Optical Coherence and Photon Statistics

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FOREWORD

The lectures on which these notes are based were intended to serve as an elementary introduction to quantum optics. They were begun, for that reason, with discussions of classical experiments, and the introduction of quantum mechanical ideas was carried out fairly gradually. The most advanced knowledge of quantum electrodynamics which they require at any point is some acquaintance with the connection between the quantization of harmonic oscillators and that of fields. This is material which is covered in the first two of Professor Kroll's lectures, or in the initial chapters of a number of elementary texts on field theory.

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Lecture I. INTRODUCTION

The field of optics, after seeming to have reached a sort of maturity, is beginning to undergo some rapid and revolutionary changes. These changes are connected with things which we have, as a matter of principle, known about for many years, but the extent to which we could put our knowledge into practice has, until just a few years ago, been extremely limited. Thus the electromagnetic character of light waves has been familiar knowledge since the last century. A vast body of theory and technique concerning the generation of electromagnetic waves has been built up during these years, but virtually all of it has dealt with radio frequency fields. Light waves of course, are of the same electromagnetic character as radio waves. But because the only ways we had of generating them in the past were extremely clumsy (in a sense we shall presently discuss at some length) there has been very little occasion until recently to apply the insights of radio-frequency theory in optics. A simple physical reason, as we shall see, lies at the bottom of this: all of the traditional types of optical sources possess a certain chaotic quality in common. They are what a radio engineer would refer to as noise generators, and all of the delicate and ingenious techniques of optics are exercises in the constructive use of noise. The invention of the optical maser has removed this barrier with almost a single stroke. It allows us to presume that we will some day be able to control fields oscillating at optical or higher frequencies with the same sort of precision and versatility that have become familiar in radio frequency technology.

Another recent change is the development of detectors which respond strongly to individual quanta of light. These have permitted us to explore the corpuscular character of optical fields. All of the traditional optical experiments have not only dealt with extremely crude sources, but have paid very little attention to the detection of individual light quanta. The detectors used were typically sensitive only to substantial numbers of photons and were quite slow in action so that we measured only intensities which had been averaged over relatively long periods of time. The new light detectors enable us to ask more subtle questions than just ones about average intensities; we can, for example, ask questions about the counting of pairs of quanta, and can make measurements of the probability that the quanta are present at an arbitrary pair of space points, at an arbitrary pair of times.

If the instrumentation in optics has made long strides in the direction of dealing with photons, it is worth mentioning that the instrumentation in the radio frequency field is leading in that direction as well. The energies of radio frequency

photons are extremely small, much smaller than the thermal fluctuation energy κT (T = noise temperature ~ room temperature for most amplifiers). There has consequently not been much need in radio frequency technology to date to pay attention to the corpuscular structure of the field. The recent invention, however, of low noise amplifiers, such as the microwave maser, has lowered the noise temperature of the detecting device to such a degree that with further progress it seems not impossible that individual photons may be detected. So, even in the microwave region, there is now a certain amount of attention being paid to the corpuscular structure of light.

It is interesting, in any case, to investigate the corpuscular nature of electromagnetic fields, because it will set the ultimate limitation to the possibility of transmitting information by means of fields. We will not discuss information theory in these lectures, but we will have some things to say which are related to noise theory. Noise theory is the classical form of the theory of fluctuations of the electromagnetic field and is quite naturally related to the theory of quantum fluctuations of the field. All of these subjects fall under a general heading which we might call photon statistics. Coherence theory too, is properly speaking, a rather small area of the same general subject. Its purpose is simply to formulate some useful ways of classifying the statistical behaviour of fields.

The problem to which we shall address ourselves in these lectures is the construction of a fairly rigorous and general treatment of the problems of photon statistics. There is no need, in doing it, to make any material distinction between radio frequency and optical fields (or between these and X-ray fields for that matter). A part of the formalism, that which has to do with the definition of coherence, is suggested in fact as a way of unifying the rather different concepts of coherence, which have characterized these areas in the past.

We have already remarked that optical experiments have only rarely dealt with individual photons. Much the same observation can be made for optical theory as well. If the photon has to such a remarkable degree remained a stranger to optical theory some justification for that fact surely lies in the great success of the simple wave models in the analysis of optical experiments. Such models are usually spoken of as being classical in character since they proceed typically from some kind of analogy to classical electromagnetic theory and pay as little attention to the corpuscular character of the radiation as the experimental arrangement will permit.

In these approaches one talks typically about some kind of "optical disturbance function" which is assumed to obey the wave equation and perhaps certain boundary conditions as well. The function may represent the components of the electric vector or possibly other field quantities such as the vector potential, or the magnetic field. In many applications in fact one does not need to be very specific about what it really does or does not represent.

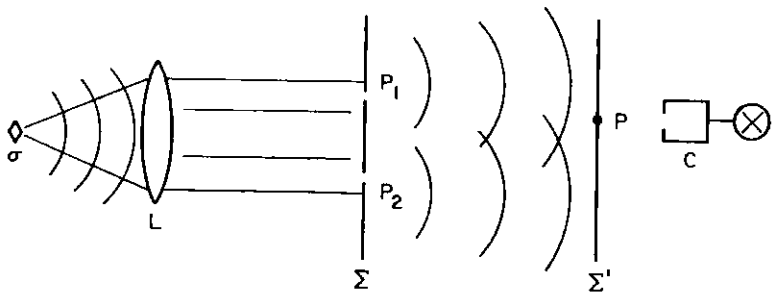

Figure 1

Let us consider the Young interferometer (Fig. 1) in order to illustrate the elementary approaches we are discussing. A plane, quasi-monochromatic wave coming from a point source σ impinges on the screen Σ with two parallel slits at the positions P_1 and P_2.

The two waves emerging from the slits give rise to an interference pattern on the screen Σ', which we can often see with the unaided eye. The simplest way of predicting the form of the interference pattern is to ignore the vector character of the electromagnetic field and introduce a scalar field φ which is presumed to describe the "optical disturbance." We then try to find a function φ which satisfies the wave equation together with a set of boundary conditions which we take to represent the effect of the screen Σ. That problem, as you remember, is in general a good deal too difficult to be solved exactly, and it is customary to make a number of simplifying approximations such as dealing very crudely with the boundary conditions, and making use of the Huyghens principle. By these familiar methods we reach a simple evaluation of the field distribution φ on the screen Σ'.

Of course, if we are to predict the form of the interference pattern, we must at some stage face the question of attaching a physical interpretation to the field φ. The most familiar approach is to regard $\varphi(\mathbf{r},t)$ as a real field and to identify it, perhaps, with one of the components of the electric field vector. The experimental fringe pattern is then predicted quite accurately, as we all know, if the light intensity on the screen is identified with φ^2, the square of our optical field. The identification possesses the justification, from the standpoint of classical theory, that the Poynting vector, which tells us the energy flux, is indeed quadratic in the field strength. In spite of this evident support the identification is not a unique one, however; it pays too little attention to the way in which the light is detected.

Let us suppose that the light intensity is measured by using a photon counter at the position of the screen. We then ask how we may predict the response of the counter as it is used to probe the pattern. Although the use of the wave equation to find the field amplitude φ did not introduce any distinctions between the classical and the quantum theoretical approaches to the diffraction problem, the use of a photon counter as a detector does introduce a distinction. The photon counter is an intrinsically quantum mechanical instrument. Its output is only predictable in terms of statistical averages even when the state of the field is specified precisely. If we are to predict this average response we must be rather more specific than we have thus far been about the field which the counter sees and we must treat the detection mechanism in a fully quantum mechanical way. What we find when we do these things is that the counter may be more accurately thought of as responding to a complex field φ^+ rather than the real field φ, and as having an output proportional, not to φ^2, but to $|\varphi^+|^2$, (The distinction is not a trivial one physically, since in a monochromatic field φ^2 oscillates rapidly in magnitude while $|\varphi^+|^2$ remains constant.) Once this answer is known it can be used as a crude rule for bypassing the explicit discussion of the detection mechanism in applications to other detection problems.

The use of such rules as a means of avoiding the explicit use of quantum mechanics has several times been called the "semi-classical approach". While approaches of this type clearly need a rule of some sort to bridge the gap between their descriptions of the wave and particle behaviors of photons they may remain perfectly correct approaches in a quantum mechanical sense as long as the rule has been chosen correctly. The fact that a mistaken form of this rule has been used repeatedly in "semi-classical" discussions is a good indication that the fully quantum mechanical discussion is not entirely beside the point.

One of the properties of the "semi-classical" approaches that makes them elementary is that they deal with ordinary numbers and functions. They make no use of the apparatus of non-commuting operators which, it may appear, ought to be part of any formal quantum mechanical description of the field. Later in these

lectures we shall show that for a certain class of fields there need be no error in a statistical description of the field which is based upon such ordinary functions as we find by solving the wave equation. It is possible to describe these fields fully by means which are rather similar to those used in the classical theory of noise. Where such a description is available it means that there need be nothing incorrect about the so-called "classical" or "semi-classical" approaches except their names, which then become totally misleading. It has recently been claimed that the class of states of the field for which the simple statistical description we have mentioned is available includes all states of the field, and that consequently the quantum theory and the "classical" theory will always yield equivalent results. We shall have to return to this point later in the lectures when we are better equipped to discuss it, but for the present we may remark that this claim seems to be based more upon wishful thinking than upon accurate mathematics. The quantum theory still offers the only complete and logically consistent basis for discussing field phenomena.

The general subject we shall be discussing, to give it its most imposing name, is quantum electrodynamics. It is an extremely well developed subject. Although it has long been clear that classical electrodynamics is the limit of quantum electrodynamics for $h \to 0$, there have never been any very powerful methods available for discussing electrodynamical problems near the classical limit.

All of quantum electrodynamics has historically been developed in terms of the stationary states $| n >$ of the field hamiltonian $\mathcal{H}$. These correspond to the presence of an integer number n of quanta, i. e. they obey the equation

$$\mathcal{H} \, | \, n > \; = \; (n + \tfrac{1}{2}) \; \hbar \, \omega \; | \, n > \; . \tag{1.1}$$

The n-quantum states form a complete set which has usually been regarded as the "natural" basis for the development of all states of the field. To the extent that virtually all electrodynamical calculations have been done by means of expansions in powers of the field strengths, the numbers of photons which have been dealt with in the calculations have usually been very small integers. The classical limit of quantum electrodynamics, on the other hand, is one in which the quantum numbers are typically quite large. Not only are they large but they are typically quite uncertain. If, for example, a harmonic oscillator is vibrating in a state with a relatively well defined phase, it is necessary that it not only be in a state with a large quantum number, but that the quantum number of the state also be quite uncertain, $(\Delta n \, \Delta \varphi \gtrsim 1)$. When we must deal with quantum states of the electromagnetic field for which the phase of the field is well defined, they can likewise only be states in which the occupation number n is intrinsically rather indefinite. In such cases the description of expectation values in terms of the n- quantum states becomes rather awkward and untransparent.

One of the mathematical tools we shall use in these lectures is a set of quantum states rather better suited to the description of amplitude and phase variables than the n-quantum states. The use of these states makes the relationship of the classical and quantum mechanical forms of electrodynamics considerably clearer than it has been before.

CLASSICAL THEORY

It may help to underscore the close connection between the quantum theory we shall develop and the classical theory if we begin by discussing the classical theory alone for a while. We shall describe the classical field in terms of the familiar field vectors, the electric field $\mathbf{E}(\mathbf{r}, t)$ and the magnetic field $\mathbf{B}(\mathbf{r}, t)$. We will take these to obey the source-free Maxwell equations

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0, & \nabla \times \mathbf{E} &= -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} &= 0, & \nabla \times \mathbf{B} &= \frac{1}{c} \frac{\partial \mathbf{E}}{\partial t} \end{aligned} \tag{1.2}$$

by assuming that whatever source has radiated the fields has ceased to radiate further.

Since our detectors are usually sensitive to electric rather than magnetic fields, we shall confine ourselves to a discussion of the field $\mathbf{E}(\mathbf{r}, t)$. One of the first things which is done in many classical calculations is to use a Fourier series or integral to expand the time dependence of the field and in that way to separate the field into two complex terms:

$$\mathbf{E}(\mathbf{r}, t) = \mathbf{E}^{(+)}(\mathbf{r}, t) + \mathbf{E}^{(-)}(\mathbf{r}, t) \tag{1.3}$$

The first of these terms, which we shall call the positive frequency part, $\mathbf{E}^{(+)}$, contains all the amplitudes which vary as $e^{-i\omega t}$ for $\omega > 0$. The other, the negative frequency part, contains all amplitudes which vary as $e^{i\omega t}$. These terms are complex conjugates of one another

$$\mathbf{E}^{(-)} = \mathbf{E}^{(+)*} \tag{1.4}$$

and contain equivalent physical information. Either one or the other is frequently used in classical calculations and called either the complex field strength or the complex signal. The use of these complex fields in classical contexts is usually regarded as a mathematical convenience rather than a physical necessity since classical measuring devices tend to respond only to the real field, $\mathbf{E} = 2 \, \mathrm{Re} \, \mathbf{E}^{(+)}$.

Quantum mechanical detectors, as we have noted, behave rather differently from classical ones, and for the discussion of these the separation of the field into its positive and negative frequency parts takes on a much deeper significance than it does for classical detectors. As we shall later see, an ideal photon counter (one which has zero size and is equally sensitive to all frequencies) measures the product $\mathbf{E}^{(-)}(\mathbf{r}, t) \, \mathbf{E}^{(+)}(\mathbf{r}, t) = |\mathbf{E}^{(+)}(\mathbf{r}, t)|^2$. That, at least, is what the detector would measure if we were capable of preparing fields with precisely fixed field strengths. But of course we are never capable of controlling the motions of the charges in our sources with very great precision. In practice all fields are radiated by sources whose behavior is subject to considerable statistical uncertainty. The fields are then correspondingly uncertain and what we require is a way of describing this uncertainty in mathematical terms.

It is more convenient, in describing the randomness of the fields, to deal with a discrete set of variables than to deal with the whole continuum at once. We shall therefore only attempt to describe the field lying inside a certain volume of space within which we can expand it in terms of a discrete set of orthogonal mode functions. We shall take the set of vector mode functions $\{ \mathbf{u}_k(\mathbf{r}) \}$ to obey the wave equations

$$\left(\nabla^2 + \frac{\omega_k^2}{c^2} \right) \mathbf{u}_k(\mathbf{r}) = 0 \, , \tag{1.5}$$

which define a set of frequencies $\{\omega_k\}$ when they are satisfied together with the constraint

$$\nabla \cdot \mathbf{u}_k(\mathbf{r}) = 0 \tag{1.6}$$

and a suitable set of boundary conditions. These functions may be assumed to form an orthonormal set

$$\int \mathbf{u}_k^*(\mathbf{r}) \cdot \mathbf{u}_{k'}(\mathbf{r}) \, d\mathbf{r} = \delta_{kk'} \tag{1.7}$$

which is complete within the volume being studied. They may then be used to express the electric field vector in the form

$$\mathbf{E}(\mathbf{r},t) = \Sigma_k C_k u_k(\mathbf{r}) e^{-i\omega_k t} + \Sigma_k C_k^* u_k^*(\mathbf{r}) e^{i\omega_k t}, \tag{1.8}$$

The two sums on the right are then evidently $\mathbf{E}^{(+)}$ and $\mathbf{E}^{(-)}$, respectively.

