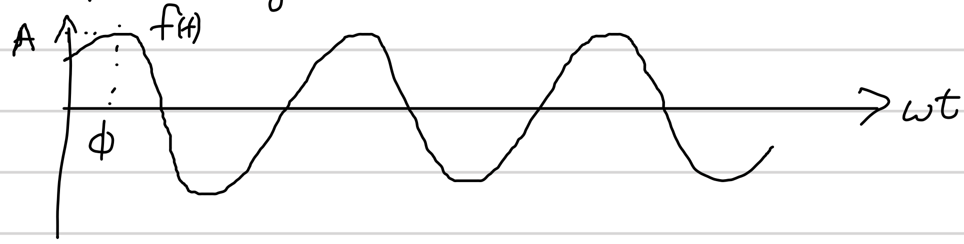
Physics 566: Quantum Optics I

Lecture 2: Oscillators and Coherence

Coherence, the physics at the center of quantum optics, is the capacity of a system to exhibit interference. In classical physics, interference is a wave phenomenon, and the fundamental waves are functions oscillating with a definite frequency. To understand the nature of coherence, it is thus essential to understand basic features of oscillators:

Fundamentals of oscillating functions



Any function the oscillates with a definite frequency, ω, is uniquely specified by two parameters:

$$f(t) = \underset{\text{Amplitude}}{A} \cos(\omega t - \underset{\text{phase}}{\phi}) = (A\cos\phi)\cos\omega t + (A\sin\phi)\sin\omega t$$

$$= a\cos\omega t + b\sin\omega t$$

Quadratures (90° phase-separated amplitudes)

It is thus convenient to characterize the signal by a single complex amplitude:

$$\tilde{f} \equiv a + ib = Ae^{i\phi}$$

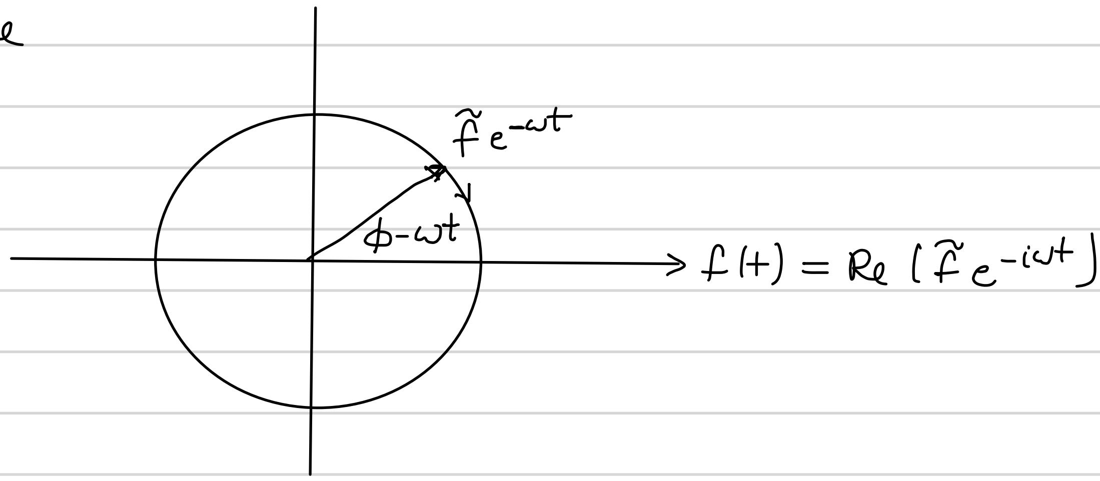
$a = \mathrm{Re}(\tilde{f}) = A\cos\phi$, $b = \mathrm{Im}(\tilde{f}) = A\sin\phi$, $A = |\tilde{f}| = \sqrt{a^2+b^2}$, $\phi = \mathrm{Arg}(\tilde{f}) = \tan^{-1}\left(\frac{b}{a}\right)$

