Physics 566. Quantum Optics I

Lecture 4: Two Level Quantum Systems

The Two-level Atom

We now begin our foray in the the quantum world, launching off our discussion of the classical Lorentz model. We are interested in the quantum description of absorption / emission of light. To start, we will treat the light classically as an electromagnetic wave; we will return to the quantized field soon enough, and understand when this is a good description. We will also focus our attention on atomic gases, though much of the essential physics in quantum optics is the same, whether we talk about atoms, molecules, or condensed phase liquids or solids. Treating the matter (atoms) quantumly, while the field (wave) classically is known as the semiclassical model.

The total Hamiltonian is of the form

$$\hat{H} = \hat{H}_A + \hat{H}_{int}(t)$$

- $\hat{H}_A = \sum_k E_k |k\rangle\langle k|$ is the bare atomic Hamiltonian, with $|k\rangle$ and energy eigenstate, with k containing a complete set of quantum numbers, E_k = energy eigenvalue.

- $\hat{H}_{int}$ describes the interaction of the atom with an electromagnetic field, taken here to be a monochromatic wave (laser field), $\vec{E}(\vec{r},t) = \mathrm{Re}(\underset{\sim}{\vec{E}}(\vec{r})e^{-i\omega_L t})$, $\vec{B}(\vec{r},t) = \mathrm{Re}(\underset{\sim}{\vec{B}}(\vec{r})e^{-i\omega_L t})$

Where the size of the atom, $a_0 \ll \frac{c}{\omega_L} \sim \lambda_L$, we can make a multipole expan.

$$\Rightarrow \hat{H}_{int} \approx \underbrace{-q\vec{R}\cdot\vec{E}(R,t)}_{\text{monopole}} \underbrace{-\hat{\vec{d}}\cdot\vec{E}(\vec{R},t)}_{\text{electric dipole}} \underbrace{-\hat{\vec{\mu}}\cdot\vec{B}(\vec{R},t)}_{\text{magnetic dipole}} + \text{quadrupole} + \cdots$$

Here $\vec{R}$ is the center of mass of the atom, which we treat here as a classical variable, but we will return to think about the quantized motion of the whole atom when we study laser cooling next semester. Note: the monopole term is zero for a neutral atom but not for ions. We will return to this.

Recall, from time-dependent perturbation theory, given a harmonic interaction

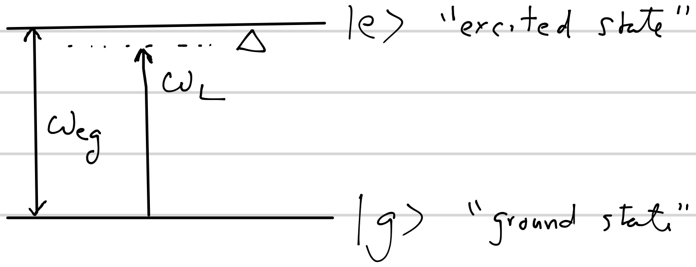
$$\hat{H}_{int}(t) = \hat{H}^{(+)} e^{-i\omega t} + \hat{H}^{(-)} e^{+i\omega t}$$

to lowest order, the transition probability from intial state $|i\rangle$ to final state $|f\rangle$ is

$$P_{f\leftarrow i} = \frac{|\langle f|\hat{H}^{(+)}|i\rangle|^2}{(E_f - E_i - \hbar\omega_L)^2} \sin^2\left[\left(\frac{E_f - E_i}{\hbar} - \omega_L\right)t\right] + \frac{|\langle f|\hat{H}^{(-)}|i\rangle|^2}{(E_i - E_f - \hbar\omega_L)^2} \sin^2\left[\left(\frac{E_i - E_f}{\hbar} - \omega_L\right)t\right]$$

From this expression, it is clear that resonances dominate. That is, if the applied force is oscillating at a frequency ω_L close to the Bohr frequency $\omega_{fi} \equiv \frac{E_f - E_i}{\hbar}$, only the transition probability $P_{f\leftarrow i}$ is nonnegligible. This

leads us to the concept of the "two-level atom."