When the expansion in orthogonal modes is used the field is evidently specified completely by the set of complex Fourier amplitudes $\{C_k\}$. To describe random fields we must regard these numbers as random variables in general. Usually the most we can state about these coefficients can be expressed through a probability distribution $p(\{C_k\}) = p(C_1, C_2, C_3, \ldots)$. Then, if we measure some function of $\mathbf{E}$ or of $\mathbf{E}^{(+)}$, the most we can hope to predict is its mean value, i.e., if we measure $F(\mathbf{E}^{(+)})$ we can only hope to find the average

$$\langle F(\mathbf{E}^{(+)})\rangle = \int p(\{C_k\}) F[\mathbf{E}^{(+)}(\{C_k\})] \Pi_k d^2 C_k , \tag{1.9}$$

where the differential element of area is given by $d^2C_k = d(\mathrm{Re}C_k)\, d(\mathrm{Im}C_k)$.

It is important to remember that this average is an ensemble average. To measure it we must in principle repeat the experiment many times by using the same procedure for preparing the field over and over again. That may not be a very convenient procedure to carry out experimentally but it is the only one which represents the precise meaning of our calculation. The fields we are discussing may vary with time in arbitrary ways. As an example we might take the field generated by a radio transmitter sending some arbitrarily chosen message. There is therefore no possibility in general of replacing the ensemble averages by time averages. The theory of non-stationary statistical phenomena can only be developed in terms of ensemble averages.

The solution of problems in statistical thermodynamics has accustomed us to thinking of statistical fluctuations about the ensemble average as being very small. We are thus usually willing to forget about the need in principle to make an ensemble of thermodynamic measurements and are content to compare just a single measurement with the predicted ensemble average. While the justification of such shortcuts may be excellent in thermodynamic contexts, it is not always so good in statistical optics. Thus when we speak later of the interference patterns produced by superposing light from independent sources we shall find that individual measurements yield results wholly unlike their ensemble averages. The distinction between particular measurements and their averages may thus be quite essential.

Lecture II. INTERFERENCE EXPERIMENTS

One of the classic experiments which exhibits the coherence properties of light is the Young experiment (Fig. 2).

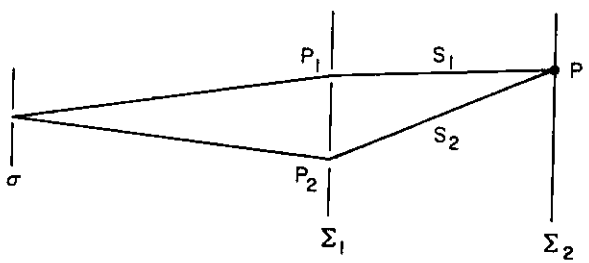

Figure 2

The field present at P at time t may be approximated by a certain linear superposition of the fields present at the two pinholes at earlier times:

$$\mathbf{E}^{(+)}(\mathbf{r},t) = \lambda_1 \mathbf{E}^{(+)}(\mathbf{r}_1, t_1) + \lambda_2 \mathbf{E}^{(+)}(\mathbf{r}_2, t_2) \tag{2.1}$$

where the times are given by $t_{1,2} = t - S_{1,2}/c$. The coefficients λ_1, λ_2 depend on the geometry of the arrangement, but are taken to be independent of the properties of the field.

We shall assume, to begin the discussion, that a photodetector placed at P measures the squared absolute value of some component of the complex field strength. (At a later point we shall discuss the validity of the assumption in some detail.) If we write the measured field component as $\mathbf{E}^{(+)}(\mathbf{r},t)$, we then have

$$\begin{aligned}|\mathbf{E}^{(+)}(\mathbf{r},t)|^2 = \mathbf{E}^{(-)}(\mathbf{r},t)\mathbf{E}^{(+)}(\mathbf{r},t) &= |\lambda_1|^2 \mathbf{E}^{(-)}(\mathbf{r}_1,t_1)\mathbf{E}^{(+)}(\mathbf{r}_1,t_1) \\ &+ |\lambda_2|^2 \mathbf{E}^{(-)}(\mathbf{r}_2,t_2)\mathbf{E}^{(+)}(\mathbf{r}_2,t_2) \\ &+ 2\,\mathrm{Re}\{\lambda_1^*\lambda_2 \mathbf{E}^{(-)}(\mathbf{r}_1,t_1)\mathbf{E}^{(+)}(\mathbf{r}_2,t_2)\}.\end{aligned} \tag{2.2}$$

Now since our preparation of the source rarely fixes the Fourier coefficients C_k very precisely we must in principle perform the experiment repeatedly and then average in order to find a non-random result. The only thing we can really predict is the ensemble average of $|\mathbf{E}^{(+)}(\mathbf{r},t)|^2$ taken over the set of random coefficients $\{C_k\}$,

$$\begin{aligned}\langle |\mathbf{E}^{(+)}(\mathbf{r},t)|^2\rangle &= |\lambda_1|^2 \langle |\mathbf{E}^{(+)}(\mathbf{r}_1,t_1)|^2\rangle + |\lambda_2|^2 \langle |\mathbf{E}^{(+)}(\mathbf{r}_2,t_2)|^2\rangle \\ &+ 2\mathrm{Re}\,\lambda_1^*\lambda_2 \langle \mathbf{E}^{(-)}(\mathbf{r}_1,t_1)\mathbf{E}^{(+)}(\mathbf{r}_2,t_2)\rangle .\end{aligned} \tag{2.3}$$

If we introduce the first order correlation function

$$G^{(1)}(\mathbf{r}t, \mathbf{r}'t') = \langle \mathbf{E}^{(-)}(\mathbf{r}t)\mathbf{E}^{(+)}(\mathbf{r}'t')\rangle , \tag{2.4}$$

we can rewrite Eq. (2.3) in the following way

$$\begin{aligned}\langle |\mathbf{E}^{(+)}(\mathbf{r},t)|^2\rangle &= |\lambda_1|^2 G^{(1)}(\mathbf{r}_1t_1, \mathbf{r}_1t_1) + |\lambda_2|^2 G^{(1)}(\mathbf{r}_2t_2, \mathbf{r}_2t_2) \\ &+ 2\mathrm{Re}\{\lambda_1^*\lambda_2 G^{(1)}(\mathbf{r}_1t_1, \mathbf{r}_2t_2)\}.\end{aligned} \tag{2.5}$$

We have omitted consideration of vector and tensor indices of the fields and correlation functions, respectively, since the vector properties of the field are not too important in this experiment. We would have to take careful account of them if somehow a rotation of the plane of polarisation were induced behind one pinhole, or if the polarisation were in any way made to play a more active role.

A particular case which occurs almost universally in classic optics is that in which the incident field is stationary. The term "stationary" does not mean that nothing is happening. On the contrary, the field is ordinarily oscillating quite rapidly. It means that our knowledge about the field does not change with time. More formally, we associate stationarity with invariance of the statistical description of the beam under displacements of the time variable. The correlation function $G^{(1)}$ for such fields can therefore only depend on the difference $t - t'$

$$G^{(1)}(t, t') = G^{(1)}(t - t') \quad \text{(stationary field)}, \tag{2.6}$$

(Note that by discussing only a single type of correlation function we are stating a necessary condition for stationarity, but not a sufficient one. All average properties of a stationary field must be unchanged by time displacements.) When random classical fields are represented by means of stationary stochastic processes the models used usually have the ergodic property. That property means that the function $G^{(1)}(t - t')$ which is defined as an ensemble average, has the same value as the time averaged correlation function $\Gamma^{(1)}(t - t')$,

$$G^{(1)}(r_1, r_2, \tau) = \Gamma^{(1)}(r_1, r_2, \tau) = \lim_{T\to\infty} \frac{1}{T} \int_0^T E^{(-)}(r_1, t_1+\tau)\, E^{(+)}(r_2, t_1)\, dt_1 . \tag{2.7}$$

The properties of the time-averaged correlation functions $\Gamma^{(1)}$ for classical fields have been discussed in detail in Chapter X of the text of Born and Wolf.

It may be of some help in the lectures that follow to have some more concrete applications of interference experiments in mind. Let us take a brief look at one of the fundamental techniques of interferometry by considering a case in which the field incident upon a detector is a superposition of two plane waves. We assume that the propagation vectors of the two plane waves are only very slightly different. This might be the case for example for monochromatically filtered light from the two members of a double star. If we assume that the frequencies of both waves are equal we may write

$$E^{(+)}(r, t) = A\, e^{i(k\cdot r - \omega t)} + B\, e^{i(k'\cdot r - \omega t)} . \tag{2.8}$$

The question we now ask is: what kind of measurement can be performed to determine that we are receiving radiation from two sources and not just one?

Before answering the question let us specify the statistical character of the coefficients A and B. They are, of course, particular examples of the coefficients C_k previously introduced. We will assume A and B to be distributed independently of one another. This means that the probability function p(A, B) factorizes,

$$p(A, B) = p_1(A)\, p_2(B). \tag{2.9}$$

We will assume further more as properties of the distributions p_1 and p_2, that the phases of the complex amplitudes A and B are individually random. We then have $\langle A\rangle = \langle B\rangle = 0$, More generally the mean values of various powers of the amplitudes and their complex conjugates such as $\langle A\, B^*\rangle$, $\langle |A|^2\, A^*\, B\rangle$, etc. will vanish. Averages in which the amplitudes are paired with their complex conjugates however, take on positive values,

$$\langle |A|^{2n}\rangle \neq 0, \qquad \langle |B|^{2n}\rangle \neq 0, \qquad n = 1, 2 \ldots \tag{2.10}$$

A famous device invented to answer the question we have asked is the Michelson stellar interferometer (Fig. 3).

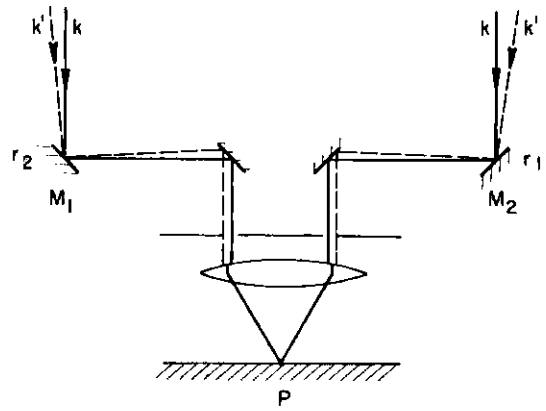

Figure 3

The field at the point P and time t is, in effect, the sum of the two fields impinging on the mirrors M_1, M_2 at the same instant t' (if the optical paths $\overline{M_1P}$ and $\overline{M_2P}$ are equal). Each of these two fields is of the form (2.8) evaluated at the points r_1 and r_2 respectively. The average intensity at P will therefore be

$$\langle E^{(-)}(r, t)\, E^{(+)}(r, t)\rangle = 2\,\mathrm{Re}\{\langle |A|^2 + |B|^2\rangle + \langle |A|^2\rangle e^{-ik\cdot(r_1 - r_2)} + \langle |B|^2\rangle e^{-ik'\cdot(r_1 - r_2)}\}, \tag{2.11}$$

where we have used $\langle AB^*\rangle = \langle A\rangle\langle B^*\rangle = 0$ in reaching this expression. If we introduce the correlation function (2.4),

$$G^{(1)}(r_1 t', r_2 t') = \langle E^{(-)}(r_1 t')\, E^{(+)}(r_2 t')\rangle = \langle |A|^2\rangle e^{-ik\cdot(r_1 - r_2)} + \langle |B|^2\rangle e^{-ik'\cdot(r_1 - r_2)}, \tag{2.12}$$

then intensity may be written as

$$\langle E^{(-)}(r, t)\, E^{(+)}(r, t)\rangle = 2\mathrm{Re}\{\langle |A|^2 + |B|^2\rangle + G^{(1)}(r_1 t', r_2 t')\}. \tag{2.13}$$

The correlation function which describes the interference effect is time independent, because of the stationary character of the field we are treating.

We see from Eq. (2.12) that the correlation function contains two spatially oscillating terms. The way in which these terms reinforce or cancel one another will depend on the displacement $r_1 - r_2$. If $\langle |A|^2\rangle = \langle |B|^2\rangle$, Eq. (2.11) yields

$$\langle E^{(-)}(r, t)\, E^{(+)}(r, t)\rangle = 4\langle |A|^2\rangle\{1 + \cos[\tfrac{1}{2}(k + k')\cdot(r_1 - r_2)]\times \cos[\tfrac{1}{2}(k - k')\cdot(r_1 - r_2)]\}. \tag{2.14}$$

The interference intensity which we see at the point r will be part of a pattern of parallel fringes which we see at the focus of the telescope. Although we have not attempted to describe the fringe pattern in detail, the expression (2.14) for the intensity does indicate one of the characteristic properties of the pattern, that it will vanish altogether when the displacement $r_1 - r_2$ is adjusted so that

$$\cos[\tfrac{1}{2}(k - k')\cdot(r_1 - r_2)]$$

passes through the value zero. By observing the fringes we know that we are dealing with two sources rather than one, and by finding the values of $r_1 - r_2$ at which the fringes disappear we determine their angular separation. The Michelson interferometer has indeed been used to measure the angular separations of double stars, and for measuring angular diameters of stars as well. Only a few stellar diameters have been measured in this way, however, because of the difficulties inherent in working with a large interferometer. An unusually great mechanical stability is clearly required of the apparatus. Furthermore random variations of the index of refraction along the optical path can wash out the pattern.

Instruments quite similar to the Michelson stellar interferometer have been used in radio-astronomy to determine the angular size of celestial radio sources. They consist of two separated antennas supplying signals to a common detector system. In the case of these instruments, as well, it is technically difficult to increase the separation of the antennas without introducing random phase differences in the path between the antennas and the detector. To overcome this difficulty Hanbury Brown and Twiss have devised another form of radio interferometer (Fig.4).

The signals at the antennas are detected individually and then the detector outputs, which are of much lower frequency, are transmitted to a central correlating device where they are multiplied together and the product is averaged. The angular size of the source is obtained from measurements of the way in which the correlation of the intensity fluctuations of the signals varies with the separation of the antennas. An equivalent arrangement may be used with visible light.

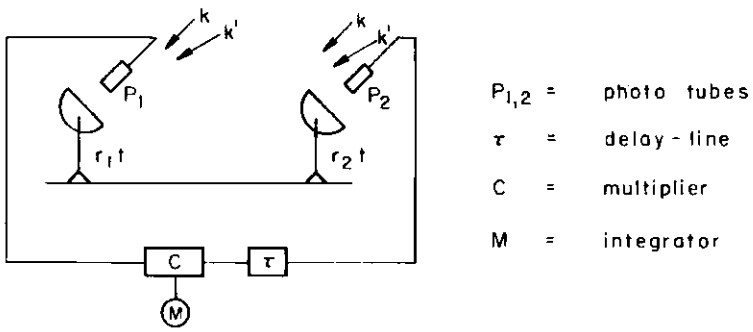

Figure 4

The essence of the trick used by Hanbury Brown, and Twiss was to detect the signals first and by taking away the high frequency components of the incoming radiation, to transmit to the central observation point just a measure of the fluctuations of the intensities arriving at the receivers. Since the detector signals are of relatively low frequency they are easy to transmit faithfully over distances large compared to the limiting dimensions of Michelson interferometers. This experiment is quite different in nature from the interferometer experiment we described earlier because it deals with the average of the product of two random intensities rather than with a single intensity.