$$\Rightarrow \boxed{f(t) = \mathrm{Re}(\tilde{f}e^{-i\omega t})} = \mathrm{Re}([a+ib]e^{-i\omega t}) = \mathrm{Re}(Ae^{i\phi}e^{-i\omega t})$$

$$= \mathrm{Re}(ae^{-i\omega t}) + \mathrm{Re}(ibe^{-i\omega t}) = \mathrm{Re}(Ae^{-i(\omega t - \phi)})$$

$$= a\cos\omega t + b\sin\omega t = A\cos(\omega t - \phi)$$

The real signal is thus the real part of a "phasor" in the complex plane



Simple Harmonic Oscillator

The physics of an oscillating function is simple harmonic motion

Hamiltonian: $H = \frac{p^2}{2m} + \frac{1}{2} m \omega^2 x^2$

Hamilton's Eqns: $\dot{x} = \frac{p}{m}$, $\dot{p} = -m\omega^2 x \Rightarrow \ddot{x} = -\omega^2 x$

Define characteristic units: E_c, x_c, p_c

Dimensionless variables: $\underline{X} = \frac{x}{x_c}$, $\underline{P} = \frac{p}{p_c}$, $h = \frac{H}{E_c}$

$$\Rightarrow E_c h = \frac{\underline{P}^2 p_c^2}{2m} + \frac{1}{2} m \omega^2 x_c^2 \underline{X}^2$$

Natural choice of units: $E_c = \frac{p_c^2}{m} = m\omega^2 x_c^2$

$$\Rightarrow h = \frac{1}{2}(\underline{X}^2 + \underline{P}^2)$$

Contours of constant energy: Circles of radius $\frac{\sqrt{\underline{X}^2 + \underline{P}^2}}{\sqrt{2}}$

$$\dot{\underline{X}} = +\omega \underline{P}, \quad \dot{\underline{P}} = -\omega \underline{X}, \quad \ddot{\underline{X}} + \omega^2 \underline{X} = 0$$

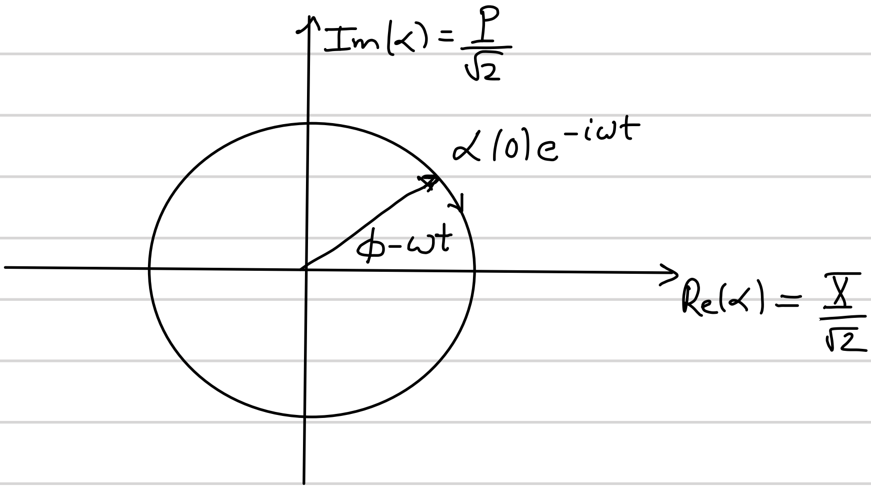
$$\underline{X}(t) = \underline{X}(0)\cos\omega t + \frac{\dot{\underline{X}}(0)}{\omega}\sin\omega t = \underline{X}(0)\cos\omega t + \underline{P}(0)\sin\omega t$$

$$\underline{P}(t) = \underline{P}(0)\cos\omega t + \frac{\dot{\underline{P}}(0)}{\omega}\sin\omega t = \underline{P}(0)\cos\omega t + \underline{X}(0)\sin\omega t$$