When $|\Delta| << \omega_{eg}$ from any other nearby transition we can restrict our attention to just these two levels: if the population starts in the ground state $|g\rangle$, then population can transition to the excited state or back to the ground state, but no where else. We can thus restrict our attention to a 2D subspace of the otherwise infinite dimensional Hilbert space. The total state of the atom is then

$$|\Psi(t)\rangle = C_g(t)|g\rangle + C_e(t)|e\rangle$$

Evolving under the Hamiltonian, $\hat{H} = E_g|g\rangle\langle g| + E_e|e\rangle\langle e| + \hat{H}_{int}(t)$ ← projected on $\{|g\rangle, |e\rangle\}$

Typically, we consider electric dipole allowed transitions $|g\rangle \leftrightarrow |e\rangle$, so

$$\hat{H}_{int} = -\hat{\vec{d}}\cdot\vec{E}(\vec{R},t) = e\hat{\vec{r}}\cdot\vec{E}(\vec{R},t), \quad \hat{\vec{r}} = \text{electron position relative to center-of-mass}$$

Note: The atomic eigenstates are eigenstates of parity, so $\langle e|\hat{\vec{r}}|e\rangle = \langle g|\hat{\vec{r}}|g\rangle = 0$

$\therefore$ In the 2D space: $\hat{H}_{int} = -(\langle e|\hat{\vec{d}}|g\rangle\, |e\rangle\langle g| + \langle g|\hat{\vec{d}}|e\rangle\, |g\rangle\langle e|)\cdot\vec{E}(\vec{R},t)$

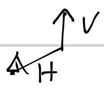
$$= -\langle e|\vec{d}|g\rangle\cdot\vec{E}\; |e\rangle\langle g| + h.c.$$

Two-level Quantum Mechanics. Qubits

The take away message from the discussion above is that when the driving field is close to resonance between to (nondegenerate) energy levels, all of the quantum dynamics is restricted to a two-dimensional Hilbert space. The beautiful thing is that all two-dimensional Hilbert spaces are isomorphic! That means that any 2D Hilbert space maps onto another with the same mathematics, even if the physics is completely different!

In modern parlance, an 2D Hilbert space defines a quantum bit, or "qubit" where we define a logical basis $\{|0\rangle, |1\rangle\}$. Quantum bits can exist in superposition of $|0\rangle$ and $|1\rangle$. This is the foundation of quantum information and why quantum optics is the natural arena in which QI is explored.

Examples of qubits:

- Two-level atom $|0\rangle = |e\rangle$, $|1\rangle = |g\rangle$
- Polarization of light  $|H\rangle = |0\rangle$, $|V\rangle = |1\rangle$
- Spin of an electron $|0\rangle = |\uparrow_z\rangle$, $|1\rangle = |\downarrow_z\rangle$

The last example is the quintissential paradigm: All two-level systems (qubits) are isomorphic to the physics of spin-½. We thus map the wide variety of physical systems: atom-laser interactions, polarization optics, etc., into an equivalent spin-½ paradigm. In the remainder of the lecture, we will review the foundations of this description.

<u>Spin-½</u>. Angular momentum $\hat{\vec{S}}$ $\quad [\hat{S}_i, \hat{S}_j] = i\hbar\,\epsilon_{ijk}\,\hat{S}_k$

Standard basis, simultaneous eigenstates of $\hat{S}^2$, $\hat{S}_z$

$$\hat{S}^2|s,m_s\rangle = s(s+1)|s,m_s\rangle, \quad \hat{S}_z = \hbar m_s|s,m_s\rangle, \quad s = 1/2 \quad m_s = \pm\frac{1}{2}$$