It is easy to see that in the average of the product of the two signals there is an interference term, which permits us to resolve the two incoming waves. First we note that a square-law detector placed at P_1 gives a response proportional to

$$|E^{(+)}(r_1,t)|^2 = |A|^2 + |B|^2 + AB^* e^{i(k-k')\cdot r_1} + A^*B\, e^{-i(k-k')\cdot r_1} \quad . \tag{2.16}$$

This output no longer contains the rapid oscillations of the incoming wave. An average of this detected signal, however, would have no interference term (since $< AB^* > = 0$). What Hanbury Brown, and Twiss did is multiply together the two detected signals and then, and only then, to measure the statistical average. The average of the product of two intensities of the form of Eq. (2.16) is

$$<|E^{(+)}(r_1t)|^2\,|E^{(+)}(r_2t)|^2> = <(|A|^2+|B|^2)^2> + 2<|A|^2|B|^2>\cos[(k-k')\cdot(r_1-r_2)], \tag{2.17}$$

where we have used the fact that $< |A|^2A^*B > = 0$, etc. The cosine term clearly represents an interference effect. We can use it to resolve the two sources by observing its behavior as $r_1 - r_2$ is varied. It is important to note that the interference effect has been found by considering the average of a quantity quartic in the field amplitudes. In the case of Michelson's interferometer we deal only with expressions quadratic in the field amplitudes.

Although we have discussed the interferometer experiments in terms of ensemble averages, it is clear that they are not ordinarily performed in this way, but rather as time averages. The calculation of time averages, however, is typically at least a little more difficult than the calculation of ensemble averages (and often it is incomparably more difficult). To consider the interferometer measurements as time averages we should have to note that the two plane waves are not, in general, perfectly monochromatic. It follows then that the coefficients A and B, which we were content earlier to evaluate only at a particular instant of time, actually vary with time. To proceed further we should have to adopt models to represent A(t) and B(t) as stochastic functions of time. As we shall see presently, there are extremely persuasive reasons, when we are dealing with natural light sources, to take these models to be Gaussian stochastic processes. Then, since such processes have the ergodic property, we are justified in identifying time averages with ensemble averages.

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Lecture III INTRODUCTION OF QUANTUM THEORY

When we describe the electromagnetic field in quantum mechanical terms we must think of the field vectors E and B as operators which satisfy the Maxwell equations. The states, $|>$, on which these operators act and their adjoints, $<|$, contain the information which specifies the field. When measurements are made of the physical quantity which correspond to an operator $\mathcal{O}$, we can not expect in general to find the same results repeatedly. What we find instead is that the measured values fluctuate about the average value given by the product $<|\mathcal{O}|>$. The fluctuation is only absent if the state, $|>$, happens to be an eigenstate of $\mathcal{O}$, i.e., if we have

$$\mathcal{O}|> = \mathcal{O}'|>, \tag{3.1}$$

where $\mathcal{O}'$ is an ordinary number rather than an operator. In that case it is convenient to use Dirac's convention and let the eigenvalue $\mathcal{O}'$ be a label for the state by writing the latter as $|\mathcal{O}'>$.

As in classical electromagnetic theory, it is convenient to separate the field operator, $\mathbf{E}(\mathbf{r},t)$, which is naturally Hermitian, into the sum of its positive frequency and negative frequency parts:

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}^{(+)}(\mathbf{r},t) + \mathbf{E}^{(-)}(\mathbf{r},t) \tag{3.2}$$

These parts, as we have already noted classically, represent complex rather than real fields. The operators $\mathbf{E}^{(\pm)}$ are therefore not Hermitian, but they are Hermitian adjoints of one another

$$\mathbf{E}^{(-)}(\mathbf{r},t) = \{\mathbf{E}^{(+)}(\mathbf{r},t)\}^\dagger \quad . \tag{3.3}$$

While the fields $\mathbf{E}^{(+)}$ and $\mathbf{E}^{(-)}$ play essentially indistinguishable roles in classical theory, they tend to play quite dissimilar roles in the quantum theory. The

operator $E^{(+)}$ describes the annihilation of a photon while $E^{(-)}$ describes the creation of one. This identification of the operators is virtually the only fact we shall have to borrow from more formal developements of quantum field theory.

We must think fundamentally of all electric field measurements as being made on the Hermitian operator $E(r, t)$ given by Eq. (3.2). In the classical limit it is usually true that the complex fields $E^{(+)}$ and $E^{(-)}$ make contributions of equal magnitude to our measurements. From a quantum mechanical standpoint that is because quantum energies are so small in the classical limit ($\hbar\omega \to 0$), that test charges emit quanta as readily as they absorb them. In the quantum domain, on the other hand, we must expect that the fields $E^{(+)}$ and $E^{(-)}$ will make contributions of altogether different magnitudes to the quantities we measure, such as transition amplitudes.

If we are using atomic systems in their ground states as probes of the electric field for example, then the atoms have no energy to emit photons and can only absorb them. In this case, which corresponds in principle to that of a typical photo-detector, only the annihilation operator $E^{(+)}$ figures significantly in determining the transition amplitudes. More exactly, if we do a calculation of the transition amplitude using first order perturbation theory, we easily find that the creation operator $E^{(-)}$ contributes only an extremely small amplitude which varies so rapidly with time that it leads to no observable effect at all. The creation operator can only contribute materially if the detector contains excited atoms. (Thermal energies are a great deal too small to furnish atoms excited to optical energies, but at microwave frequencies it may be necessary to take thermally excited atoms into account.)

In the third and higher orders of perturbation theory, the creation operator can indeed play a tiny role in an absorption experiment. The effect in question is a radiative correction to the first order absorption probability which all estimates indicate will be quite small. We see, therefore, that it is fairly accurate to say that a typical photodetector detects the field $E^{(+)}$ rather than the field E. Although this statement is clearly an approximate one rather than a rigorous one it is none the less important since it furnishes us a reason for formulating the theory in terms of a set of non-Hermitian operators. The formulation, as we shall see, allows in turn a great deal of insight into the way the theory passes to the classical limit.

To gain some further insights into the kinds of quantities measured in photon counting experiments, let us examine the role played by the field operator in the calculation of the appropriate transition probabilities. In the next lecture we shall indicate how these transition probabilities are calculated in some detail by taking due account of the atomic nature of the detector. Let us for the moment, however, ignore the detailed dynamics of the detector and assume simply that it is an ideally selective device, one which is sensitive to the field $E^{(+)}(rt)$ at a single point of space r at each instant of time t. We may take the transition probability of the detector for absorbing a photon from the field at position r and time t to be proportional to

$$w_{i \to f} = |\langle f| E^{(+)}(rt) |i\rangle|^2 , \tag{3.4}$$

where $|i\rangle$ is the initial state of the field before the detection process, and $|f\rangle$ is a final state in which the field could be found after the process. In fact we never measure the final state of the field. The only thing we do measure is the total counting rate. To calculate the total rate we have to sum Eq. (3.4) over all the final states of the field that can be reached from $|i\rangle$ by an absorption process. We can, however, extend the sum over a complete set of final states since the the states which cannot be reached (e.g., states $|f\rangle$ which differ from $|i\rangle$ by two or more photons) simply will not contribute to the result since they are orthogonal to the state $E^{(+)}(rt)|i\rangle$.

When the final state summation is carried out the counting rate becomes, in effect,

$$w = \sum_f |\langle f| E^{(+)}(r, t) |i\rangle|^2 = \langle i| E^{(-)}(r, t) E^{(+)}(r, t) |i\rangle , \tag{3.5}$$

where the completeness relation $\Sigma |f\rangle\langle f| = 1$ has been used. The counting rate w is proportional to the probability per unit time that an ideal photocounter, placed at r, absorbs a photon from the field at time t. It is, according to Eq. (3.5), given by the expectation value of the positive definite Hermitian operator $E^{(-)}(r,t) E^{(+)}(r,t)$, taken in the state $|i\rangle$ which the field was in prior to the measurement. Eq. (3.5) shows explicitly that the photocounter is not sensitive to the square of the real field (as has been assumed in many "semi-classical" calculations), but rather to an operator which corresponds to the squared absolute magnitude the complex field-strength.

We have thus far supposed that we know the state $|i\rangle$ of the field. That does not mean, of course, that we can predict the result of a single measurement made with our counter. If we repeat the measurement another result will quite likely turn out, and Eq. (3.5) gives us only the mean value of many repeated measurements. So quantum mechanics forces us to talk about ensemble averages even if we know the state of the field precisely.

In practice, of course, we almost never know the state $|i\rangle$ very precisely. Radiation sources are usually complicated systems with many degrees of freedom, so the states $|i\rangle$ depend, as a rule, on many uncontrollable parameters. Since we have no possibility of knowing the exact state of a field, we must resort to a statistical description. This description summarizes our knowledge of the field, by averaging over the unknown parameters. The predictions that we make by using this description must therefore, in principle, be compared experimentally with ensemble averages. With this understanding we may write the counting rate as an ensemble average of Eq. (3.5) over all random variables involved in the state $|i\rangle$,

$$w = \{\langle i| E^{(-)}(r, t) E^{(+)}(r, t) |i\rangle\}_{\text{av. over } i} . \tag{3.6}$$

If we introduce the density operator $\rho = \{|i\rangle\langle i|\}_{\text{av. over } i}$, we may write this average as

$$w = \mathrm{Tr}\{\rho E^{(-)}(r, t)\, E^{(+)}(r, t)\} , \tag{3.7}$$

where Tr stands for the trace of the operator which follows. The density operator is the average of the projection operators on the initial field states. It is obviously Hermitian, $\rho^\dagger = \rho$. Furthermore, it also has the property of positive definiteness, $\langle j|\rho|j\rangle \geq 0$ for any state $|j\rangle$. It is worth emphasizing that a two-fold averaging process is implied by Eq. (3.7). That we must average the measurements made upon a pure state is an intrinsic requirement of quantum mechanics which has no classical analogue. The ensemble average over initial states, on the other hand, is analogous to the averaging over the set of random coefficients $\{C_k\}$ which we described in the classical theory.

Equation (3.7) gives the counting rate of a single ideal photodetector in terms of the quantum mechanical correlation function

$$G^{(1)}(x, x') = \mathrm{Tr}\{\rho E^{(-)}(x) E^{(+)}(x')\}, \qquad x \equiv \{r, t\} , \tag{3.8}$$

which is analogous to the correlation function introduced to describe classical interference experiments. To describe more sophisticated experiments, e.g., the coincidence experiment of Hanbury Brown and Twiss, it is useful to define a more general set of correlation functions

$$G^{(n)}(x_1 \cdots x_n, x_{n+1} \cdots x_{2n}) = \mathrm{Tr}\{\rho E^{(-)}(x_1) \cdots E^{(-)}(x_n) E^{(+)}(x_{n+1}) \cdots E^{(+)}(x_{2n})\} . \tag{3.9}$$

The function $G^{(n)}$ will be referred to as the n-th order correlation function. The analytical properties of this set of functions and their relation to experimental measurements will be discussed later.

We could, of course, have chosen to define a somewhat larger class of correlation functions than the $G^{(n)}$ by dealing with averages such as $\mathrm{Tr}\{\rho E^{(-)}E^{(+)}E^{(+)}E^{(+)}\}$, which contain unequal numbers of creation and annihilation operators. If we have chosen not to set down any special notation for such averages it is because they are not of the types which are measured in typical photon counting experiments. Such averages may, in principle, be measured in other kinds of experiments but they will always vanish in stationary states of the field and, much more generally, whenever the absolute phases of the fields are random. Random absolute phases are, of course, rather characteristic of optical and other extremely high frequency fields.

Lecture IV THE ONE-ATOM PHOTON DETECTOR

Let us now consider the photodetection process in somewhat more detail. We shall imagine, for the present, that our photon counter is a rather idealized type of device which has as its sensitive element a single atom which is free to undergo photoabsorption transitions such as the photoelectric effect. We assume that the atom is shielded from the radiation field we are investigating by a shutter of some sort which opens at time t_o and closes again at time t. Our problem will be to calculate the probability that a photoabsorption process takes place during this interval and that it is recorded by our apparatus.

The detector will be assumed to be far enough from the radiation source so that the field behaves as a free field. The hamiltonian of the system (field + detector) can then be written as

$$\mathcal{H} = \mathcal{H}_0 + \mathcal{H}_I \quad ; \quad \mathcal{H}_0 = \mathcal{H}_{0,\mathrm{at}} + \mathcal{H}_{0,F} \ ,$$

where $\mathcal{H}_0$ is the sum of Hamiltonians of the free field and the atom. The interaction term $\mathcal{H}_I$ is time independent in the Schrödinger picture. In the interaction representation, however, it becomes time dependent. If we make use of the electric dipole approximation, which is quite accurate at optical frequencies, we can write the time dependent interaction Hamiltonian as

$$\mathcal{H}_I = e^{\frac{i}{\hbar}\mathcal{H}_0 t}\ \mathcal{H}_I\ e^{-\frac{i}{\hbar}\mathcal{H}_0 t} = -e\sum_\gamma q_\gamma(t)\cdot \mathbf{E}(\mathbf{r},t) \ . \qquad (4.1)$$

In this expression $\mathbf{r}$ represents the position of the atomic nucleus and q_γ the position operator of the γ-th electron relative to the nucleus. The time dependence of the field $\mathbf{E}(\mathbf{r},t)$ which occurs in Eq. (4.1) is that of the free field uninfluenced by the presence of the atom.

The Schrödinger equation of the combined system of field and atom in the interaction representation is

$$i\hbar\frac{\partial}{\partial t}\,|t> = \mathcal{H}_I(t)\,|t> \ . \qquad (4.2)$$

Its solution can be written in the general form

$$|t> = U(t,t_o)\,|t_o> \ ,$$

where $U(t,t_o)$ is the unitary time development operator which describes the way in which the initial state changes under the influence of the perturbation. In the first order of perturbation theory the solution has the well-known form

$$|t> = \left\{1 + \frac{1}{i\hbar}\int_{t_o}^{t}\mathcal{H}_I(t')dt'\right\}|t_o> \ . \qquad (4.3)$$

Let us suppose that the system is initially in the state $|gi> = |g>|i>$, where $|i>$ is some known state of the field, and $|g>$ is the ground state of the atom. We ask now for the probability that the system at time t is in a specified state $|af> = |a>|f>$, where $|a>$ is an excited state of the atom and $|f>$ is the final state of the field. This probability is given by the squared absolute value of the matrix element

$$<af|U(t,t_o)|gi> = \frac{1}{i\hbar}\int_{t_o}^{t}<af|\mathcal{H}_I(t')|gi>dt' \ . \qquad (4.4)$$

(The zeroth order term in $U(t,t_o)$ of Eq. (4.3) does not contribute because of the orthogonality of the electron states $|a>$ and $|g>$.) By substituting the interaction operator from Eq. (4.1) we can separate the matrix element into two parts, a matrix element for the atom and one for the field:

$$<af|U(t,t_o)|gi> = \frac{ie}{\hbar}\sum_\gamma\int_{t_o}^{t}<a|q_\gamma(t')|g>\cdot<f|\mathbf{E}(\mathbf{r},t')|i>dt' \ . \qquad (4.5)$$

To evaluate the atomic matrix element we recall that

$$q_\gamma(t') = e^{\frac{i}{\hbar}\mathcal{H}_0 t'}\ q_\gamma(o)\ e^{-\frac{i}{\hbar}\mathcal{H}_0 t'} = e^{\frac{i}{\hbar}\mathcal{H}_{0,\mathrm{at}} t'}\ q_\gamma(o)\ e^{-\frac{i}{\hbar}\mathcal{H}_{0,\mathrm{at}} t'} \ .$$

The latter relation holds because the field hamiltonian $\mathcal{H}_{0,F}$ commutes with the atomic Hamiltonian $\mathcal{H}_{0,\mathrm{at}}$ and with the electron coordinate $q_\gamma(o)$ as well. We may write the matrix element as

$$<a|\Sigma_\gamma q_\gamma(t')|g> = M_{ag}\,e^{i\omega_{ag}t'}$$

with

$$M_{ag} = <a|\Sigma_\gamma q_\gamma(o)|g> \quad \text{and} \quad \hbar\omega_{ag} = E_a - E_g \ .$$

The matrix element M_{ag} occurs simply as a time independent coefficient in the transition amplitude

$$<af|U(t,t_o)|gi> = \frac{ie}{\hbar}\int_{t_o}^{t}e^{i\omega_{ag}t'}\ M_{ag}\cdot<f|\mathbf{E}(\mathbf{r},t')|i>dt' \ . \qquad (4.6)$$

We can now replace $\mathbf{E}(\mathbf{r},t')$ in this expression by the sum of the two operators $\mathbf{E}^{(+)}(\mathbf{r},t)$ and $\mathbf{E}^{(-)}(\mathbf{r},t)$. The emission operator $\mathbf{E}^{(-)}(\mathbf{r},t)$ contains only negative frequencies, i.e., exponential time dependences of the form $e^{i\omega t}$ for $\omega>0$. The time integrals of these terms clearly oscillate rapidly with increasing t. They are furthermore quite small in amplitude compared with the terms contributed by the annihilation operator $\mathbf{E}^{(+)}(\mathbf{r},t)$. What we are describing, in fact, is the way in which the transitions are restricted by the conservation of energy. In order to find that the atomic transitions conserve the energy of the field quanta with an accuracy $\Delta E = \hbar\Delta\omega$, we must leave our shutter open for a length of time $t - t_o \gg 1/\Delta\omega$. In practice we always have $\Delta\omega \ll \omega_{ag}$, i.e., the shutter is open for a great many periods of oscillation and then the contribution of the emission term $\mathbf{E}^{(-)}(\mathbf{r},t)$ is entirely negligible. (We are assuming that the detector is at a relatively low temperature, as we have remarked in the preceding lecture.)