Define $\alpha(t) = \dfrac{X(t) + i P(t)}{\sqrt{2}} = \alpha(0) e^{-i\omega t}$ (Complex amplitude)

$$\left.\begin{aligned} X(t) &= \sqrt{2}\, \mathrm{Re}(\alpha(t)) = \frac{\alpha(t) + \alpha^*(t)}{\sqrt{2}} \\ P(t) &= \sqrt{2}\, \mathrm{Im}(\alpha(t)) = \frac{\alpha(t) - \alpha^*(t)}{i\sqrt{2}} \end{aligned}\right\} \text{Quadratures}$$

Phase space trajectory



Poly chromatic signal

A sufficiently well behaved signal $f(t)$ can always be expressed in terms of a superposition of monochromatic oscillators via Fourier's theorem

$$f(t) = \int_{-\infty}^{\infty} \tilde{f}(\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi}, \quad \text{where} \quad \tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{+i\omega t}\, dt$$

When $f(t)$ is real, $\tilde{f}^*(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t} dt = \tilde{f}(-\omega)$

$$\Rightarrow f(t) = \int_{-\infty}^{0} \tilde{f}^*(-\omega)\, e^{i\omega t} \frac{d\omega}{2\pi} + \int_0^{\infty} d\omega\, \tilde{f}(\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi}$$

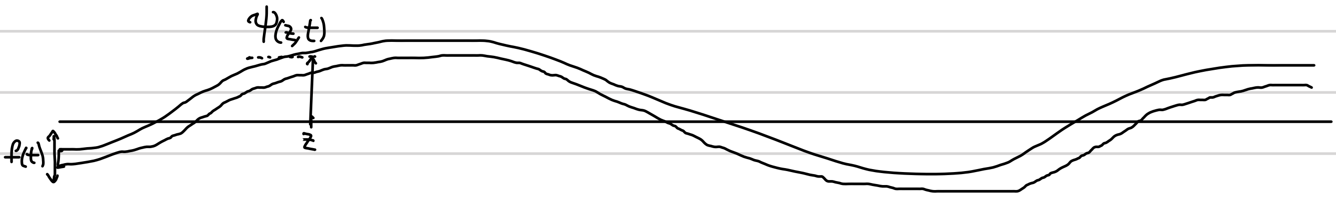
$$= \underbrace{\int_0^{\infty} \tilde{f}(\omega)\, e^{-i\omega t} \frac{d\omega}{2\pi}}_{\equiv f^{(+)}(t) \text{ positive frequency component}} + \underbrace{\int_0^{\infty} d\omega\, \tilde{f}^*(\omega)\, e^{+i\omega t} \frac{d\omega}{2\pi}}_{\equiv f^{(-)}(t) : \text{ negative freq. component}}$$

$$f^{(-)}(t) = \left[f^{(+)}(t)\right]^*, \quad f(t) = f^{(+)}(t) + f^{(-)}(t) = \mathrm{Re}\left(2 f^{(+)}(t)\right)$$

Aside: $2f^{(+)}(t)$ is sometimes called the "Analytic signal" the factor of 2 is annoying and pops up here and there

Wave propagation

Consider the simplest case: One dimension wave propagation for a scalar field



The wave amplitude satisfies the 1D wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v^2}\frac{\partial}{\partial t^2}\right)\psi(z,t) = 0 \quad , \quad v = \text{phase velocity.}$$

Given boundary condition $\psi(z=0,t) = f(t)$, $\psi(z,t) = \begin{cases} f(t - z/v) , & \text{for } z>0 \\ f(t+z/v), & \text{for } z<0 \end{cases}$