Raising and lowering operators: $\hat{S}_\pm \equiv \hat{S}_x + i\hat{S}_y$, $[\hat{S}_z, \hat{S}_\pm] = \pm\hat{S}_\pm$, $[\hat{S}_+, \hat{S}_-] = 2\hat{S}_z$

$$\hat{S}_+|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2}, \tfrac{1}{2}\rangle, \quad \hat{S}_-|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \hbar|\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$

$$\hat{S}_+|\tfrac{1}{2}, \tfrac{1}{2}\rangle = \text{null}, \quad \hat{S}_-|\tfrac{1}{2}, -\tfrac{1}{2}\rangle = \text{null}$$

Pauli-spin operators / matrices

Define $\hat{S} \equiv \frac{\hbar}{2}\hat{\sigma}$ (Note: $\hat{S}_{\pm} = \hbar\hat{\sigma}_{\pm}$, factor of 2) (Reduce qubit to its essence: not units)

Define $|0\rangle = |\frac{1}{2}, \frac{1}{2}\rangle = |\uparrow_z\rangle$, $|1\rangle = |\frac{1}{2}, -\frac{1}{2}\rangle = |\downarrow_z\rangle$

$\hat{\sigma}_z|0\rangle = |0\rangle$, $\hat{\sigma}_z|1\rangle = -|1\rangle$ $\quad \hat{\sigma}_z|z\rangle = (-1)^z|z\rangle$, $z = \{0,1\}$

$\hat{\sigma}_+|1\rangle = |0\rangle$, $\hat{\sigma}_-|0\rangle = |1\rangle$, $\hat{\sigma}_+|0\rangle = \hat{\sigma}_-|1\rangle = \text{null}$

$\hat{\sigma}_+ = |0\rangle\langle 1|$, $\hat{\sigma}_- = |1\rangle\langle 0|$, $\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$, $\hat{\sigma}_y = \frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i}$

Matrix representation in the standard "computational" basis:

$$\hat{\sigma}_z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \hat{\sigma}_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \hat{\sigma}_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \hat{\sigma}_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

$[\hat{\sigma}_i, \hat{\sigma}_j] = 2i\epsilon_{ijk}\hat{\sigma}_k$, $\{\hat{\sigma}_i, \hat{\sigma}_j\} = 2\delta_{ij}$ $\Rightarrow$ Different $\hat{\sigma}_i$ anticommute, $\hat{\sigma}_i^2 = \hat{\mathbb{1}}$ (square to identity)

$$\hat{\sigma}_i\hat{\sigma}_j = \delta_{ij}\hat{\mathbb{1}} + i\epsilon_{ijk}\hat{\sigma}_k$$

$$[\hat{\sigma}_z, \hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm, \qquad [\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$$

$\hat{\sigma}_i^\dagger = \hat{\sigma}_i$ (Hermitian) and thus $\hat{\sigma}_i^\dagger\hat{\sigma}_i = \hat{\sigma}_i^2 = \hat{\mathbb{1}}$ (also unitary).

The eigenvalues of $\hat{\sigma}_i$ are ± 1 $\Rightarrow$ $\mathrm{Tr}(\hat{\sigma}_i) = \sum$ eigenvalues $= 0$

$\det(\hat{\sigma}_i) = \prod$ eigenvalues $= -1$

The Pauli matrices together with the 2X2 identity matrix form an "orthogonal basis" for all 2x2 matrices with complex entries with the trace inner product.

Aside: The set of $d \times d$ matrices are themselves form a vector space $\mathbb{C}^{d^2}$. $c_1\hat{A} + c_2\hat{B} \in \mathbb{C}^{d^2}$

Inner produce $(\hat{A}|\hat{B}) \equiv \mathrm{Tr}(\hat{A}^\dagger\hat{B})$ (Projection $\hat{A}$ onto $\hat{B}$)

Basis operators $\hat{T}_{nm} \equiv |n\rangle\langle m|$ $\Rightarrow$ $\mathrm{Tr}(\hat{T}_{nm}\hat{A}) = \langle n|\hat{A}|m\rangle$

$$\hat{A} = \sum A_{nm}\hat{T}_{nm}$$

identity $\quad x \quad y \quad z$

$\{\hat{\sigma}_0, \hat{\sigma}_1, \hat{\sigma}_2, \hat{\sigma}_3\}$ is an orthogonal basis: $\quad Tr(\hat{\sigma}_i \hat{\sigma}_j) = Tr(\hat{\mathbb{1}})\delta_{ij} = 2\delta_{ij}$