We must next sum the squared modulus of the amplitude (4.6) over all final states $|f>$ of the field, since no observations are ordinarily made of those states. One of the virtues of working with the expression (4.6) for the amplitude is that in the final state summation we can sum over all the states of a complete set; those final states which cannot be reached by the field for physical reasons are present in the sum but contribute nothing, either because the matrix elements leading to them vanish identically, or because the time integrals of the matrix elements vanish.

Thus the constraint represented by the conservation of energy, for example, is actually implicit in the structure of the time integrals in the sum of the squared amplitudes,

$$\sum_f |\langle af|U(t,t_o)|gi\rangle|^2 = \left(\frac{e}{\hbar}\right)^2 \int_{t_o}^{t}\int_{t_o}^{t} dt'dt'' \, e^{i\omega_{ag}(t''-t')} \sum_{\mu,\nu} M^*_{ag,\mu} M_{ag,\nu} \langle i|E^{(-)}_\mu(r,t') \times E^{(+)}_\nu(r,t'')|i\rangle, \tag{4.7}$$

which has been derived by using the relation

$$\langle f|\mathbf{E}^{(+)}|i\rangle^* = \langle i|\mathbf{E}^{(-)}|f\rangle$$

and the completeness relation $\Sigma_f |f\rangle\langle f| = 1$.

We have already discussed the need to average such expression as Eq. (4.7) over an ensemble of initial states $|i\rangle$ since the initial state is rarely known accurately in practice. We then find for the transition probability the expression

$$\begin{aligned} p_{g\to a}(t) &= \{\Sigma|\langle af|U(t,t_o)|gi\rangle|^2\}_{\text{av over i}} \\ &= \left(\frac{e}{\hbar}\right)^2 \sum_{\mu,\nu} \int_{t_o}^{t}\int_{t_o}^{t} dt'\, dt''\, e^{i\omega_{ag}(t''-t')} \quad M^*_{ag,\mu} M_{ag,\nu} \\ &\qquad \mathrm{Tr}\{\rho E_\mu^{(-)}(r,t') E_\nu^{(+)} r, t'')\} \\ &= \left(\frac{e}{\hbar}\right)^2 \sum_{\mu,\nu} \int_{t_o}^{t}\int_{t_o}^{t} dt'\, dt''\, e^{i\omega_{ag}(t''-t')} M^*_{ag,\mu} M_{ag,\nu}\, G^{(1)}_{\mu\nu}(rt',\, rt'') . \end{aligned} \tag{4.8}$$

The definitions of the density operator ρ of the field and of the first order correlation function $G^{(1)}$ have been given in the preceding lecture.

The foregoing discussion has assumed that the atom makes a transition to a specified final state $|a\rangle$. Counters employing discrete final states have received a certain amount of discussion recently. Bloembergen and Weber, for instance, have proposed using a scheme illustrated by Fig. 5.

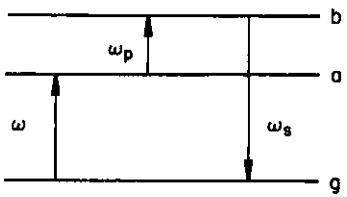

Figure 5

When the atom is excited to the state a by an incident field of frequency ω it is then raised to a higher level b by a pumping field at frequency ω_p. The emission of a photon with the sum frequency $\omega_s = \omega + \omega_p$ indicates the absorption of a photon from the incident field.

In the detectors used to date, however, the final states $|a\rangle$ of the atoms form an extremely dense set, or a continuum; the atoms are simply ionized, for instance. Since a counter of photoelectrons has only a limited ability to select among final atomic states (e.g., the counting of photoelectrons places only weak restrictions on their momenta), we have to sum the probability given by Eq. (4.8) over at least part of the continuum of final states $|a\rangle$. But not all ejected electrons can really be counted. Often they are ejected in directions for which the counter is insensitive or they are stopped by matter. The device might furthermore be built so as to introduce some explicit selection according to energies before detecting photoelectrons.

We shall not discuss the actual means used for detecting the photoelectrons in any detail here. Instead we shall assume simply that the probability that an electron ejected by photoabsorption is really registered is given by some function R(a). The way in which this function varies with the final state $|a\rangle$ of the electron-ion system will depend, in general, on the geometrical and physical properties of the actual counting device. If we now sum the probabilities given by Eq. (4.8) over the final states $|a\rangle$ using the probability R(a) as a weight, we find for the probability of detecting a photon absorption in our one-atom detector

$$\begin{aligned} p^{(1)}(t) &= \sum_a R(a)\, p_{g\to a}(t) \\ &= (e/\hbar)^2 \sum_{\mu,\nu} \int_{t_o}^{t}\int_{t_o}^{t} dt'\, dt''\, \sum_a R(a)\, M^*_{ag,\mu}\, M_{ag,\nu} \\ &\qquad \times e^{i\omega_{ag}(t''-t')}\, G^{(1)}_{\mu\nu}(rt',\, rt'') . \end{aligned} \tag{4.9}$$

We now separate the sum over the final states into two parts, a sum over the final electron energies and one over all other variables such as momentum directions, spin, etc. We do this by introducing the sensitivity function,

$$s_{\nu\mu}(\omega) = 2\pi\left(\frac{e}{\hbar}\right)^2 \sum_a R(a) M_{ag,\nu} M^*_{ag,\mu}\, \delta(\omega - \omega_{ag}) , \tag{4.10}$$

which contains contributions only from transitions with a fixed energy transfer, $\hbar\omega$. (Note that $s_{\nu\mu}(\omega)$, although it is written as a sum of delta functions, is actually a well-behaved function for the case we are considering since the sum over states $|a\rangle$ is really an integration over states with a continuum of energies.)

By making use of the sensitivity function and of the properties of the delta-function it contains we may write the counting probability in Eq. (4.9) in the form

$$p^{(1)}(t) = \frac{1}{2\pi} \int_{t_o}^{t} dt' \int_{t_o}^{t} dt'' \int_{-\infty}^{\infty} d\omega \sum_{\mu,\nu} s_{\nu\mu}(\omega)\, e^{i\omega(t''-t')}\, G^{(1)}_{\mu\nu}(rt', rt'') . \tag{4.11}$$

Since $s_{\nu\mu}(\omega) = 0$ for $\omega < 0$ we have extended the integral over the variable ω from $-\infty$ to $+\infty$. If we define the Fourier transform of the sensitivity function by

$$S_{\nu\mu}(t) = (1/2\pi) \int_{-\infty}^{\infty} s_{\nu\mu}(\omega)\, e^{i\omega t} d\omega , \tag{4.12}$$

we finally obtain

$$p^{(1)}(t) = \int_{t_o}^{t} dt' \int_{t_o}^{t} dt'' \sum_{\nu\mu} S_{\nu\mu}(t''-t')\, G^{(1)}_{\mu\nu}(rt', rt'') . \tag{4.13}$$

Eq. (4.13) represents the total transition probability when our shutter is open from time t_o to t. To obtain the rate at which transitions occur we must differentiate with respect to t.

In general there is nothing very localizable in time about the absorption process. It is not possible to say that the photon has been absorbed in a particular interval of time small compared to the total period during which the shutter has been open. This becomes quite clear if we assume that the sensitivity $s_{\nu\mu}(\omega)$ is sharply peaked with a small width $\Delta\omega$. Then $S_{\nu\mu}(t''-t')$ takes on nonvanishing values for $|t''-t'| \leq 1/\Delta\omega$, which may be an arbitrarily long interval of time for small $\Delta\omega$. The degree of non-locality in time which enters the integral in Eq. (4.13) is, roughly speaking, just the reciprocal $1/\Delta\omega$ of the bandwidth of our device. If the bandwidth is narrow the counter measures an average of values of $G^{(1)}(rt', rt'')$ with t' quite different from t''. In optical experiments a narrow band sensitivity is usually reached by putting narrow band light filters in front of broad band counters, i.e., by "filtering" the correlation function $G^{(1)}$ rather than by discriminating between photoelectrons. Broad band counters are therefore, in this sense, somewhat more basic than narrow band ones.

In the limiting case of extremely broadband detection the detection process

becomes approximately local in time. We have already made some mention in the preceding lecture of an ideal photodetector. Such a detector, we shall assume, has a sensitivity function $s_{\nu\mu}(\omega)$ which is constant for all frequencies. To gain a quick insight into the meaning of this assumption we note that when the sensitivity function is a constant, $s_{\nu\mu}$, independent of frequency, Eq. (4.12) reduces to

$$S_{\nu\mu}(t) = s_{\nu\mu}\,\delta(t). \tag{4.14}$$

The photon absorption process then becomes, in effect, localized in time, and the transition probability given by Eq. (4.13) reduces to

$$p^{(1)}(t) = \sum_{\mu,\nu} s_{\nu\mu} \int_{t_0}^{t} G^{(1)}(rt', rt')\,dt'. \tag{4.15}$$

Now the assumption that $s_{\nu\mu}(\omega)$ is independent of frequency would be quite a difficult one to meet in practice for $\omega > 0$. When we take negative values of ω into account it becomes, strictly speaking, an impossible condition to meet since $s_{\nu\mu}(\omega) = 0$ for $\omega < 0$. But in fact neither of these troubles stands in the way of our constructing actual devices which approximate the behavior of ideal detectors arbitrarily well, as long as we agree to use them on radiation fields of restricted frequency bandwidth. Once we assume that the field excitations have finite bandwidth all we really require of our detector is that its sensitivity be constant over the excited frequency band. The detector then functions in an ideal way no matter how much the sensitivity varies outside the excited band.

To show that we need only be concerned to have the sensitivity remain constant over the band which is actually excited, we shall examine Eq. (4.11) for the transition probability in a little more detail. Let us begin by imagining that the time interval $t - t_0$ is exceedingly great, e.g., we let $t \to \infty$ and $t_0 \to -\infty$. Then if we let $K_{\mu\nu}(\omega)$ be the Fourier integral

$$K_{\mu\nu}(\omega) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, e^{i\omega(t''-t')}\, G_{\mu\nu}^{(1)}(rt', rt''), \tag{4.16}$$

it is clear that $K_{\mu\nu}$ vanishes for frequencies ω lying outside the excited band. (e.g., The diagonal elements $K_{\mu\mu}(\omega)$ are simply proportional to the power spectra of the three field components.) We may then make use of $K_{\mu\nu}(\omega)$ to rewrite Eq. (4.11) as

$$p^{(1)}(t) = (1/2\pi) \int_{-\infty}^{\infty} \sum_{\mu\nu} s_{\nu\mu}(\omega) K_{\mu\nu}(\omega)\, d\omega \tag{4.17}$$

Now as long as $s_{\nu\mu}(\omega)$ takes on the constant value $s_{\nu\mu}$ over the excited band (and no matter how it behaves elsewhere) we may write Eq. (4.17) as

$$\begin{aligned} p^{(1)}(t) &= \sum_{\mu\nu} s_{\nu\mu} (1/2\pi) \int_{-\infty}^{\infty} K_{\mu\nu}(\omega)\, d\omega \\ &= \sum_{\mu\nu} s_{\nu\mu} \int_{-\infty}^{\infty} G_{\mu\nu}^{(1)}(rt', rt')\, dt', \end{aligned} \tag{4.18}$$

and the latter of these expressions again shows the locality in time of the photon absorption process which we noted earlier in Eq. (4.15), i.e., the two arguments of the correlation function in the integrand are the same.

In order to derive the foregoing result we imagined that the time interval $t - t_0$ was allowed to become infinite. To see the influence of the fact that the time interval has a finite length, let us define a time-dependent step function

$$\eta(t') = \begin{cases} 0 \text{ for } t' < t_0 \\ 1 \text{ for } t_0 < t' < t \\ 0 \text{ for } t' > t \end{cases}$$

Then the limits of the time integrations in Eq. (4.11), for example, may be extended from $-\infty$ to ∞ if we first multiply the correlation function in the integrand by $\eta(t')\,\eta(t'')$. This extension of the limits of the time integrations means that we may use once more an argument of the type which led to Eq. (4.18). But the difference is that the function $K_{\mu\nu}(\omega)$ must now be regarded as the Fourier transform

$$K_{\mu\nu}(\omega) = \int_{-\infty}^{\infty} dt' \int_{-\infty}^{\infty} dt''\, e^{i\omega(t''-t')}\, \eta(t')\, G_{\mu\nu}^{(1)}(rt', rt'')\, \eta(t''). \tag{4.19}$$

The bandwidth of this function will in general be different from that of the radiation present but the difference will only be significant if the period during which the shutter is open is extremely brief.

Let us suppose the bandwidth of the radiation present, i.e., of the function $G^{(1)}$, is $\delta\omega$. The bandwidth associated with the functions η is of order $(t - t_0)^{-1}$. The frequency width characteristic of $K_{\mu\nu}(\omega)$ is presumably of the magnitude of the larger of these two widths. Then if we assume that the sensitivity function of our detector only varies appreciably over an interval $\Delta\omega$, we shall secure an expression for the transition probability which reduces to the form of Eq. (4.15) as long as $\Delta\omega$ satisfies the two conditions

$$\Delta\omega \gg \delta\omega \qquad \text{and} \qquad \Delta\omega \gg (t - t_0)^{-1}.$$

The second of these conditions sets a lower bound $1/\Delta\omega$ to the length of time our shutter can be open if we want the behavior of our counter to remain ideal.