$t \mp z/v$: retarded time

For an oscillating signal $f(t) = \mathrm{Re}(\tilde{f} e^{-i\omega t})$

$$\psi(z,t) = \mathrm{Re}(\tilde{f} e^{-i\omega(t\mp z/v)}) = \mathrm{Re}(\tilde{f} e^{i(\pm kz - \omega t)}) \quad , \quad k = \frac{\omega}{v} = \frac{2\pi}{\lambda}$$

For a polychromatic signal the generated wave

$$\psi(z,t) = \underbrace{\int_0^\infty d\omega \left[\tilde{f}(\omega) + \tilde{f}^*(\omega)\right] e^{i(kz-\omega t)}}_{\text{forward propagating waves}} + \underbrace{\int_0^\infty d\omega \left[\tilde{f}(\omega) + \tilde{f}^*(\omega)\right] e^{i(-kz-\omega t)}}_{\text{Backward propagating}}$$

$$= \underbrace{\int_{-\infty}^{\infty} dk\, \tilde{f}(\omega_k)\, e^{i(kz-\omega_k t)}}_{\psi^{(+)}(z,t)} + \underbrace{\int_{-\infty}^{\infty} dk\, \tilde{f}^*(\omega_k)\, e^{-i(kz-\omega_k t)}}_{\psi^{(-)}(z,t)}$$

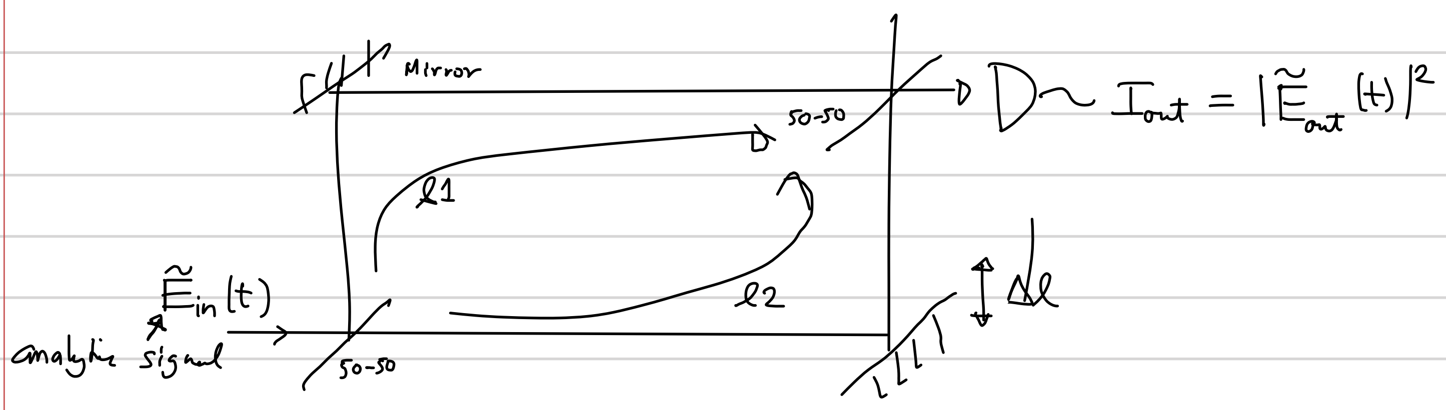
$$\omega_k = v|k| \quad (\text{Dispersion relation})$$

For electric field, in 3D, including polarization: $\vec{k}\cdot\vec{\epsilon}_\mu = 0$

$$\vec{E}(\vec{r},t) = \underbrace{\sum_\mu \int d^3\vec{k}\, \tilde{E}(\omega_{\vec{k}})\, \vec{\epsilon}_\mu(\vec{k})\, e^{i(\vec{k}\cdot\vec{r} - \omega_{\vec{k}} t)}}_{\vec{E}^{(+)}(\vec{r},t)} + \underbrace{\sum_\mu \int d^3\vec{k}\, \tilde{E}^*(\omega_k)\, e^{-i(\vec{k}\cdot\vec{r} - \omega_k t)}}_{\vec{E}^{(-)}(\vec{r},t)}$$

