Coherence, the physics at the heart of quantum optics, is the capacity of a system to exhibit interference. In classical physics, interference is a wave phenomenon, and the fundamental wave is a function oscillating with a definite frequency. To understand the nature of coherence, it is thus essential to understand basic features of oscillators.

Fundamentals of oscillating function

\[ A \downarrow \phi \quad f(t) = \cos(\omega t - \phi) = (A \cos \phi) \cos \omega t + (A \sin \phi) \sin \omega t \]

- Amplitude
- Phase

\[ = a \cos \omega t + b \sin \omega t \]

- Quadratures
- (90° phase separation, amplitudes)

It is thus convenient to characterize the signal by a single complex amplitude:

\[ f(z) = a + ib = A e^{i\phi} \]

\[ A = \text{Re}(f) = A \cos \phi, \quad b = \text{Im}(f) = A \sin \phi, \quad A = |f| = \sqrt{a^2 + b^2}, \quad \phi = \text{Arg}(f) = \tan^{-1}(\frac{b}{a}) \]

\[ \Rightarrow f(t) = \text{Re}(f e^{-i\omega t}) = \text{Re}((a + ib) e^{-i\omega t}) = \text{Re}(A e^{i\phi} e^{-i\omega t}) \]

\[ = \text{Re}(a e^{-i\omega t}) + \text{Re}(ib e^{-i\omega t}) = \text{Re}(A e^{-i(\omega t - \phi)}) \]

\[ = a \cos \omega t + b \sin \omega t = A \cos(\omega t - \phi) \]
The real signal is thus the real part of a “phasor” in the complex plane

\[ f(t) = \text{Re} \left( \hat{f} e^{-i\omega t} \right) \]

**Simple Harmonic Oscillator**

The physics of an oscillating function is simple harmonic motion.

**Hamiltonian:** \[ H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2 \]

**Hamilton's Eqs:** \[ \dot{x} = \frac{p}{m}, \quad \dot{p} = -m \omega^2 x \Rightarrow \ddot{x} = -\omega^2 x \]

**Define characteristic units:** \( E_c, x, p \)

**Dimensionless variables:** \[ \bar{x} = \frac{x}{x_c}, \quad \bar{p} = \frac{p}{p_c}, \quad \bar{\lambda} = \frac{H}{E_c} \]

\[ E_c \bar{\lambda} = \frac{p^2 x_c^2}{2m} + \frac{1}{2} m \omega^2 x_c^2 \bar{x}^2 \]

**Natural choice of units:** \[ E_c = \frac{p_c^2}{2m} = m \omega^2 x_c^2 \]

\[ \Rightarrow \bar{\lambda} = \frac{1}{2} (\bar{x}^2 + \bar{p}^2) \]

**Contours of constant energy:** Circle of radius \( \sqrt{\bar{x}^2 + \bar{p}^2} \)

\[ \dot{\bar{x}} = +\omega \bar{p}, \quad \dot{\bar{p}} = -\omega \bar{x} \]

\[ \ddot{x} + \omega^2 x = 0 \]

\[ x(t) = x(0) \cos \omega t + \frac{\dot{x}(0)}{\omega} \sin \omega t = x(0) \cos \omega t + p(0) \sin \omega t \]

\[ p(t) = p(0) \cos \omega t + \frac{\dot{p}(0)}{\omega} \sin \omega t = p(0) \cos \omega t + x(0) \sin \omega t \]
Define \( \alpha(t) = \frac{X(t) + i P(t)}{\sqrt{2}} = \alpha(0) e^{-i \omega t} \) (Complex amplitude)

\[
\begin{align*}
X(t) &= \sqrt{2} \text{ Re}(\alpha(t)) = \frac{\alpha(t) + \alpha^*(t)}{\sqrt{2}} \\
P(t) &= \sqrt{2} \text{ Im}(\alpha(t)) = \frac{\alpha(t) - \alpha^*(t)}{i \sqrt{2}}
\end{align*}
\]