If we differentiate Eq. (4.15) with respect to time we find that the rate of increase of the transition probability, i.e., the counting rate, is

$$w^{(1)}(t) = \frac{dp^{(1)}(t)}{dt} = \sum_{\nu\mu} s_{\nu\mu} G_{\nu\mu}^{(1)}(rt, rt). \tag{4.20}$$

Having carried the tensor indices of the sensitivity and correlation functions far enough to illustrate their role in determining the transition probabilities we shall now eliminate them by imagining the field to possess a specified polarization $\hat{e}$. This can be accomplished in practice, of course, by putting a polarization filter in front of the counter. With the notation

$$\begin{aligned} E^{(-)}(r,t) &= \hat{e} \cdot E^{(-)}(r,t) \\ E^{(+)}(r,t) &= \hat{e}^* \cdot E^{(+)}(r,t) \\ G^{(1)}(rt, r't') &= \mathrm{Tr}\{\rho\, E^{(-)}(r,t)\, E^{(+)}(r',t')\} \\ s &= \sum_{\mu\nu} \hat{e}_\nu\, s_{\nu\mu}\, \hat{e}_\mu^*, \end{aligned} \tag{4.21}$$

Equation (4.20) may be rewritten as

$$w^{(1)}(t) = s\, G^{(1)}(rt, rt). \tag{4.22}$$

We have thus justified the assumption, made in the course of the simplified discussions given earlier, that an ideal photon counter can be constructed to respond, in effect, to the field at a given instant of time. Its counting rate is proportional to the first order correlation function evaluated at a single point and a single time.

In deriving the foregoing results we have employed the electric dipole approximation. The use of that approximation has been much more a matter of convenience than one of necessity. We could as well have retained the general coupling between the momentum of the atomic electrons and the vector potential. We would then have made use of correlation functions for the vector potential rather than for the electric field. The only difference in the calculations would then be a matter of

taking account of the finite size of the atom. Instead of having atomic matrix elements simply occurring as constant factors in the transition probabilities we would have to integrate products of the atomic wave functions and the correlation functions. The transition probabilities, in other words, would be integrals which involve the correlation functions for finite spatial as well as temporal intervals. Fortunately these unilluminating complications are not too necessary quantitatively at optical and lower frequencies.

Lecture V THE n-ATOM PHOTON DETECTOR

The photon counter we have thus far discussed has as its sensitive element only a single atom. Since that is hardly a very realistic picture of an actual detector, we must generalize our arguments to deal with detectors containing arbitrarily many atoms which may undergo photoabsorption processes. We shall carry out this generalization in two stages. In the present lecture we consider detectors which consist of a relatively modest number of atoms and show how these can be used to investigate the higher order correlation properties of the fields. We shall postpone until the last lecture a full discussion of the statistical properties of actual photon counting experiments, since it will be useful to discuss the coherence properties of fields first.

The one-atom detector, as we have seen, furnishes us with measurements of the first-order correlation function of the field, $G^{(1)}$. There exist, however, more general correlation properties of fields; some of these are related, for example, to experiments in which we measure coincidences of photon absorption processes taking place at different points in space and time. Such an experiment has been performed for example by Hanbury Brown and Twiss, and we shall discuss it in some detail in the later lectures.

Let us suppose that n similar atoms are placed at different positions $\mathbf{r}_1, \mathbf{r}_2 \ldots \mathbf{r}_n$ in the field. These atoms, we assume, form the sensitive element of a species of compound detector. A shutter in front of all of the atoms will be opened during the time interval from t_o to t. We ask for the probability that each of the atoms has absorbed a photon from the field during that time interval. Though this problem is still rather artificial in nature, its solution will be an essential part of the general discussion of photon counting we shall undertake later.

The process in question involves the absorption of n photons, and therefore, to calculate its probability, we must, strictly speaking, apply n-th order perturbation theory. Needless to say, a number of simplifications are available to us in doing this.

In order to solve the Schrödinger equation in the interaction representation

$$i\hbar \frac{\partial}{\partial t} |t> = \mathscr{H}_I(t) |t> , \tag{5.1}$$

we have already introduced the unitary time development operator $U(t, t_o)$ which transforms the states according to the scheme

$$|t> = U(t, t_o) |t_o>.$$

A formal solution for $U(t, t_o)$ may be written in the form

$$U(t, t_o) = \left\{ e^{-\frac{i}{\hbar}\int_{t_0}^{t} \mathscr{H}_I(t')dt'} \right\}_+ \tag{5.2a}$$

$$= \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n \int_{t_o}^{t} \cdots \int_{t_o}^{t} \{\mathscr{H}_I(t_1) \cdots \mathscr{H}_I(t_n)\}_+ \prod_{p=1}^{n} dt_p , \tag{5.2b}$$

where the bracket symbol $\{\ \}_+$ stands for a time ordering operation to be carried out on all the operators inside the bracket. It requires that the products of operators be rearranged so that their time arguments increase from right to left. The representations (5.2a and b) for the solution are perhaps most easily derived by writing the Schrödinger equation (5.1) as an integral equation and solving the integral equation by means of a power series.

The interaction Hamiltonian $\mathscr{H}_I(t)$ for the n atoms interacting with the field is given by

$$\mathscr{H}_I(t) = \sum_{j=1}^{n} \mathscr{H}_{I,j}(t) , \tag{5.3}$$

where $\mathscr{H}_{I,j}(t)$ represents the coupling of the j-th atom to the field. The individual coupling terms take the form

$$\mathscr{H}_{I,j}(t) = -e \sum_{\gamma} q_{j\gamma}(t) \cdot \mathbf{E}(\mathbf{r}_j, t) , \tag{5.4}$$

which we have already discussed. We shall assume, for simplicity, that the atoms are dynamically independent of one another, i.e., that their zeroth order Hamiltonians are separable and commute.

The n-fold absorption process is described, to lowest order, by the n-th order term $U^{(n)}(t, t_o)$ of $U(t, t_o)$, i.e., the n-th order term of the series in Eq. (5.2b). By inserting the Hamiltonian given by Eq. (5.3) into Eq. (5.2b), we obtain for $U^{(n)}(t, t_o)$ an expression containing n^n terms, which represent all of the ways in which n atoms can participate in an n-th order process. Many of these terms, however, have nothing to do with the process we are considering, since we require each atom to participate by absorbing a photon once and only once. Terms involving repetitions of the Hamiltonian for a given atom describe processes other than those we are interested in. The only terms which do contribute are those in which each of the $\mathscr{H}_{I,j}$ appears only once. There are n! such terms, and all of them contribute equally since the bracket $\{\ \}_+$ is a symmetric function of the operators it contains. Therefore, the part of $U^{(n)}(t, t_o)$ we must consider reduces to

$$\left(\frac{-i}{\hbar}\right)^n \int_{t_o}^{t} \cdots \int_{t_o}^{t} \{\mathscr{H}_{I,1}(t_1)\mathscr{H}_{I,2}(t_2) \cdots \mathscr{H}_{I,n}(t_n)\}_+ \prod_{p=1}^{n} dt_p . \tag{5.5}$$

Since none of the n atoms can emit a photon (each of them is in the ground state initially), only the positive frequency part of the electric field operator in each $\mathscr{H}_{I,j}$ will contribute to the transition amplitude. When the electric field operator in Eq. (5.4) is replaced by $\mathbf{E}^{(+)}(\mathbf{r}_j, t)$ we shall write the resulting interaction Hamiltonian as $\mathscr{H}_{I,j}^{(+)}$. The operators $\mathscr{H}_{I,j}^{(+)}$ commute with each other since the atoms are dynamically independent and the fields $\mathbf{E}^{(+)}(\mathbf{r}_j, t)$ commute. We can therefore drop the ordering bracket $\{\ \}_+$ in the expression (5.5), and write the desired part of $U^{(n)}(t, t_o)$ as an n-fold product of single integrals

$$\left(\frac{-i}{\hbar}\right)^n \prod_{j=1}^{n} \int_{t_0}^{t} \mathscr{H}_{I,j}^{(+)}(t')\, dt' . \tag{5.6}$$

The result is a simple one. The operator which induces the transitions which interest us is simply a product of the operators which induce the individual absorption processes. This does not mean, however, that the matrix of the transition operator factorizes.

In evaluating the matrix element of the operator (5.6) between two states of the entire system we must note that the individual atoms which are all in the same ground state initially may make transitions to final states a_j which are different for different atoms. If we indicate these initial and final states for the atoms with

$|\{g\}>$ and $|\{a_j\}>$, and use the symbols $|i,\{g\}>$ and $|f,\{a_j\}>$ for the initial and final states of the entire system, then the matrix element of (5.6) or of $U^{(n)}(t,t_o)$ takes the form

$$<f,\{a_j\}|U^{(n)}(t,t_o)|i,\{g\}> = \left(\frac{ie}{\hbar}\right)^n \int_{t_o}^{t}\cdots\int_{t_o}^{t} e^{i\Sigma_j \omega_{a_j g} t_j} <f|E^{(+)}(r_n t_n)\cdots E^{(+)}(r_1 t_1)|i> \prod_{j=1}^{n} M_{a_j g} \prod_{p=1}^{n} dt_p , \quad (5.7)$$

where we have introduced notation for the atomic matrix elements and frequencies analogous to that of the preceding lecture, and have eliminated tensor indices by assuming the field to have a unique polarization as in Eq. (4.21).

We must next carry out upon the amplitude (5.7) the now familiar procedures of squaring, summing over final states of the field and averaging over initial states of the field. The expression we derive in that way is a transition probability for each of the atoms to reach a specified final state $|a_j>$. Since each of these final states is in general part of a continuum we must sum the probability we have derived over all the relevant final atomic states. We shall again assume that our counting device does not record all of these final states with equal likelihood, but is characterized by a certain probability $R(a_j)$ that any particular photoabsorption process is recorded. For simplicity we shall take this recording probability to be the same function for each of the n atoms of the detector. We may then carry out the final state summations for the atoms by introducing the same sensitivity functions we discussed in Eqs. (4.10) and (4.12) of the preceding lecture. When these simple sums and averages are all carried out we find for the n-fold counting probability

$$p^{(n)}(t) = \int_{t_o}^{t}\cdots\int_{t_o}^{t} \prod_{j=1}^{n} S(t'_j - t'_j)\, G^{(n)}(r_1 t'_1 \cdots r_n t'_n, r_n t''_n \cdots r_1 t''_1) \times \prod_{j=1}^{n} dt'_j\, dt''_j . \quad (5.8)$$

In this expression $G^{(n)}$ is the n-th order correlation function for the field defined by

$$G^{(n)}(x_1 \cdots x_{2n}) = \mathrm{Tr}\{\rho E^{(-)}(x_1)\cdots E^{(-)}(x_n) E^{(+)}(x_{n+1})\cdots E^{(+)}(x_{2n})\}$$

with $x_j = \{r_j, t\}$.

For broad band detectors eq. (5.8) reduces to the simpler form

$$p^{(n)}(t) = s^n \int_{t_o}^{t}\cdots\int_{t_o}^{t} G^{(n)}(r_1 t'_1 \cdots r_n t'_n, r_n t''_n \cdots r_1 t''_1) \prod_{j=1}^{n} dt'_j . \quad (5.9)$$

An ideal n-atom counter thus measures a time integral of the n-th order correlation function.

We have thus far considered the n atoms which undergo photoabsorption to be part of a single detector. But a detector constructed in this way is not very different, really, from a set of n detectors of the one-atom variety we discussed in the last lecture. If we regard the n atoms as the sensitive elements of a set of n independent detectors, then the n-fold photoabsorption process we have been discussing furnishes the basis of a primitive technique for n-fold coincidence counting of photons.

The technique may be refined a little if we imagine that there is a separate shutter in front of each one-atom detector. Then we may assume that all the shutters open at the same time t_o but that the time at which each of them is closed may be varied arbitrarily. Let us suppose that the time at which the j-th shutter is closed is t_j. Then the j-th atom only sees the field from time t_o to t_j. The effect of closing the shutter may be simulated by assuming that the atom is decoupled from

the field at time t_j. For this purpose we may introduce the step function

$$\theta(t) = \begin{cases} 0 \text{ for } t<0 \\ 1 \text{ for } t>0 \end{cases} \quad (5.10)$$

and write an effective interaction Hamiltonian (i.e., one which takes account of the closing of all the shutters) as

$$\mathcal{H}_I(t) = \sum_{j=1}^{n} \theta(t_j - t)\, \mathcal{H}_{I,j}(t) . \quad (5.11)$$

The calculation of the probability that a photoabsorption takes place in each detector is essentially the same with the effective Hamiltonian (5.11) as the calculation we have described earlier. The only real difference, besides the one of interpretation, is that the answer for the total detection probability is now an n-fold time integral in which the upper limits of integration are the times t_j. For the broad band case the answer is, for example

$$p^{(n)}(t_1\cdots t_n) = s^n \int_{t_o}^{t_1} dt'_1 \cdots \int_{t_o}^{t_n} dt'_n\, G^{(n)}(r_1 t'_1 \cdots r_n t'_n, r_n t'_n \cdots r_1 t'_1) . \quad (5.12)$$

The times $t_1 \cdots t_n$ may be varied independently. An n-fold delayed coincidence rate, i.e., a counting rate per (unit time)^n, may therefore be defined as

$$w^{(n)}(t_1\cdots t_n) = \frac{\partial^n}{\partial t_1 \cdots \partial t_n}\, p^{(n)}(t_1\cdots t_n) = s^n G^{(n)}(r_1 t_1 \cdots r_n t_n, r_n t_n \cdots r_1 t_1) . \quad (5.13)$$

This result verifies the statement we made earlier that coincidence experiments performed with ideal detectors furnish measurements of the higher order correlation functions.

It may be worth emphasizing that the kinds of measurement processes we have been describing differ both in method and in spirit from those that are customarily discussed in the formal quantum mechanical theory of measurement. The formal theory of measurement has been useful in establishing the physical interpretation of quantum mechanical expressions. But because there are few areas in which exact statements meeting the required assumptions of the theory can be made, the applications of the formal theory have been quite restricted to date.

The kinds of field measurements we have discussed are, by contrast, explicitly approximate in character. We have only calculated the transition probabilities to the lowest order in which the transitions occur. While this approximation would not be too difficult to remedy for individual atomic transitions, the higher order effects in multi-atom detectors would be found to have quite a complicated mathematical structure. It is implicit in the approximation we have used that the electromagnetic influences (as well as other influences) of one atom on another are ignored. That can be seen, for example, from the fact that the $E^{(+)}$ operators which occur in the correlation function $G^{(n)}$ all commute. The transition rate (5.13), for example, does not depend on the ordering of the times $t_1\cdots t_n$ even though the points $r_j t_j$ may have time-like relationships to one another and electromagnetic disturbances can indeed pass from one point to another.

While the atoms may influence one another electromagnetically in ways not described by our lowest-order results, those influences are typically extremely small and are sometimes of a kind that can be eliminated experimentally. To take a specific example, let us suppose, that instead of a simple photoabsorption process in atom 1, we have a type of Raman effect which produces another photon as well as a photoelectron (Fig. 6). The emmited photon may then be absorbed by atom 2, producing a second photoelectron. Not only does this type of process have an

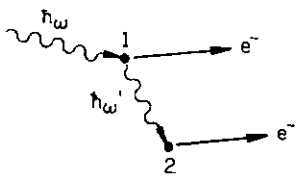

Figure 6

extremely small cross section, but it may be eliminated entirely by choosing detector atoms with ionization potentials greater than $(1/2)\hbar\omega$.

We have mentioned the electromagnetic influences of the atoms upon one another just to underscore the fact that we have not been describing an exact theory of measurement. It may none the less be an extremely useful and accurate theory.