Temporal Coherence of Classical Electromagnetic Waves

We have seen that coherence of an electromagnetic field is seen in an interferometer

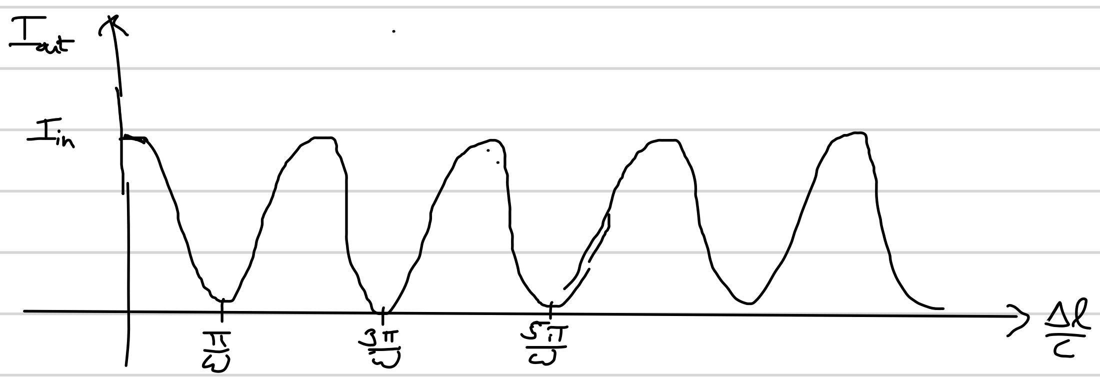


$$\tilde{E}_{out}(t) = -\frac{1}{2}\left(\tilde{E}_{in}\left(t-\frac{\ell_1}{c}\right) + \tilde{E}_{in}\left(t-\frac{\ell_2}{c}\right)\right)$$

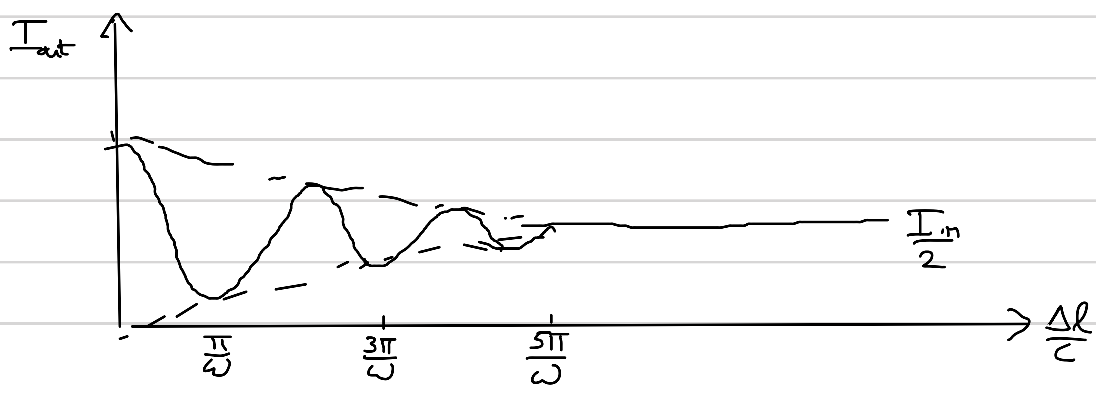
$$\tilde{I}_{out}(t) = \frac{1}{4} I_{in}\left(t-\frac{\ell_1}{c}\right) + \frac{1}{4} I_{in}\left(t-\frac{\ell_2}{c}\right) + \frac{1}{2} \mathrm{Re}\left[\tilde{E}_{in}\left(t-\frac{\ell_1}{c}\right), \tilde{E}^*_{in}\left(t-\frac{\ell_2}{c}\right)\right]$$