Quadratures

Phase space trajectory

Polychromatic signal

A sufficiently well-behaved signal \( f(t) \) can always be expressed in terms of a superposition of monochromatic oscillators via Fourier's theorem

\[
f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t} \frac{d\omega}{2\pi}, \quad \text{where} \quad \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{i\omega t} \, dt
\]

When \( f(t) \) is real, \( \tilde{f}^*(-\omega) = \tilde{f}(\omega) \), \( \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{-i\omega t} \, dt = \tilde{f}(-\omega) \)

\[
\Rightarrow f(t) = \int_{-\infty}^{\infty} \tilde{f}(-\omega)e^{i\omega t} \frac{d\omega}{2\pi} + \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t} \frac{d\omega}{2\pi}
\]

\[
= \int_{-\infty}^{\infty} \tilde{f}(\omega)e^{-i\omega t} \frac{d\omega}{2\pi} + \frac{i\omega}{2\pi} \tilde{f}(\omega)e^{-i\omega t} \frac{d\omega}{2\pi}
\]

\[
= f^+(t) + f^-(t) \quad \text{positive frequency component} \quad \text{negative frequency component}
\]

\[
f^-(t) = [f^+(t)]^*, \quad f(t) = f^+(t) + f^-(t) = \text{Re} \left( 2f^+(t) \right)
\]

Aside: \( 2f^+(t) \) is sometimes called the "analytic signal," the factor of 2 is annoying and pops up here and there
Wave propagation

Consider the simplest case: One dimension wave propagation for a scalar field

\[
\left( \frac{\partial^2}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2}{\partial t^2} \right) \psi(z,t) = 0, \quad v = \text{phase velocity}
\]

Given boundary condition \( \psi(z=0,t) = f(t) \), \( \psi(z,t) = f(t - \frac{2z}{v}), \) for \( z > 0 \)
\( t = \frac{2z}{v} : \) retarded time \( \{ f(t + \frac{2z}{v}), \) for \( z < 0 \)

For an oscillating signal \( f(t) = \text{Re} \left( \hat{f} e^{-i\omega t} \right) \)

\( \psi(z,t) = \text{Re} \left( \hat{f} e^{-i\omega(t + \frac{2z}{v})} \right) = \text{Re} \left( \hat{f} e^{i(2kz - \omega t)} \right), \quad k = \frac{2\pi}{\lambda} \)

For a polychromatic signal the generated wave

\[
\psi(z,t) = \int_{-\infty}^{\infty} \left[ \hat{f}(\omega) + \hat{f}^*(\omega) \right] e^{i(kt - \omega t)} + \int_{-\infty}^{\infty} \left[ \hat{f}(\omega) + \hat{f}^*(\omega) \right] e^{-i(kt + \omega t)}
\]

\[= \sum_{k=\infty}^{0} \hat{f}(k) e^{i(kt - \omega t)} + \sum_{k=\infty}^{0} \hat{f}^*(k) e^{-i(kt + \omega t)}
\]

\[\psi^{(1)}(z,t) \quad \psi^{(2)}(z,t)
\]

\[\omega_k = v|k| \quad \text{(Dispersion relation)}
\]

For electric field, in 3D, including polarization: \( \mathbf{E} \cdot \mathbf{E} = 0 \)

\[
\mathbf{E}(\mathbf{r},t) = \sum_{k} \int d^3k \mathbf{E}(k) \mathbf{E}^*(k) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega k t)} + \sum_{\mathbf{m}} \int d^3k \mathbf{E}(k) \mathbf{E}^*(k) e^{i(\mathbf{k} \cdot \mathbf{r} - \omega k t)}
\]

\[\mathbf{E}^{(1)}(\mathbf{r},t) \quad \mathbf{E}^{(2)}(\mathbf{r},t)
\]
Temporal Coherence of Classical Electromagnetic Waves