$\Rightarrow$ Any operator on $\mathbb{C}^2$ $\quad \hat{A} = \frac{1}{2}\sum_{i=0}^{3} a_i \hat{\sigma}_i = \frac{1}{2}(a_0 \hat{\mathbb{1}} + \vec{a}\cdot\hat{\vec{\sigma}})$

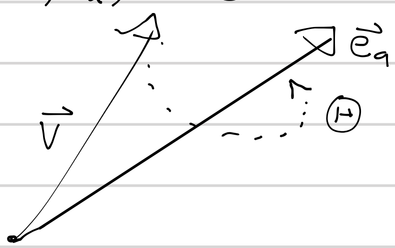
where $a_0 = Tr(\hat{A}\hat{\sigma}_0) = Tr(\hat{A})$

$i = 1,2,3 \qquad a_i = Tr(\hat{A}\hat{\sigma}_i)$

SU(2). Rotations in a 2D Hilbert Space

Angular momentum is the generator of rotations in 3D space:

Rotation operator: $\hat{D}(\Theta, \vec{e}_a) = e^{-\frac{i}{\hbar}\Theta \vec{e}_a\cdot\hat{\vec{S}}} = e^{-i\frac{\Theta}{2}\vec{e}_a\cdot\hat{\vec{\sigma}}}$



Rotation of a vector $\vec{V}$ about an direction (unit vector $\vec{e}_a$)

E.g $\quad \hat{D}(\Theta, \hat{e}_z)\,\hat{\sigma}_x\,\hat{D}^{\dagger}(\Theta, \hat{\sigma}_z) = \cos\Theta\,\hat{\sigma}_x + \sin\Theta\,\hat{\sigma}_y = \vec{e}_n\cdot\hat{\vec{\sigma}}$ $\quad$ (Passive rotation)

where $\vec{e}_n = \cos\Theta\,\vec{e}_x + \sin\Theta\,\vec{e}_y$

In general, for an arbitrary direction $\vec{e}_n$ there exists a rotation that maps $\vec{e}_z \Rightarrow \vec{e}_n$, and thus there exists a Θ and $\vec{e}_a$ such that

$\hat{D}(\Theta, \hat{e}_a)\,\hat{\sigma}_z\,\hat{D}^{\dagger}(\Theta, \vec{e}_a) = \vec{e}_n\cdot\hat{\vec{\sigma}} \Rightarrow \vec{e}_n\cdot\hat{\vec{\sigma}}$ and $\hat{\sigma}_z$ are similar matrices.

$\Rightarrow$ In eigenvalues of $\vec{e}_n\hat{\sigma}$ are ± 1, with eigenvectors $|\uparrow_{\vec{e}_n}\rangle = \hat{D}(\Theta, \vec{e}_a)|\uparrow_{\vec{e}_z}\rangle$

$|\downarrow_{\vec{e}_n}\rangle = \hat{D}(\Theta, \vec{e}_a)|\downarrow_{\vec{e}_z}\rangle$

Using the properties of the Pauli matrices

$$\hat{D}(\Theta, \vec{e}_a) = \cos\left(\frac{\Theta}{2}\right)\hat{\mathbb{1}} - i\sin\left(\frac{\Theta}{2}\right)\vec{e}_a\cdot\hat{\vec{\sigma}}$$

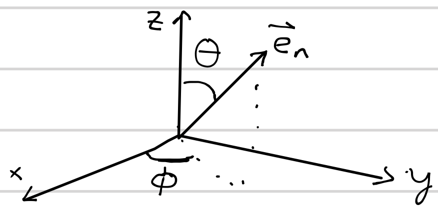
The eigenvalues of this matrix as $\cos\left(\frac{\Theta}{2}\right) \mp i\sin\left(\frac{\Theta}{2}\right) = e^{\pm i\frac{\Theta}{2}}$

$\Rightarrow \det[\hat{D}(\Theta, \vec{e}_a)] = e^{+i\frac{\Theta}{2}}e^{-\frac{i}{2}\Theta} = 1 \Rightarrow$ "Special unitary matrix"

$\{\hat{D}(\Theta, \vec{e}_a)\} = SU(2)$. $\quad$ General $U(2) = e^{i\varphi}SU(2)$.