Lecture VI PROPERTIES OF THE CORRELATION FUNCTIONS

The n-th order correlation function was defined as the expectation value

$$G^{(n)}(x_1 \cdots x_{2n}) = \mathrm{Tr}\{\rho E^{(-)}(x_1) \cdots E^{(-)}(x_n) E^{(+)}(x_{n+1}) \cdots E^{(+)}(x_{2n})\} . \tag{6.1}$$

The averaging process we carry out to evaluate this expression is the quantum mechanical analogue of the classical procedure introduced in the first lecture. There we spoke of averages over a set of random Fourier coefficients. The resemblance between the two approaches is not yet a very persuasive one, but it will become more so as we proceed.

As a first property of the correlation functions we note that when we have an upper bound on the number of photons present in the field then the functions $G^{(n)}$ vanish identically for all orders higher than a fixed order M. To state the property more explicitly, if $|n>$ is an n-quantum state and the density operator is written in the form

$$\rho = \sum_{m,n} c_{nm} |n><m| , \tag{6.2}$$

then if we have $c_{nm} = 0$ whenever $n > M$ or $m > M$, it follows from the nature of the annihilation operators $E^{(+)}$, that

$$E^{(+)}(x_1) \cdots E^{(+)}(x_p)\rho = 0 \tag{6.3}$$

$$\text{for } p > M .$$

Furthermore, the conjugate relation

$$\rho E^{(-)}(x_1) \cdots E^{(-)}(x_p) = 0 \tag{6.4}$$

also holds for $p > M$. Thus it follows that

$$G^{(p)} \equiv 0 \tag{6.5}$$

for $p > M$.

This property of the correlation functions must be regarded as a rather strange one when viewed from the standpoint of classical theory. There the correlation functions are essentially sums of moments of the probability distribution for the Fourier coefficients, and it would be quite difficult to imagine a case for which the moments higher than a certain order vanish identically. We have, in fact, constructed states which have no classical analogue by imposing an upper bound on the number of photons present. However, that should not be surprising since in the limit $\hbar \to 0$ these are states whose total energy goes to zero.

A further property of the correlation functions can be derived from the general statement

$$\mathrm{Tr}(A^\dagger) = (\mathrm{Tr}\, A)^* , \tag{6.6}$$

which holds for all linear operators A. Applying this identity to the correlation function (6.1), we find

$$[G^{(n)}(x_1 \cdots x_{2n})]^* = \mathrm{Tr}\{E^{(-)}(x_{2n}) \cdots E^{(-)}(x_{n+1}) E^{(+)}(x_n) \cdots E^{(+)}(x_1)\rho^\dagger\}$$
$$= \mathrm{Tr}\{\rho E^{(-)}(x_{2n}) \cdots E^{(-)}(x_{n+1}) E^{(+)}(x_n) \cdots E^{(+)}(x_1)\} = G^{(n)}(x_{2n} \cdots x_1) . \tag{6.7}$$

Here we have made use of the Hermitian character of ρ and of the invariance of the trace of a product of operators under a cyclic permutation.

As a consequence of the commutation properties of the $E^{(+)}$ and $E^{(-)}$ we can freely permute the arguments $(x_1 \cdots x_n)$ and $(x_{n+1} \cdots x_{2n})$ without altering the value of $G^{(n)}(x_1 \cdots x_n, x_{n+1} \cdots x_{2n})$. We cannot, however, interchange any of the first n arguments with any of the remaining n, unless suitable terms are added, since the corresponding operators do not commute.

A number of interesting inequalities can be derived from the general statement

$$\mathrm{Tr}\{\rho A^\dagger A\} \geq 0. \tag{6.8}$$

This relation, which follows from the positive definite character of the operator in the brackets, holds for any linear operator A. To prove the inequality we note that ρ is Hermitian and therefore can be diagonalized. Thus, in some representation it has the form

$$<k|\rho|m> = \delta_{km} p_k . \tag{6.9}$$

It follows immediately from the definition of the density operator that

$$p_k = <k|\rho|k> = \{<k|i><i|k>\}_{av.} = \{|<i|k>|^2\}_{av.} \geq 0. \tag{6.10}$$

(Furthermore, since $\mathrm{Tr}\,\rho = \sum_k p_k = 1$, not all the p_k vanish.) Now a simple application of the completeness relation gives

$$\mathrm{Tr}\{\rho A^\dagger A\} = \sum_k p_k <k|A^\dagger A|k>$$
$$= \sum_k p_k \sum_m <k|A^\dagger|m><m|A|k> = \sum_k p_k \sum_m |<m|A|k>|^2 \geq 0. \tag{6.11}$$

Of course this value for the trace is independent of the particular representation used. Hence the proof of the inequality (6.8) is completed.

A number of results may be derived from the general inequality (6.8) by means of various substitutions. For example the choice $A = E^{(+)}(x)$ gives at once

$$G^{(1)}(x, x) \geq 0. \tag{6.12}$$

Similarly the substitution $A = E^{(+)}(x_1) \cdots E^{(+)}(x_n)$ give us

$$G^{(n)}(x_1 \cdots x_n, x_n \cdots x_1) \geq 0. \tag{6.13}$$

These two relations are also evident from the physical meaning of the "diagonal" forms of the $G^{(n)}$. The forms are interpretable as photon intensities and coincidence rates respectively, and are thus intrinsically positive.

These results and all of our later ones can be generalized immediately to deal with vector fields $E_\mu^{(+)}(x)$ rather than the scalar field $E^{(+)}(x)$. We need only associate a vector index μ_j with each coordinate x_j. We can thus consider x_j as a shorthand for the set of variables $\{r_j, t_j, \mu_j\}$ instead of simply $\{r_j, t_j\}$.

Another possible choice for the operator A is

$$A = \sum_{j=1}^{n} \lambda_j E^{(+)}(x_j) \quad , \tag{6.14}$$

where the λ_j are a set of arbitrary complex numbers. For this case (6.8) gives us

$$\sum_{ij} \lambda^*_i \lambda_j G^{(1)}(x_i, x_j) \geq 0. \tag{6.15}$$

Thus the set of correlation functions $G^{(1)}(x_i, x_j)$ forms a matrix of coefficients for a positive definite quadratic form. Such a matrix has, of course, a positive determinant,

$$\det [G^{(1)}(x_i, x_j)] \geq 0. \tag{6.16}$$

For n = 1 this is simply the relation (6.2). For n = 2 we find

$$G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2) \geq | G^{(1)}(x_1 x_2) |^2, \tag{6.17}$$

which is a simple generalization of the Schwarz inequality.

By proceeding along the same line we can derive an infinite sequence of inequalities. We shall confine ourselves however, to mentioning the quadratic ones for the higher order correlation functions. If we write

$$A = \lambda_1 E^{(+)}(x_1) \cdots E^{(+)}(x_n) + \lambda_2 E^{(+)}(x_{n+1}) \cdots E^{(+)}(x_{2n}), \tag{6.18}$$

then the positive-definiteness of the related quadratic form requires that we have

$$\begin{aligned} &G^{(n)}(x_1 \cdots x_n, x_n \cdots x_1)\, G^{(n)}(x_{n+1} \cdots x_{2n}, x_{2n} \cdots x_{n+1}) \\ &\geq | G^{(n)}(x_1 \cdots x_n, x_{n+1} \cdots x_{2n}) |^2. \end{aligned} \tag{6.19}$$

SPACE AND TIME DEPENDENCE OF THE CORRELATION FUNCTIONS

We note that the operators $E^{(+)}(r, t)$ occurring in the correlation functions, obey the Maxwell equations and furthermore satisfy whatever boundary conditions we ordinarily require of the electric field vector (e.g., periodic boundary conditions or the conditions for conducting walls). As a result the functions $G^{(n)}(x_1 \cdots x_{2n})$ obey 2n wave equations and 2n sets of boundary conditions, one for each of the space-time variables.

Let us now consider the structure of the functions $G^{(n)}$ in stationary fields. The best way to define stationarity in quantum mechanics is to require that the density operator commute with the Hamiltonian. This criterion is equivalent to the statement that ρ is independent of time in the Schrodinger picture. (In the Heisenberg picture, however, the density operator for isolated systems is always time-independent.) If we use this definition and the familiar interpretation of the Hamiltonian as an infinitesimal time-displacement operator we may write

$$\begin{aligned} &\mathrm{Tr}\{\rho E^{(-)}(x_1) \cdots E^{(+)}(x_{2n})\} = \mathrm{Tr}\{e^{\frac{iH\tau}{\hbar}} \rho E^{(-)}(x_1) \cdots E^{(+)}(x_{2n}) e^{-\frac{iH\tau}{\hbar}}\} \\ &= \mathrm{Tr}\{e^{\frac{iH\tau}{\hbar}} \rho e^{-\frac{iH\tau}{\hbar}} e^{\frac{iH\tau}{\hbar}} E^{(-)}(x_1) e^{-\frac{iH\tau}{\hbar}} \cdots e^{\frac{iH\tau}{\hbar}} E^{(+)}(x_{2n}) e^{-\frac{iH\tau}{\hbar}}\} \\ &= \mathrm{Tr}\{\rho E^{(-)}(r_1, t_1, +\tau) \cdots E^{(+)}(r_{2n}, t_{2n} + \tau)\}, \end{aligned}$$

where τ is an arbitrary time parameter. We have thus shown that for stationary fields the correlation functions obey the identity

$$G^{(n)}(r_1 t_1 \cdots r_{2n} t_{2n}) = G^{(n)}(r_1 t_1 + \tau, \cdots r_{2n} t_{2n} + \tau) \ , \tag{6.20}$$

i.e. they are not changed by a common time displacement of all the arguments. As a result, the $G^{(n)}$ may be thought of as depending only on (2n-1) time differences. The same sort of argument can also be constructed for dealing with spatial displacements. When the density operator commutes with the components of the momentum of the field, the correlation functions are invariant under displacement of the spatial coordinates in the corresponding directions.

One further mathematical property of the correlation functions is a consequence of the way in which the functions are constructed from the positive and negative frequency parts of the fields. The function $G^{(n)}(t_1 \cdots t_n, t_{n+1} \cdots t_{2n})$ has a time dependence which, according to our convention, contains only positive frequencies for the variables $t_{n+1} \cdots t_{2n}$ and only negative frequencies for $t_1 \cdots t_n$. Thus, for example, if we ignore the spatial dependences we may write

$$G^{(1)}(t, t') = \sum_{kk'} c_{kk'} e^{i\omega_{k'} t} e^{-i\omega_k t'} \tag{6.21}$$

with ω_k and $\omega_{k'} > 0$.

Now if we consider $G^{(1)}(t, t')$ as a function of two complex time variables, t and t', it is clearly an analytic function of t' in the half plane Im t' $\leq$ 0, and an analytic function of t in the half-plane Im t $\geq$ 0.

We can therefore use the Cauchy theorem of complex function theory to construct identities such as

$$G^{(1)}(t, t') = \frac{1}{2\pi i} \int_C \frac{G^{(1)}(t, t'')}{t'' - t'} dt'' \ , \tag{6.22}$$

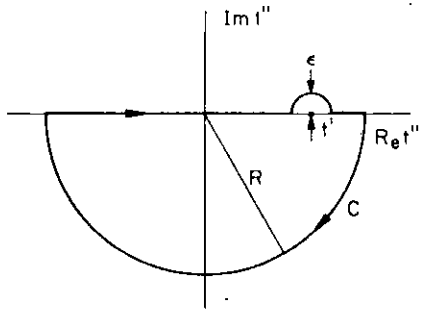

Figure 7.

where C is the contour in the complex t''-plane which is shown in Fig. 7.

Now from the boundedness of the coefficients $c_{kk'}$ in Eq. (6.21) we may see that the semi-circular part of the contour in the lower half plane gives no contribution in the limit as the radius R goes to infinity. Furthermore we note that the contribution of the infinitesimal semi-circular contour in the upper half-plane is just $-\pi i$ times the residue at the pole. In this way we find

$$G^{(1)}(t,t') = \frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{G^{(1)}(t,t'')}{t'' - t'} dt'' \quad , \tag{6.23}$$

where the integration is performed along the real axis and the symbol P denotes the Cauchy principal value. When we take the real and imaginary parts of Eq. (6.23), we obtain the pair of relations

$$\mathrm{Im}\, G^{(1)}(t,t') = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Re}\, G^{(1)}(t,t'')}{t'' - t'} dt'' \tag{6.24}$$

$$\mathrm{Re}\, G^{(1)}(t,t') = -\frac{i}{\pi} P \int_{-\infty}^{\infty} \frac{\mathrm{Im}\, G^{(1)}(t,t'')}{t'' - t'} dt'' . \tag{6.25}$$

These relations enable us in principle to calculate the imaginary part of the correlation functions once we know the real part and vice versa.

Hilbert transform relationships of this type have received a considerable amount of attention in physics and electrical engineering in connection with the requirement that linearly responding systems behave causally. The relations such as (6.24) and (6.25) which are obeyed by the correlation functions, however, have nothing to do with causality. They are simply consequences of the way in which the functions have been defined.

Lecture VII DIFFRACTION AND INTERFERENCE

From a mathematical standpoint, the quantum mechanical treatment of diffraction problems need not differ too greatly from the classical treatment. The field operators are required in general to obey the same linear differential equations and boundary conditions as the classical fields. The problem of constructing such operators may be reduced to the problem of finding a suitable set of mode functions in which to expand them (i.e., a set of mode functions which satisfies the wave equation together with suitable boundary conditions on any surfaces present). To find these modes we naturally resort to the familiar methods of the classical theory of boundary value problems. The solution for the mode functions is not a quantum dynamical problem at all. On the other hand, the fact that it is a well-explored "classical" problem does not mean, as we all know, that it is necessarily a simple one.

Let us return, for example, to the discussion of Young's experiment, illustrated in Fig. 2. When we said that the field at points on the screen Σ_2 is simply a linear combination of the fields at the two pinholes P_1 and P_2, evaluated at appropriate times, we were not solving the diffraction problem exactly, but making a number of physical approximations. One approximation, for example, was an implicit neglect of the fact that transmission of light through the pinholes has a slightly dispersive character. (This effect can be quite small if the bandwidth of the incident radiation is not too broad.) Approximations such as these are essentially classical in character. They are present simply because we have not taken the trouble to solve the classical diffraction problem more precisely.

With this understanding we can now discuss Young's experiment in fully quantum mechanical terms. The positive frequency part of the field $E^{(+)}(r,t)$ when evaluated on the screen Σ_2 will be given, just as in the classical theory, by a linear combination of the fields $E^{(+)}$ evaluated at the pinholes and having the form of Eq. (2.1). The only difference is that the fields $E^{(+)}$ are now operators. If we assume that the two pinholes are not only quite tiny compared with their separation but equal in size then we shall have $\lambda_1 = \lambda_2$ in Eq. (2.1) and we may let the constant λ stand for both coefficients. Now if our observations of the interference pattern on the screen Σ_2 are made with an ideal photon detector, the counting rate of the detector will be proportional to $G^{(1)}(r\,t, r\,t)$. In other words, the intensity observed will be proportional to

$$I = \mathrm{Tr}\{\rho E^{(-)}(rt)\, E^{(+)}(rt)\} = \mathrm{Tr}\{\rho |\lambda|^2 [E^{(-)}(x_1) + E^{(-)}(x_2)]\,[E^{(+)}(x_1) + E^{(+)}(x_2)]\} \quad , \tag{7.1}$$

where we have again let x_j stand for the point (r_j, t_j). This intensity may be expressed in terms of first order correlation functions by expanding the product in Eq. (7.1). We then find

$$I = |\lambda|^2 \{G^{(1)}(x_1,x_1) + G^{(1)}(x_2,x_2) + 2\,\mathrm{Re}\, G^{(1)}(x_1,x_2)\} . \tag{7.2}$$

The first two terms on the right side of this equation are the intensities which would be contributed by either pinhole in the absence of the other. These are, according to the assumptions we have made, rather slowly varying functions of x_1 and x_2. The third term on the right side of Eq. (7.2) is the interference term, as we have already noted in the classical discussion. The correlation function for $x_1 \neq x_2$ in general takes on complex values. If we write it as

$$G^{(1)}(x_1 x_2) = |G^{(1)}(x_1 x_2)|\, e^{i\varphi(x_1,x_2)} \quad ,$$

then the intensity becomes

$$I = |\lambda|^2 \{G^{(1)}(x_1x_1) + G^{(1)}(x_2x_2) + 2|G^{(1)}(x_1x_2)|\cos\varphi(x_1x_2)\} \quad , \tag{7.3}$$

and we see in the oscillation of the cosine term the origin of the familiar interference fringes.