We have seen that coherence of an electromagnetic field is seen in an interferometer.

\[ \tilde{E}_{\text{out}}(t) = -\frac{1}{2} \left( \tilde{E}_{\text{in}}(t - \frac{k_1}{c}) + \tilde{E}_{\text{in}}(t - \frac{k_2}{c}) \right) \]

\[ \tilde{I}_{\text{out}}(t) = \frac{1}{4} \tilde{I}_{\text{in}}(t - \frac{k_1}{c}) + \frac{1}{4} \tilde{I}_{\text{in}}(t - \frac{k_2}{c}) + \frac{1}{2} \Re \left[ \tilde{E}_{\text{in}}(t - \frac{k_1}{c}) \tilde{E}^{*}_{\text{in}}(t - \frac{k_2}{c}) \right] \]

For a monochromatic signal \( \tilde{E}_{\text{in}}(t) = E_0 e^{-i\omega t} \), \( \tilde{I}_{\text{in}}(t) = |E_0|^2 \) (constant)

\[ \tilde{I}_{\text{out}} = \frac{1}{2} \tilde{I}_{\text{in}} + \frac{1}{2} \tilde{I}_{\text{in}} \Re \left( e^{i\omega \Delta l/c} \right) = \tilde{I}_{\text{in}} \left( \frac{1 + \cos \left( \frac{\omega \Delta l}{c} \right)}{2} \right) \]

Such a field is said to be perfectly coherent in time. Such a field does not exist in the universe. A more realistic interferometry signal
Why doesn't the interference go on forever as a function of $\Delta t$?

$\implies$ Field has only finite coherence, because it is fluctuating

Classical noise: "Natural light," e.g. Sunlight or light from a bulb is not arising from a single perfectly oscillating source. It is radiated from an ensemble of "random sources," oscillating with random phases with respect to one another. Moreover, this phase can fluctuate due to other random events.

E.g. "Collision broadened" light source: Consider a gas of radiators and a given pressure, density, and temperature. A molecule in the gas radiates at some given frequency. However, due to collisions between the molecules, the phase is interrupted. We thus have a sequence of random wave trains.

$\sqrt{\text{collision}_1} \text{ collision}_2 \sqrt{\text{collision}_3} \text{ collision}_4 \sqrt{\text{collision}_5}$

Monochromatic oscillations between collisions at random times

From the kinetic theory of gases, the mean time between collisions

$$\left(\frac{1}{\sigma_0}\right)^{-1} = \frac{1}{\sigma_0} N \sqrt{\langle v^2 \rangle}, \quad \sigma_0 = \text{collision cross section}, \quad N = \text{density}$$

And the probability the the molecule has a free oscillating for time $t$ without a collision is

$$P(t) = \frac{1}{\sigma_0} e^{-\frac{t}{\sigma_0}} dt$$

The electric field is thus a random variable, because we don't have full information about the trajectory of the molecules. For a single radiator:

$$\hat{E}_1(t) = E_0 e^{-i\omega t + i\phi(t)} \quad \phi(t) \text{ is random}$$

The total field from the gas is the is the sum of the field radiated from all of the atoms, each with its own random phase.

$$\hat{E}(t) = \hat{E}_1(t) + \hat{E}_2(t) + \cdots + \hat{E}_N(t) = E_0 e^{-i\omega t} \sum_{i=1}^{N} e^{i\phi_i(t)} \quad \text{random complex number}$$

$\phi_i(t)$
The random phases add up like a random walk in phase space

\[
\text{Im} \alpha
\]

\[
\text{Re} \alpha
\]

The probability distribution is thus Gaussian:

\[
p(\alpha) = \frac{1}{\pi N} e^{-|\alpha|^2 / N} : \text{N-step random walk}
\]

The intensity is not constant, it fluctuates.

