^2\rangle = \frac{\hbar}{2}e^{-2r} \quad . \tag{1.33}$$

If we use a normalization, $\langle\psi|\psi\rangle = \int_{-\infty}^{\infty} |\psi(q')|^2 dq' = 1$, we can determine the coefficient C_3, as $C_3 = \left(2\pi\langle\Delta\hat{q}^2\rangle\right)^{\frac{1}{4}}$. Therefore, the Schrödinger wavefunction in q-representation is expressed by the Gaussian wavepacket:

$$\psi(q') = \frac{1}{(2\pi\langle\Delta\hat{q}^2\rangle)^{\frac{1}{4}}} \exp\left[-\frac{(q'-\langle\hat{q}\rangle)^2}{4\langle\Delta\hat{q}^2\rangle} + \frac{i}{\hbar}\langle\hat{p}\rangle q'\right] \quad . \tag{1.34}$$

The Fourier transform of (1.34) provides the Scrödinger wavefunction in $p-$representation:

$$\begin{aligned} \varphi(p') &= \frac{1}{\sqrt{2\pi\hbar}} \int dq' \exp\left(-\frac{i}{\hbar}p'q'\right)\psi(q') \\ &= \frac{1}{(2\pi\langle\Delta\hat{p}^2\rangle)^{\frac{1}{4}}} \exp\left[-\frac{(p'-\langle\hat{p}\rangle)^2}{4\langle\Delta\hat{p}^2\rangle} - \frac{i}{\hbar}\langle\hat{q}\rangle(p'-\langle\hat{p}\rangle)\right] \quad . \end{aligned} \tag{1.35}$$

We can express $\psi(q')$ in terms of the inverse Fourier transform of $\varphi(p)$ as

$$\begin{aligned} \psi(q') &= \frac{1}{\sqrt{2\pi\hbar}} \int dp' \exp\left(\frac{i}{\hbar}p'q'\right)\varphi(p') \\ &= \frac{\exp\left(\frac{i}{\hbar}\langle\hat{q}\rangle\langle\hat{p}\rangle\right)}{\sqrt{2\pi\hbar}} \int dp' \exp\left[i\frac{p'}{\hbar}(q'-\langle\hat{q}\rangle)\right] \exp\left[-\frac{(p'-\langle\hat{p}\rangle)^2}{4\langle\Delta\hat{p}^2\rangle}\right] \quad . \end{aligned} \tag{1.36}$$

The Schrödinger wavefunction $\psi(q')$ consists of the linear superposition of de Broglie waves with a wavelength $\lambda_{dB} = \frac{h}{p'}$ and its Gaussian distribution centered at $\langle\hat{p}\rangle$ and with a variance $\langle\Delta\hat{p}^2\rangle$. These plane waves interfere with each other. At positions close to $\langle\hat{q}\rangle$, the phase rotation is slow so that "constructive interference" occurs. At positions far from $\langle\hat{q}\rangle$, the phase rotation is fast so that "destructive interference" occurs. In this way, a particle is localized to the vicinity of the average position $\langle\hat{q}\rangle$.

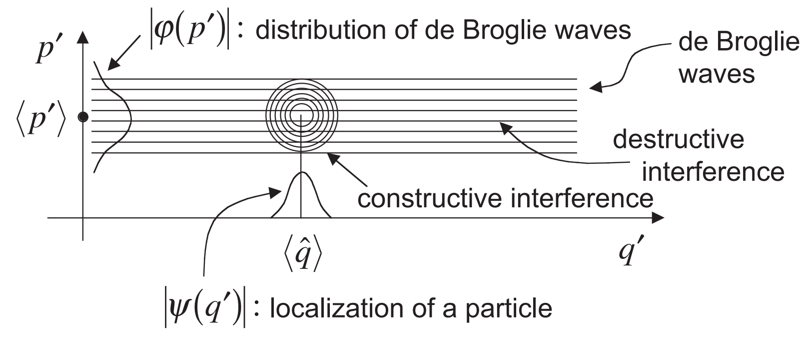

Figure 1.6: Phase space distribution of a minimum uncertainty wavepacket.