SOME GENERAL REMARKS ON INTERFERENCE

The discussion we have given of Young's experiment is so closely related to the usual classical analysis that it may not be too clear in what way the interference phenomenon is a quantum mechanical one. A few general remarks about the quantum mechanical interpretation of interferences may therefore be in order. Interference phenomena characteristically occur in quantum mechanics whenever the probability amplitude for reaching a given final state from a given initial one is the sum of two or more partial amplitudes which have well defined phase relations. The individual partial amplitudes are usually contributed by alternative ways in which the system can evolve from its initial state to the final one.

The Young experiment furnishes a simple illustration of these generalities. We may consider as the initial state of the system one in which a wave packet representing a single incident photon lies to the left of the first screen σ (Fig. 2.) which has the single pinhole. We assume that initially all atoms of our photodetector are in the ground state. The final state of the system will be taken to be one in which the photon has been absorbed and one of the atoms of the counter has been correspondingly excited. The amplitude for reaching this final state is the sum of

two amplitudes, each associated with the passage of the photon through one of the two pinholes in the screen Σ_1.

It is interesting to note that the existence of the interference effect is linked quite essentially with our inability to tell which of the possible paths the photon actually takes. Neils Bohr has shown, in a famous argument, that any attempt to determine which of the two paths the photon has followed will wipe out the interference fringes. One way of making such an attempt, for example, is by trying to measure the recoil of the screen Σ_1 when it deflects the photon. The photon may transfer either of two different recoil momenta to the screen (if it excites the counter). However, if we are to make sufficiently accurate measurements of the momentum of the screen we must be prepared to accept an uncertainty in its position which will mean that no fringes appear when the experiment is performed repeatedly.

This lesson is one which can be generalized to apply to all of the quantum mechanical situations we have described earlier. The different paths by which a system may evolve will contribute amplitudes with well-defined phase relations only as long as we have no way of telling which path the system takes. When we make observations to determine the path we characteristically alter the system by making the phases of the partial amplitudes random relative to one another, i.e., we wipe out any interference of the amplitudes on the average.

The alternative paths we have been speaking of are evolutionary paths or histories. For single particle systems such histories may often be identified with spatial trajectories, but for systems with many particles or variable numbers of particles the concept is a much more general one. It is important to emphasize that the quantities which interfere in quantum mechanics are amplitudes associated with particular histories, since the terminology which has been used has often invited confusion on this score.

An example of a statement which is often quoted and easily misinterpreted is made by Dirac in the first chapter of his classic text, The Principles of Quantum Mechanics (Oxford, Clarendon Press, 3rd edition, 1947, p. 9.) There Dirac points out that the interference of the two component beams of the Michelson Interferometer cannot be interpreted as taking place because the photons of one beam sometimes annihilate photons from the other and sometimes combine to produce four photons. "This would contradict the conservation of energy. The new theory, which connects the wave functions with probabilities for one photon, gets over the difficulty by making each photon go partly into each of the two components. Each photon then interferes only with itself. Interference between two different photons never occurs." These remarks were only intended to refer to an experimental situation generically similar to that of Young's experiment, one in which the interference pattern is revealed by detecting single photons. To attempt to apply Dirac's remarks as a general doctrine for dealing with other types of interference experiments may lead to contradictions, as we shall presently see.

FIRST-ORDER COHERENCE

The word "coherence" is used not only in optics, but in a variety of quantum mechanical and communication theoretical contexts as well. We shall not attempt to construct an encyclopedia of these usages here. We shall try instead to give the term a precise meaning when applied to electromagnetic fields. The meaning we shall adopt is in fact one which links several of these conventional usages together.

The familiar concept of optical coherence is associated with the possibility of producing interference fringes when two fields are superposed. Let us return to the expression (7.3) for the intensity observed in Young's experiment. It is clear that no fringes will be observed if the correlation function $G^{(1)}(x_1, x_2)$ vanishes, and we may describe that condition by saying that the fields at x_1 and x_2 are incoherent.

It is only natural, on the other hand, to associate the highest degree of coherence with a field which exhibits the strongest possible interference fringes. Now, in the last lecture, we have derived a general inequality, (Eq. 6.17), which states

$$|G^{(1)}(x_1x_2)| \leq \{G^{(1)}(x_1x_1)\, G^{(1)}(x_2x_2)\}^{\frac{1}{2}} .$$

When we keep the intensities $G^{(1)}(x_1x_1)$ and $G^{(1)}(x_2x_2)$ fixed, the strongest contrast of the fringe intensities which is possible corresponds to using the equality sign in this relation. Thus we have established the necessary condition for coherence

$$|G^{(1)}(x_1, x_2)| = \{G^{(1)}(x_1,x_1)\, G^{(1)}(x_2,x_2)\}^{\frac{1}{2}} . \tag{7.4}$$

If we introduce the normalized correlation function

$$g^{(1)}(x_1,x_2) = \frac{G^{(1)}(x_1, x_2)}{\{G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2)\}^{\frac{1}{2}}} , \tag{7.5}$$

the condition (7.4) becomes

$$|g^{(1)}(x_1, x_2)| = 1 \tag{7.6}$$

or, in other words,

$$g^{(1)}(x_1, x_2) = e^{i\varphi(x_1,x_2)} .$$

Substitution in (7.3) now gives for the intensity in Young's experiment

$$|\lambda|^{-2} I = G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2) + 2\{G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2)\}^{\frac{1}{2}} \cos\varphi(x_1, x_2)$$
$$= \left|\{G^{(1)}(x_1, x_1)\}^{\frac{1}{2}} e^{i\varphi(x_1,x_2)} + \{G^{(1)}(x_2, x_2)\}^{\frac{1}{2}}\right|^2 . \tag{7.7}$$

This intensity varies between the limits

$$I_{min} = (\{G^{(1)}(x_1, x_1)\}^{\frac{1}{2}} - \{G^{(1)}(x_2, x_2)\}^{\frac{1}{2}})^2 \tag{7.8}$$

and

$$I_{max} = (\{G^{(1)}(x_1, x_1)\}^{\frac{1}{2}} + \{G^{(1)}(x_2, x_2)\}^{\frac{1}{2}})^2 . \tag{7.9}$$

The parameter which is usually called the visibility of the fringes is given by

$$v = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} = \frac{2\{G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2)\}^{\frac{1}{2}}}{G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2)} . \tag{7.10}$$

If the fields incident on the two pinholes have equal intensity, i.e., if $G^{(1)}(x_1, x_1) = G^{(1)}(x_2, x_2)$, then the intensity varies between zero and $4G^{(1)}(x_1, x_1)$ and the visibility is $v = 1$.

The condition (7.4) is only a condition on the fields at two space-time points x_1 and x_2. When it is satisfied we might speak of the fields at these two points as being coherent with one another. That would correspond to the usage adopted by Born and Wolf in their discussion of classical fields on the basis of time-averaged correlation functions.

In quantum mechanics one characteristically thinks of the entire field as a dynamical system It will be rather more convenient, therefore, for many analytical and statistical purposes to think of coherence as an idealized property of whole fields. That property can be described in terms of the condition (7.4), but an equivalent and mathematically more useful description can be given in terms

of the requirement that the first order correlation function factorize. Let us suppose that the correlation function $G^{(1)}(x_1, x_2)$ separates into a product of two functions $A(x_1)$ and $B(x_2)$. Then from

$$G^{(1)}(x_1, x_2) = A(x_1)\, B(x_2) \tag{7.11}$$

we conclude via the symmetry relation, Eq. (6.7), that the functions A and B obey the identity

$$A(x_2)\, B(x_1) = A^*(x_1)\, B^*(x_2)$$

or

$$\frac{A(x_2)}{B^*(x_2)} = \frac{A^*(x_1)}{B(x_1)} \quad . \tag{7.12}$$

Since in the latter relation a function of x_1 is equated to one of x_2 both functions must be constant. Furthermore the constant, let us call it μ, must be real as we can see by equating x_1 and x_2. We thus have

$$A(x) = \mu\, B^*(x), \tag{7.13}$$

and from the fact that $G^{(1)}(x, x)$ is positive it becomes evident that μ is positive. Hence, if we define the function

$$\mathcal{E}(x) = \sqrt{\mu}\; B(x), \tag{7.14}$$

we see that the first order correlation function falls into the form

$$G^{(1)}(x_1, x_2) = \mathcal{E}^*(x_1)\, \mathcal{E}(x_2). \tag{7.15}$$

This explicit construction of the factorized form of the correlation function shows that, when factorization does take place, the function $\mathcal{E}(x)$ is almost uniquely determined. The only ambiguity which remains is that of a constant multiplicative phase factor.

We shall find it most convenient to use the factorization property (7.15) as our definition of optical coherence or first-order coherence of the field. It is immediately evident that this condition implies the conditions (7.4) and (7.6) on the absolute values of the correlation functions. In fact, it is also true that the latter conditions, if they hold at all points in the field, imply in turn the factorization condition (7.15). We shall demonstrate that shortly and thereby show that the two ways of discussing coherence are equivalent. But first let us discuss some examples of coherent fields.

The most elementary example of a field for which $G^{(1)}$ factorizes is any classical field for which the Fourier coefficients C_k are precisely determined, i.e., any field for which the probability distribution $P(\{C_k\})$ reduces to a product of delta-functions. In that case the function $\mathcal{E}(x)$ is simply the classical field $E^{(+)}(x)$ itself. We perceive here a first hint of the close association which exists between coherence and noiselessness, an association which we shall presently explore further. The absence of randomness or noise in the specification of the Fourier coefficients of a field has long been the criterion used by communication engineers for speaking of a "coherent" signal.

To see another illustration of coherence let us note that one of the possible ways of performing Young's experiment, though perhaps not the most practical one, is to begin with a single photon wave packet incident upon the first pinhole. Then if we repeat the experiment many times, duplicating the wave packet precisely in each repetition, we should expect to see the familar interference fringes in the statistical distribution of photons received on the final screen. That pure states for single photons are always capable of giving rise to fringes, in this statistical sense, may be seen by examining the first-order correlation function. Let us suppose that the field is in some pure single-photon state which we denote by

$|1\ \text{phot.}>$. Then the density operator for the field is

$$\rho = |1\ \text{phot.}><1\ \text{phot.}| \tag{7.16}$$

and the first order correlation function reduces to

$$G^{(1)}(x_1, x_2) = <1\ \text{phot.}|E^{(-)}(x_1)\; E^{(+)}(x_2)|1\ \text{phot.}> \quad . \tag{7.17}$$

Now since $E^{(+)}$ is a photon annihilation operator, the state $E^{(+)}(x_2)\ |1\ \text{phot.}>$ can only be a multiple of the vacuum state which we denote as $|0>$. It is therefore possible to insert the projection operator upon the vacuum state, $|0><0|$, between the $E^{(-)}$ and $E^{(+)}$ operators in Eq. (7.17) without altering the value of the correlation function. When we do that we find

$$G^{(1)}(x_1, x_2) = <1\ \text{phot.}|E^{(-)}(x_1)\ |0><0|E^{(+)}(x_2)|1\ \text{phot.}>, \tag{7.18}$$

which is exactly the factorized form required by Eq. (7.15). Hence any pure state in which the field is occupied by a single photon possesses first order coherence. (In this way the optical definition of coherence makes contact with some of the ways in which the term is used quantum mechanically in connection with pure states.)

We have, of course, only proved that a pure one photon state is coherent. If, for example, we repeat our hypothetical one-photon interference experiment without duplicating the same wave packet each time, i.e., if we consider a mixture of pure states, then we can not expect in general to observe intensity fringes of maximum contrast. Certain particular mixtures of one photon states may, however, preserve the factorization property (7.15) of the correlation function and thereby preserve the coherence property. Hence we must not think of pure states as the only ones which bring about coherence.

To give an example, let us suppose that only one mode of the field is excited, say the k - th. Then, since the other modes all remain in their ground states, it is easily seen that we may ignore them altogether in calculating the correlation function. Now if the density operator for the k - th mode assumes the general form

$$\rho = \sum_{n,m} c_{n,m}\, |n><m|, \tag{7.19}$$

where $|n>$ is the n-th quantum state for the mode, we may write the first-order correlation function as

$$\begin{aligned} G^{(1)}(r_1 t_1,\ r_2 t_2) &= \frac{1}{2}\hbar\omega_k \sum_{n,m} c_{nm} \;<m|a_k^\dagger\, a_k|n>\; u_k^*(r_1) u_k(r_2) e^{i\omega_k(t_1 - t_2)} \\ &= C^2\; u_k^*(r_1) e^{i\omega_k t_1}\; u_k(r_2) e^{-i\omega_k t_2} \quad , \end{aligned} \tag{7.20}$$

where in the first of these expressions we have anticipated some of the notation of Eq. (8.21) and in the second we have used the definition

$$C^2 = \frac{1}{2}\hbar\omega_k \sum_n n\, c_{nn} \quad . \tag{7.21}$$

It is clear from the possibility of writing

$$\mathcal{E}(r, t) = C\, u_k(r) e^{-i\omega_k t} \tag{7.22}$$

that the correlation function (7.20) falls into the factorized from (7.15). Hence the excitation of a single mode, whether it is in a pure state or an arbitrary mixture, leads to fields with first-order coherence.

Although we have been able to give some simple examples of fields which possess first order coherence, it is worth pointing out that the factorization condition (7.15) is quite a restrictive one. It is, for example, not satisfied by pure

states of the field in general as one may easily verify by calculating the correlation function for a state in which two or more photons are present and occupy different modes. Initial states such as these may lead to fringes in Young's experiment but the fringes will not, as a rule, satisfy the condition of maximum contrast. While the coherence condition is a restrictive one, we shall show presently that there exists a much broader class of states which satisfy it than those we have considered thus far.

Let us note particularly that no statement has been made requiring that coherent fields be monochromatic. The fields which satisfy the factorization condition (7.15), or for which interference fringes of maximum (instantaneous) contrast occur, can have arbitrary time dependences. The functions $\mathcal{E}(r, t)$ which determine the correlation functions of these fields may consequently have arbitrary Fourier spectra. What seems perhaps curious about these statements is that the experimental effort to produce nearly coherent beams of light has chiefly been a struggle to produce highly monochromatic ones. The reason for this connection has been that all of the effort has involved the use of stationary light sources. Such sources lead to fields for which the first order correlation function depends only on the difference of two times,

$$G^{(1)}(t_1, t_2) = G^{(1)}(t_1 - t_2) \quad . \tag{7.23}$$

If such fields are to be coherent the correlation function must factorize to the form

$$G^{(1)}(t_1 - t_2) = \mathcal{E}^*(t_1)\,\mathcal{E}(t_2), \tag{7.24}$$

but this is a functional equation which has only exponential solutions. Since the dependence of $G^{(1)}$ on the variable t_2, can only contain positive frequencies we must have $\mathcal{E}(t) \sim e^{-i\omega t}$ for some $\omega > 0$. In other words, a coherent field which is stationary can only be monochromatic.