For a monochromatic signal $\tilde{E}_{in}(t) = E_0 e^{-i\omega t}$, $I_{in}(t) = |E_0|^2$ (constant)

$$I_{out} = \frac{1}{2} I_{in} + \frac{1}{2} I_{in} \mathrm{Re}\left(e^{i\omega \frac{\Delta \ell}{c}}\right) = I_{in}\left(\frac{1+\cos\left(\frac{\omega \Delta \ell}{c}\right)}{2}\right)$$



Such a field is said to be perfectly coherent in time. Such a field does not exist in the universe. A more realistic interferometery signal

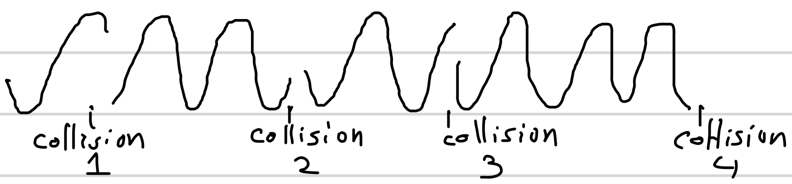


Why doesn't the interference go on for ever as a function of Δl?

$\Rightarrow$ Field has only finite coherence, because it is <u>fluctuating</u>

<u>Classical noise</u>: "Natural light", e.g. Starlight or light from a bulb is not arising from a single perfectly oscillating source. It is radiated from an ensemble of "random sources", oscillating with random phases with respect to one another. Moreover, this phase can fluctuate due to other random events

E.g. "Collision-broadened" light source: Consider a gas of radiators and a given pressure, density, and temperature. A molecule in the gas radiates at some given frequency. However, due to collisions between the molecules, the <u>phase</u> is interrupted. We thus have a sequence of random wave trains



Monochromatic oscillations between collisions at random times

From the kinetic theory of gases, the mean-time between collisions

$$(\tau_0)^{-1} = \sigma_0 \, n \sqrt{\langle v^2 \rangle} \quad , \quad \begin{array}{l} \sigma_0 = \text{collision cross section} \\ n = \text{density} \end{array}$$

And the probability the the molecule has a free oscillating for time τ without a collision is $p(\tau)\,d\tau = \frac{1}{\tau_0} e^{-\tau/\tau_0}\,d\tau$

The electric field is thus a <u>random variable</u>, because we don't have full information about the trajectory of the molecules. For a single radiator

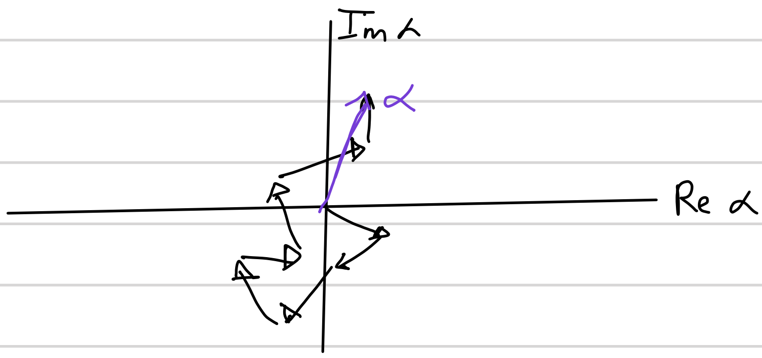
$$\tilde{E}_1(t) = E_0 \, e^{-i\omega t + i\phi_1(t)} \quad , \quad \phi_1(t) \text{ is random}$$

The total field from the gas is the is the sum of the field radiated from all of the atoms, each with its own random phase

$$\tilde{E}(t) = \tilde{E}_1(t) + \tilde{E}_2(t) + \cdots + \tilde{E}_N(t)$$
$$= E_0 e^{-i\omega_0 t} \underbrace{\sum_{i=1}^{N} e^{i\phi_i(t)}}$$

Random complex number $\alpha(t)$

The random phases add up like a random walk in phase space



The probability distribution is thus Gaussian:

$$p(|\alpha|) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}} \quad : \quad N\text{- step random walk}$$

the intensity is not constant, it <u>fluctuates</u>.