A particle-like state has an increased $\langle\Delta\hat{p}^2\rangle$ and decreased $\langle\Delta\hat{q}^2\rangle$, while a wave-like state has a decreased $\langle\Delta\hat{p}^2\rangle$ and increased $\langle\Delta\hat{q}^2\rangle$. The wave-particle duality in quantum mechanics is the result of quantum interference effect of de Broglie waves and traced back to the fact that the Schrödinger wavefunction carries not only amplitude information but also phase information.

1.2.4 Time evolution

In order to describe the time evolution of the minimum uncertainty wavepacket, we need a further postulate, which is formulated in the so-called time dependent Schrödinger equation.

A free particle prepared in a minimum uncertainty state at $t = 0$ experiences the momomentum dependent phase shift (quantum diffusion), which results in the spread of the wavepacket. The Hamiltonian of the system, $\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m}$, generate the time evolution of the vector via Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi\rangle = \hat{\mathcal{H}}|\psi\rangle \quad . \tag{1.37}$$

If we introduce the unitary evolution operator by $|\psi(t)\rangle = \hat{U}(t)|\psi(0)\rangle$ and substitute this relation into (1.37), we obtain

$$\hat{U} = \exp\left(-\frac{i}{\hbar}\frac{\hat{p}^2}{2m}t\right) \quad . \tag{1.38}$$

We multiply $\langle q|$ from the left of (1.38) and use $\int dp|p\rangle\langle p| = \hat{I}$ and $\langle q|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}\exp\left(\frac{ipq}{\hbar}\right)$, we have the time-dependent Schrödinger wavefunction as follows:

$$\begin{aligned} \psi(q,t) &= \frac{1}{\sqrt{2\pi\hbar}}\int \exp\left(i\frac{pq}{\hbar}\right)\exp\left(-\frac{i}{\hbar}\frac{p^2}{2m}\right)\varphi(p,0)dp \qquad (1.39)\\ &= \frac{1}{(2\pi)^{1/4}}\left(\Delta q + \frac{i\hbar t}{2m\Delta q}\right)^{-\frac{1}{2}}\exp\left[-\frac{q^2}{4(\Delta q)^2 + \frac{2i\hbar t}{m}}\right](\text{if}\langle\hat{p}\rangle = \langle\hat{q}\rangle = 0) \quad , \end{aligned}$$

where $\varphi(p,0) = \frac{1}{(2\pi\langle\Delta\hat{p}^2\rangle)^{1/4}}\exp\left[-\frac{(p-\langle\hat{p}\rangle)^2}{4\langle\Delta\hat{p}^2\rangle} - \frac{i}{\hbar}\langle\hat{q}\rangle\left(p - \langle\hat{p}\rangle\right)\right]$ is the initial wavepacket. As we can see, the momentum uncertainty is preserved,

$$\langle\Delta\hat{p}(t)^2\rangle = \langle\Delta\hat{p}(0)^2\rangle \quad , \tag{1.40}$$

but the position uncertainty increases,

$$\langle\Delta\hat{q}(t)^2\rangle = \langle\Delta\hat{q}(0)^2\rangle + \frac{\hbar^2 t^2}{4m^2\langle\Delta\hat{q}^2\rangle} \quad . \tag{1.41}$$

In order to preserve the minimum uncertainty wavepacket against the quantum diffusion, we need a restoring force to confine the particle position. One example of such a restoring force is a harmonic potential:

$$\hat{\mathcal{H}} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{q}^2 \quad . \tag{1.42}$$

The minimum uncertainty wavepackets preserve their features in the precense of the harmonic potential. The representative examples of such states as coherent state, phase squeezed state and amplitude squeezed state, are schematically shown in Fig. 1.8. The characteristics of those states will be described in Chapter 4.