Spin along an arbitrary direction $|\uparrow_{\vec{e}_n}\rangle$

Consider a direction in 3D space : $\vec{e}_n = \frac{\vec{r}}{|\vec{r}|} = \sin\theta\left[\cos\phi\, \vec{e}_x + \sin\phi\, \vec{e}_y\right] + \cos\theta\, \vec{e}_z$



Map $\vec{e}_z \Rightarrow \vec{e}_n$: Rotatate about y by θ, then about z by ϕ

$$\hat{D}_y(\theta) = \cos\frac{\theta}{2}\,\mathbb{1} - i\sin\frac{\theta}{2}\,\hat{\sigma}_y = \begin{bmatrix} \cos\frac{\theta}{2} & -\sin\frac{\theta}{2} \\ \sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix}$$

$$\hat{D}_z(\phi) = \cos\frac{\phi}{2}\,\mathbb{1} - i\sin\frac{\phi}{2}\,\hat{\sigma}_z = \begin{bmatrix} e^{-i\phi/2} & 0 \\ 0 & e^{i\phi/2} \end{bmatrix}$$ in standard basis

$$\Rightarrow \hat{D}_z(\phi)\,\hat{D}_y(\theta) = \begin{bmatrix} e^{-i\phi/2}\cos\frac{\theta}{2} & -e^{-i\phi/2}\sin\frac{\theta}{2} \\ e^{+i\phi/2}\sin\frac{\theta}{2} & e^{+i\phi/2}\cos\frac{\theta}{2} \end{bmatrix}$$

$$\Rightarrow |\uparrow_n\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)|\uparrow_z\rangle = e^{-i\phi/2}\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi/2}\sin\frac{\theta}{2}|\downarrow_z\rangle \equiv \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle$$

$$\Rightarrow |\downarrow_n\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)|\downarrow_z\rangle = -e^{-i\phi/2}\sin\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi/2}\cos\frac{\theta}{2}|\downarrow_z\rangle \equiv \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle$$

Example: $\hat{\sigma}_x$, $(\theta = \frac{\pi}{2}, \phi = 0)$: $|\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|\downarrow_x\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

$\hat{\sigma}_y$, $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2})$: $|\uparrow_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$

$|\downarrow_y\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$

Note: In general $|\downarrow_n\rangle = |\uparrow_{-n}\rangle$, and since $\langle\uparrow_n|\downarrow_n\rangle = 0$

$\Rightarrow$ Orthogonal spin states <u>do not</u> correspond to orthogonal directions

Finding the eigenvalues/eigenvectors of an arbitrary 2x2 matrix

Consider $\hat{A} \in \mathbb{C}^{2\times 2}$. The $\hat{A}$ can be decomposed in terms of the Pauli Matrices

$$\hat{A} = \frac{1}{2}\left(a_0 \hat{\mathbb{1}} + \vec{a}\cdot\hat{\vec{\sigma}}\right), \quad \text{where} \quad a_0 = \mathrm{Tr}(\hat{A}), \quad a_i = \mathrm{Tr}(\hat{A}\hat{\sigma}_i) \quad i=1,2,3$$
$$= \frac{1}{2}\left(a_0 \hat{\mathbb{1}} + |\vec{a}|\,\vec{e}_n\cdot\hat{\vec{\sigma}}\right), \quad \text{where} \quad \vec{e}_n = \frac{\vec{a}}{|\vec{a}|} = (\theta,\phi) \quad \therefore \tan\phi = \frac{a_y}{a_x}, \quad \cos\theta = \frac{a_z}{|\vec{a}|}$$