The interferometer thus measures the ensemble average over all the oscillators

$$I_{out} = \langle |E(t - \tfrac{l_1}{c})|^2 \rangle + \langle |E(t - \tfrac{l_2}{c})|^2 \rangle + 2\,\mathrm{Re}\,\langle E^*(t)\, E(t + \tfrac{\Delta l}{c}) \rangle$$

Note: I have assume here that the statistics are "stationary", i.e. the nature of the fluctuations is not changing in time, so the correlations depend only on the time-differences, and not the difference in times.

Consider $\langle E^*(t) E(t+\tau) \rangle = E_0^2 e^{-i\omega_0 \tau} \langle \{ e^{-i\phi_1(t)} + \dots + e^{-i\phi_N(t)} \} \{ e^{i\phi_1(t+\tau)} + \dots e^{i\phi_N(t+\tau)} \} \rangle$

The term $\langle e^{-i\phi_i(t)} e^{i\phi_j(t+\tau)} \rangle_{i \neq j} = 0$; no phase relationship between different atomic radiators

$\Rightarrow \langle E^*(t) E(t+\tau) \rangle = N \langle E_i^*(t) E_i(t+\tau) \rangle$ for any given molecule

$\Rightarrow \langle |E(t)|^2 \rangle = N E_0^2 = I_{in}$ (N times intensity of given molecule)

$$\langle E_i^*(t) E_i(t+\tau) \rangle = E_0^2 e^{-i\omega_0 \tau} N \underbrace{\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle}_{\text{probability that no collision happened for a time at least } \tau}$$

$$= \int_{\tau}^{+\infty} p(\tau)\, d\tau = e^{-\tau/\tau_0}$$

$$\Rightarrow \langle E^*(t)\, E(t+\tau)\rangle = N E_0^2\, e^{-i\omega_0 \tau}\, e^{-\tau/\tau_0} = I_{in}\, e^{\left(-i\omega_0 - \frac{1}{\tau_0}\right)\tau}$$

the output intensity: $I_{out} = I_{in}\left(\dfrac{1 + e^{-\tau/\tau_0}\cos(\omega_0 \tau)}{2}\right)$

where $\tau = \dfrac{\Delta l}{c}$

We thus see for $\Delta l/c \gg \tau_0$, there is no interference. τ_0 determines the coherence time. the field maintains coherence with itself only for time τ short compared to τ_0

the key point is that the phase relation between $E(t)$ and $E(t+\tau)$ was noisy. this made us lose coherence

$$\langle E^*(t)\, E(t+\tau)\rangle = E_0 \Big\langle |\alpha(t)|\,|\alpha(t+\tau)|\, \underbrace{e^{i\phi(t+\tau)}\, e^{-i\phi(t)}}_{\text{randomized phase}} \Big\rangle$$

The field is "temporally coherent" when there is "phase memory." the effect of the collisions is to erase the phase of oscillators. the time scale over which the oscillators maintain a well determined phase (here the mean time between collisions) sets the coherence time of the field.

the level of coherence in the field is exhibited by the degree of interference can be quantified by the visibility of the interference fringes, defined as

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \qquad 0 \le V \le 1$$

For our model of "natural light": $I_{max} = \frac{I_{in}}{2}\left(1 + e^{-\frac{\Delta l}{c\tau_0}}\right)$; $I_{min} = \frac{I_{in}}{2}\left(1 - e^{-\frac{\Delta l}{c\tau_0}}\right)$

$\Rightarrow$ the fringe visibility $V = e^{-\frac{\Delta l}{c\tau_0}}$ $\rightarrow 1$ for $\Delta l \ll c\tau_0$ (coherence length)

$\rightarrow 0$ for $\Delta l \gg c\tau_0$