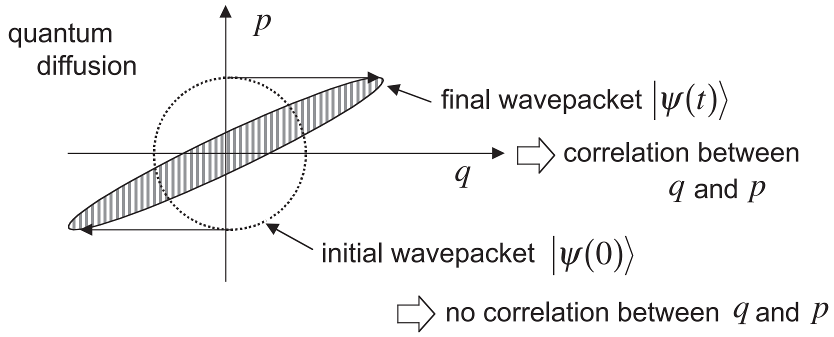

Figure 1.7: The initial and final wavepackets under the quantum diffusion.

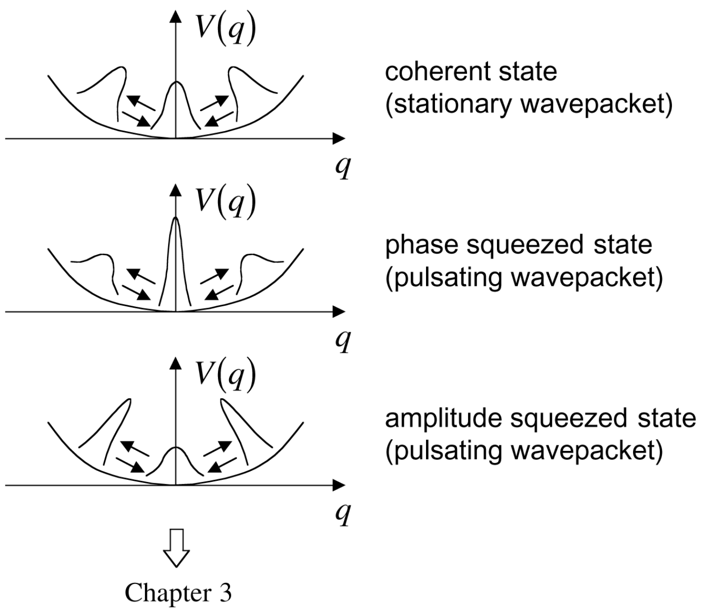

Figure 1.8: Coherent state, phase squeezed state and amplitude squeezed state in a harmonic potential.

1.2.5 Simultaneous measurement of two conjugate observables

Let us consider the simultaneous measurement of the position $\hat{q}$ and the momentum $\hat{p}$ of a quantum object. For this purpose we couple a quantum object (system) to a quantum probe which has two readout observables $\hat{x}$ and $\hat{y}$[4]. Such a compound system is described by the tensor product space : $\hat{\mathcal{H}} = \hat{\mathcal{H}}_1 \otimes \hat{\mathcal{H}}_2$, where the subscripts 1 and 2 refer to the system and probe. The two measured observables are represented by $\hat{q} = \hat{q}_1 \otimes \hat{I}_2$ and $\hat{p} = \hat{p}_1 \otimes \hat{I}_2$.

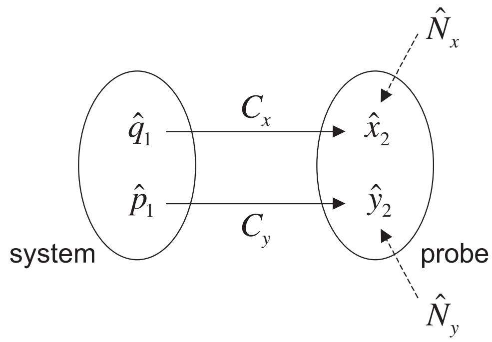

Figure 1.9: A simultaneous measurement of $\hat{q}$ and $\hat{p}$ in the coupled system-prove setup.