The interferometer thus measures the ensemble average over all the oscillators

\[
I_{\text{int}} = \langle |E(t - \frac{\Delta t}{2})|^2 \rangle + \langle |E(t + \frac{\Delta t}{2})|^2 \rangle + 2 \text{Re} \langle E(t) E(t + \Delta t) \rangle
\]

Note: I have assumed here that the statistics are "stationary", i.e.
the nature of the fluctuations is not changing in time, so the correlations depend only on the time difference, and not the difference in time.

Consider \( \langle E^d(t) E(t + \Delta t) \rangle = E_0^2 e^{-i\phi(t) \Delta t} \langle \{ e^{i\phi(t)} + \ldots + e^{-i\phi(t)} \} \{ e^{i\phi(t + \Delta t)} + \ldots + e^{-i\phi(t + \Delta t)} \} \rangle \)

The term \( \langle e^{-i\phi(t)} e^{i\phi(t + \Delta t)} \rangle \), \( i \neq j \), is 0; so phase relationship between different atomic radiators

\( \Rightarrow \langle E^d(t) E(t + \Delta t) \rangle = N \langle E^*_t(t) E_{t + \Delta t} \rangle \) for any given molecule

\( \Rightarrow \langle |E(t)|^2 \rangle = N E_0^2 = I_{\text{in}} \) \( (N \text{ times, } \text{N is } \text{ bulk of given molecule}) \)

\[
\langle E^*_t(t) E_{t + \Delta t} \rangle = E_0^2 e^{-i\delta^2 \Delta t} N \langle e^{i(\phi(t + \Delta t) - \phi(t))} \rangle
\]

\[
\text{probability that no collision happened for a time } \Delta t \text{ at heat } T
\]

\[
= \int_0^{\frac{1}{T}} p(\tau) \, d\tau = e^{-\frac{\delta^2}{T}}
\]
\[ \langle E^*(t) E(t+\tau) \rangle = N E_0^2 e^{-i\omega \tau} e^{-\tau/\tau_0} = I_{\text{in}} e^{-i\omega_0 \tau_0} e^{-\tau/\tau_0} \]

The output intensity: \( I_{\text{out}} = I_{\text{in}} \left( 1 + \frac{e^{-\tau/\tau_0}}{2} \cos(\omega_0 \tau) \right) \)

where \( \tau = \frac{\Delta k}{c} \)

We thus see for \( \Delta k \ll \tau_0 \), there is no interference. \( \tau_0 \) determines the coherence time, the field maintains coherence with itself only for time \( \tau \) short compared to \( \tau_0 \).

The key point is that the phase relation between \( E(t) \) and \( E(t+\tau) \) was noisy. This made us lose coherence.

\[ \langle E^*(t) E(t+\tau) \rangle = E_0 \langle \chi(t) | \chi(t+\tau) \rangle e^{i\phi(t+\tau)} e^{-i\phi(t)} \]

The field is "temporally coherent," when there is "phase memory." The effect of the collisions is to erase the phase of oscillators. The time scale over which the oscillators maintain a well-determined phase (here the mean time between collisions) sets the coherence time of the field.

The level of coherence in the field is exhibited by the degree of interference can be quantified by the visibility of the interference fringes, defined as

\[ V = \frac{I_{\text{max}} - I_{\text{min}}}{I_{\text{max}} + I_{\text{min}}} \quad 0 \leq V \leq 1 \]

For our model of "natural light": \( I_{\text{max}} = \frac{I_{\text{in}}}{2} (1 + e^{-\Delta k c/\tau_0}) \), \( I_{\text{min}} = \frac{I_{\text{in}}}{2} (1 - e^{-\Delta k c/\tau_0}) \)

- The fringe visibility \( V = e^{-\Delta k c/\tau_0} \rightarrow 1 \) for \( \Delta k < c \tau_0 \) (coherence)
- \( 0 \) for \( \Delta k > c \tau_0 \)