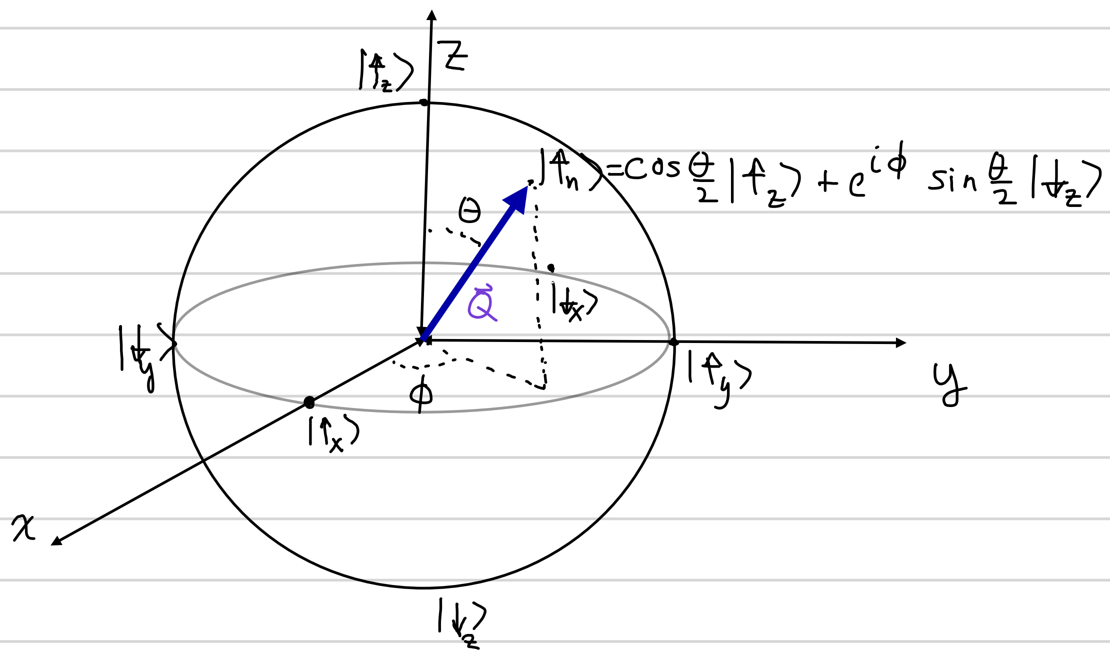
$\Rightarrow$ The eigenvalues of $\hat{A}$ are $\frac{1}{2}(a_0 \pm |\vec{a}|)$, with corresponding

$$\text{eigenvectors} \begin{cases} |\uparrow_n\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\frac{\theta}{2}|1\rangle \\ |\downarrow_n\rangle = \sin\frac{\theta}{2}|0\rangle - e^{i\phi}\cos\frac{\theta}{2}|1\rangle \end{cases}$$

Bloch Sphere

Fact: Every (pure) state in a 2D Hilbert space is equivalent (up to an overall phase) to spin-up along some direction $\vec{e}_n$ (see homework).

This provides a beautiful geometry for describing the space $\mathbb{C}^2$, for every pure state $|\uparrow_{\vec{e}_n}\rangle$ is in one-to-one correspondence with a direction on a sphere. This is known as the Bloch sphere.



The "antipodal" points on the sphere are orthogonal. A state is an equal superposition of the states on the the orthogonal axis, e.g. $|\uparrow_z\rangle$ is a 50-50 of $|\uparrow_x\rangle$ and $|\downarrow_x\rangle$; $|\uparrow_y\rangle$ is a 50-50 superposition of $|\uparrow_z\rangle, |\downarrow_z\rangle$.

The Bloch vector $\vec{Q} = \vec{e}_n = \langle\uparrow_n|\hat{\vec{\sigma}}|\uparrow_n\rangle$ (see homework)