The two readout observables are described by

$$\hat{x} = C_x\hat{q} + \hat{N}_x \quad , \tag{1.43}$$

$$\hat{y} = C_y\hat{p} + \hat{N}_y \quad , \tag{1.44}$$

where C_x and C_y are the coupling constants and $\hat{N}_x$ and $\hat{N}_y$ are the noise operators. We assume no bias condition:

$$\langle\hat{N}_x\rangle = \langle\psi_1|\langle\psi_2|\hat{I}_1 \otimes \hat{N}_{x2}|\psi_2\rangle|\psi_1\rangle = 0 \quad , \tag{1.45}$$

$$\langle\hat{N}_y\rangle = \langle\psi_1|\langle\psi_2|\hat{I}_1 \otimes \hat{N}_{y2}|\psi_2\rangle|\psi_1\rangle = 0 \quad . \tag{1.46}$$

In order to measure the two readout observables $\hat{x}$ and $\hat{y}$ simultaneously, we require

$$[\hat{x}, \hat{y}] = 0 \quad , \tag{1.47}$$

which results in

$$C_xC_y[\hat{q}, \hat{p}] + [\hat{N}_x, \hat{N}_y] + [\hat{N}_x, C_y\hat{p}] + [C_x\hat{q}, \hat{N}_y] = 0 \quad . \tag{1.48}$$

The internal noise of the quantum probe is uncorrelated with the quantum system, so we have

$$\langle[\hat{N}_x, \hat{p}]\rangle = \langle\hat{N}_x\rangle\langle\hat{p}\rangle - \langle\hat{p}\rangle\langle\hat{N}_x\rangle = 0 \quad , \tag{1.49}$$

$$\langle[\hat{q}, \hat{N}_y]\rangle = 0 \quad . \tag{1.50}$$

Thus, the noise operators must satisfy

$$\langle[\hat{N}_x, \hat{N}_y]\rangle = -C_xC_y\langle[\hat{q}, \hat{p}]\rangle = -\hbar C_xC_y \quad . \tag{1.51}$$

We now introduce the normalized readout observables by

$$\hat{q}_{\text{obs}} \equiv \frac{\hat{x}}{C_x} = \hat{q} + \frac{\hat{N}_x}{C_x} \quad , \tag{1.52}$$

$$\hat{p}_{\text{obs}} \equiv \frac{\hat{y}}{C_y} = \hat{p} + \frac{\hat{N}_y}{C_y} \quad . \tag{1.53}$$

As expected, there is no bias in the measurements of $\hat{q}$ and $\hat{p}$: $\langle \hat{q}_{\text{obs}} \rangle = \langle \hat{q} \rangle, \langle \hat{p}_{\text{obs}} \rangle = \langle \hat{p} \rangle$ The uncertainty product is now evaluated as

$$\begin{aligned} \langle \Delta \hat{q}_{\text{obs}}^2 \rangle \langle \Delta \hat{p}_{\text{obs}}^2 \rangle &= \left[\langle \Delta \hat{q}^2 \rangle + \frac{\langle \Delta \hat{N}_x^2 \rangle}{C_x^2} \right] \left[\langle \Delta \hat{p}^2 \rangle + \frac{\langle \Delta \hat{N}_y^2 \rangle}{C_y^2} \right] \\ &\geq \frac{\hbar^2}{4} + \frac{\hbar^2}{4} + 2\sqrt{\frac{\hbar^2}{4} \times \frac{\hbar^2}{4}} \\ &\geq \hbar^2 \end{aligned} \tag{1.54}$$

The equality holds when and only when the quantum system is in a minimum uncertainty state and the quantum probe has the matched internal noise : $\frac{\langle \Delta \hat{N}_x^2 \rangle}{C_x^2} = \langle \Delta \hat{q}^2 \rangle$ and $\frac{\langle \Delta \hat{N}_y^2 \rangle}{C_y^2} = \langle \Delta \hat{p}^2 \rangle$.

Inherent and irreducible lower bound for the simultaneous measurements of two conjugate observables is four times larger than the Heisenberg uncertainty product.

Bibliography

[1] P. A. M. Dirac. *The Principles of Quantum Mechanics*. Clarendon Press, Oxford, 1958.

[2] W. H. Louisell. *Quantum statistical properties of radiation*. Wiley, New York, 1973.

[3] W. Heisenberg. The actual content of quantum theoretical kinematics and mechanics. *Z. Phys.*, 43:172, 1927.

[4] E. Arthurs and J. L. Kelly. On simultaneous measurement of a pair of conjugate observables. *Bell System Technical Journal*, 44:725, 1965.