After giving so precise a definition to first order coherence we must add that it is a rather idealized condition, as is nearly any condition one places upon quantum mechanical states. We must not expect correlation functions for actual fields to obey the factorization condition (7.15) over unlimited ranges of the variables x_1 and x_2. In practice we define coherence lengths and times to describe the ranges of the spatial and temporal variables over which the factorization holds to a good approximation.

FRINGE CONTRAST AND FACTORIZATION

In the foregoing section we have defined coherence, mainly for reasons of mathematical convenience, in terms of a factorization property of the correlation function. That factorization property, we then showed, implies the condition (7.4) on the absolute value of the correlation function, i.e., the condition that the fringes show maximum contrast. Now it is possible to show that the latter condition, provided it holds for all space-time points, also implies the factorization property. The proof we present is taken from a forthcoming paper by U. Titulaer and the author.

When the relation

$$|G^{(1)}(x_1, x_2)|^2 = G^{(1)}(x_1, x_1)\, G^{(1)}(x_2, x_2) \tag{7.25}$$

holds it places severe constraints upon the density operator for the field. These constraints may be found by first noting that Eq. (7.25) implies the existence of operators A such that

$$\mathrm{Tr}(\rho\, A^\dagger A) = 0 \quad . \tag{7.26}$$

To exhibit such operators A we choose an arbitrary space-time point x_0 at which the intensity of the field is non-vanishing, $G^{(1)}(x_0, x_0) \neq 0$, and write

$$A = E^{(+)}(x) - \frac{G^{(1)}(x_0, x)}{G^{(1)}(x_0, x_0)}\, E^{(+)}(x_0) \quad . \tag{7.27}$$

It then follows that

$$\mathrm{Tr}(\rho\, A^\dagger A) = G^{(1)}(x\, x_0) - \frac{|G^{(1)}(x_0 x)|^2}{G^{(1)}(x_0\, x_0)} = 0 \tag{7.28}$$

for all points x. Now the density operator ρ can be written as an average of products of the state vectors of the system having the form

$$\rho = \sum_i p_i\, |i><i| \quad , \tag{7.29}$$

where the probabilities p_i are all positive. The vanishing of the trace given by Eq. (7.26) means that

$$\sum_i p_i <i|A^\dagger A|i> = 0 \quad . \tag{7.30}$$

Since all the terms entering the sum are intrinsically positive, we may conclude that

$$<i|\, A^\dagger A|i> = 0 \tag{7.31}$$

for all states $|i>$ for which $p_i \neq 0$. But this relation implies in turn that these states $|i>$ are eigenstates of A with eigenvalue zero

$$A|i> = 0 \quad . \tag{7.32}$$

What we have shown is that the vanishing of the trace (7.26) implies the pair of operator relations

$$A\rho = \rho A^\dagger = 0 \quad . \tag{7.33}$$

Since these relations hold when the operator A takes on the value given by Eq. (7.27), the density operator must obey the pair of identities

$$E^{(+)}(x)\, \rho = \frac{G^{(1)}(x_0, x)}{G^{(1)}(x_0, x_0)}\, E^{(+)}(x_0)\, \rho \tag{7.34}$$

$$\rho\, E^{(-)}(x) = \frac{G^{(1)}(x, x_0)}{G^{(1)}(x_0, x_0)}\, \rho\, E^{(-)}(x_0) \quad . \tag{7.35}$$

These identities may now be used to shift the arguments of correlation functions to a common reference point x_0. If we let $x = x_2$ in the first of these identities and $x = x_1$ in the second of them we may then use them to construct the relation

$$\mathrm{Tr}\{\rho\, E^{(-)}(x_1)\, E^{(+)}(x_2)\} = \frac{G^{(1)}(x_1, x_0)}{G^{(1)}(x_0, x_0)}\, \mathrm{Tr}\{\rho\, E^{(-)}(x_0)\, E^{(+)}(x_0)\}\, \frac{G^{(1)}(x_0, x_2)}{G^{(1)}(x_0, x_0)},$$

which can also be written as the functional identity

$$G^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_0)\, G^{(1)}(x_0, x_2)}{G^{(1)}(x_0, x_0)} \quad .$$

Now we have only to define the function $\mathcal{E}(x)$ as

$$\mathcal{E}(x) = \frac{G^{(1)}(x_0, x)}{\{G^{(1)}(x_0, x_0)\}^{\frac{1}{2}}} \tag{7.36}$$

in order to see that the first order correlation function takes on the factorized form

$$G^{(1)}(x_1, x_2) = \mathcal{E}^*(x_1)\,\mathcal{E}(x_2) \quad . \tag{7.37}$$

There is no need to repeat this demonstration in order to deal with the tensor structure of the correlation functions for fields which are not fully polarized. All we need to do is to consider each coordinate x as specifying a tensor index as well as a position and time.

Lecture VIII INTERPRETATION OF INTENSITY INTERFEROMETER EXPERIMENTS

In the preceding lecture we have discussed Young's experiment at some length as an example typical of the interference experiments which are based upon the measurement of a first order correlation function. While all of the older interference experiments share this character, we have discussed in the second lecture some more recent experiments which are of a fundamentally different type. These are the intensity interferometry experiments of Hanbury Brown, and Twiss which measure, in effect, the second order correlation function of the incident field.

We have given a simple classical discussion of the way in which the correlation fringes appear in the intensity interferometer when the field is produced by a pair of sources with small angular separation. It is interesting, therefore, to investigate the quantum mechanical origin of these same fringes. If we remember that the intensity interferometer functions by first detecting the incident fields in each of two receivers, we see immediately that pairs of photons must be involved in the interference effect, i.e., nothing is recorded at all unless different photons are incident on each of the two detectors at more or less the same time. It is at precisely this point that one is confronted by a serious dilemma if he attaches too great a generality to Dirac's statement that "interference between two different photons never occurs."

The general discussion of interference which we gave in the last lecture should make it clear that no such dilemma need exist. The things which should be regarded as interfering are not, strictly speaking, the photons, but alternative "histories" of the system as a whole. Let us imagine that the initial state of the system is one in which two (generally overlapping) single-photon wave packets are present in the field and the atoms of the two detectors (represented by photon counters) are in the ground state. We may take the final state of the system to be one in which both photons have been absorbed and one atom in each of the counters is correspondingly excited. If we label the photons 1 and 2, and the two counters a and b, we see that there are two alternative ways in which the final state may be reached. Either photon 1 is absorbed by counter a and 2 by b, or 1 is absorbed by b and 2 by a.

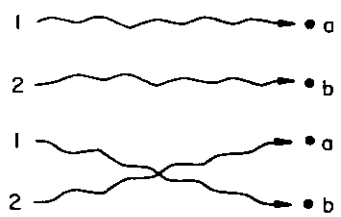

Figure 8

If the packets had altogether different average propagation vectors these alternative histories would be distinguishable by means of careful measurements

made in the counters. But the circumstances in which the fringes are observable are precisely those in which the packets have nearly the same average propagation vectors (e.g., packets with the same frequencies, small angular separation of the sources). In other words the fringes appear once again just when the alternative histories of the system become indistinguishable. Since the amplitudes for the two histories interfere, it becomes meaningless to ask which counter absorbed which photon.

HIGHER ORDER COHERENCE AND PHOTON COINCIDENCES

We recall from our classical discussions of the second lecture that the intensity interferometer measures the second order correlation function of the incident field. Radiation fields generated by natural sources tend to have a chaotic quality which allows us to construct these correlation functions from a knowledge of the first order functions. However, no such constructions are available in general for dealing with radiation from man-made sources such as the laser or radio transmitters. The fields, generated by these sources can have much higher regularity than is ever possible for natural sources. It will be useful, therefore, to sharpen the concept of coherence by defining higher order analogues of optical coherence.

We begin once more by stating conditions on the absolute values of the correlation functions. For full coherence we shall require that the normalized form of the n-th order correlation function,

$$g^{(n)}(x_1 \cdots x_{2n}) = \frac{G^{(n)}(x_1 \cdots x_{2n})}{\prod_{j=1}^{2n}\{G^{(1)}(x_j, x_j)\}^{\frac{1}{2}}} \quad , \tag{8.1}$$

have modulus unity for all n and all combinations of arguments x. If the functions have unit modulus only for $n \leq M$ we shall speak of M-th order coherence.

The concept of M-th order coherence has a simple interpretation in terms of n-fold (delayed) coincidence experiments. We know that $G^{(n)}(x_1 \cdots x_n, x_n \cdots x_1)$ is an average coincidence rate for n ideal photo-detectors registering at the points $x_1 \cdots x_n$. Since this value of the function is real and positive the condition that $g^{(n)}$ have unit modulus for $n \leq M$ implies that

$$g^{(n)}(x_1 \cdots x_n, x_n \cdots x_1) = 1$$

for $n \leq M$. Hence for fields with M-th order coherence, it is clear from the definition of $g^{(n)}$ that we have

$$G^{(n)}(x_1 \cdots x_n, x_n \cdots x_1) = \prod_{j=1}^{n} G^{(1)}(x_j\, x_j) \tag{8.2}$$

for $n \leq M$.

Expressed in experimental terms, this means that the n-fold coincidence rate is just the product of the counting rates which would be measured by each counter individually in the absence of the others. Thus there is no tendency toward statistical correlation of the photon counts. In a field with coherence of order $M \geq n$ the n photon counters register in a statistically independent way.

Several investigations of light beams using coincidence counting of photons or equivalent experimental procedures have in fact been carried out during the last few years. The first of these to detect a tendency toward statistical correlation of the arrival times of photons was performed (in addition to the other experiments we have mentioned) by Hanbury Brown, and Twiss. [1] In the experiment light from a source S (Fig. 9) passes through a pinhole P and then reaches a half-silvered mirror m, which splits it into two beams. Detectors D_1 and D_2 are placed symmetrically with respect to the mirror. Their photocurrents are multiplied together by the correlator C whose average output is the quantity measured. We may

consider the half-silvered mirror m as a device, which permits us, in effect, to place two different photodetectors at essentially the same position in the beam.

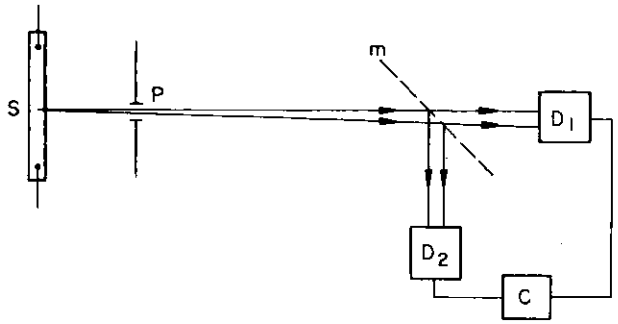

Figure 9

Shortly after the original experiment had been performed another version of it with a slightly more direct interpretation was performed by Rebka and Pound.[2] In the latter experiment D_1 and D_2 are counters of individual photons, and C is a device for registering delayed coincidences. The experiment measures the average coincidence rate as a function of delay time while the counters remain fixed in their symmetrical positions relative to the mirror. Now, even if the photon beams incident on the two counters were statistically independent of one another, there would be a certain background counting rate of accidental coincidences. This rate would, however, be independent of any time delay. Thus any observed dependence of the coincidence rate on the time delay indicates a lack of statistical independence.

The result of the experiments is indicated in Fig. 10. If the responses of the counters were statistically independent the coincidence rate would be independent of time delay. The observation of a small "bump" in the experimental curve

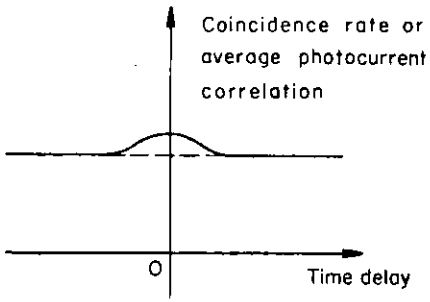

Figure 10

indicates that the photons have a distinct tendency to arrive in pairs. Although the effect was at first difficult to observe it is, as we shall show, not necessarily a small one at all. The small magnitude of the observed "bump" and its particular shape in these experiments were determined almost entirely by the relatively slow response times of the counters.

Let us note that, if the counters are placed symmetrically with respect to the mirror, the fields which are incident upon them are essentially identical, apart from a constant multiplicative factor. It follows then that if $\mathbf{r}_1$ and $\mathbf{r}_2$ are mirror-image points in the two detectors we have

$$|g^{(1)}(\mathbf{r}_1 t,\ \mathbf{r}_2 t)| = 1 \tag{8.3}$$

i.e., the fields which fall on the two detectors have essentially perfect first order coherence. The observation of a positive correlation in the coincidence rate demonstrates, on the other hand, that the fields are not coherent in the second order sense. We shall show presently that this result is a characteristic one for all experiments performed with natural light sources. These have a random character which destroys second order coherence.

FURTHER DISCUSSION OF HIGHER ORDER COHERENCE

Let us return now to the definition of higher order coherence. We have, by analogy with first order coherence, defined M-th order coherence in terms of the succession of conditions

$$|G^{(n)}(x_1 \cdots x_{2n})|^2 = \prod_{j=1}^{2n} G^{(1)}(x_j,\ x_1) \tag{8.4}$$

on the absolute values of the correlation functions for $n \leq M$. Just as in the first order case we found it convenient to express the coherence condition in an alternative way, as a factorization property of the correlation function, we shall find it even more convenient here to do much the same thing. We shall therefore state as an alternative definition the requirement that there exist a single complex function $\mathcal{E}(x)$ such that

$$G^{(1)}(x_1 \cdots x_{2n}) = \prod_{j=1}^{n} \mathcal{E}^*(x_j) \prod_{j=n+1}^{2n} \mathcal{E}(x_j) \tag{8.5}$$

for all $n \leq M$. If this factorization holds for all n we shall speak of full coherence.

If we note that the definition (8.5) contains the statement

$$G^{(1)}(x,\ x) = |\mathcal{E}(x)|^2, \tag{8.6}$$

then we see immediately that it requires that the correlation functions obey the absolute value conditions (8.4).

It is possible, on the other hand, to show that the absolute value conditions also imply the factorization properties. To do that we note that M-th order coherence always requires first order coherence. We may therefore make use of the identities which were shown in the last lecture to be consequences of first order coherence. In particular, since the operators $E^{(-)}(x_j)$ for $j = 1, \cdots n$ all commute with one another, as do the operators $E^{(+)}(x_j)$ for $j = n+1, \cdots 2n$, we can use each of the two identities (7.34) and (7.35) n times in order to shift all of the arguments of the n-th order correlation function to a particular reference point x_0. More specifically, we write

$$\mathrm{Tr}\{\rho\, E^{(-)}(x_1) \cdots E^{(-)}(x_n)\, E^{(+)}(x_{n+1}) \cdots E^{(+)}(x_{2n})\}$$

$$= \prod_{j=1}^{n} \frac{G^{(1)}(x_j, x_0)}{G^{(1)}(x_n, x_0)}\ \mathrm{Tr}\{\rho\, E^{(-)}(x_0) \cdots E^{(-)}(x_0)\, E^{(+)}(x_0)\, E^{(+)}(x_0)\}$$

$$\prod_{j=n+1}^{2n} \frac{G^{(1)}(x_0, x_j)}{G^{(1)}(x_0, x_0)},$$

which is the identity

$$G^{(n)}(x_1 \cdots x_{2n}) = \frac{G^{(n)}(x_0 \cdots x_0)}{\{G^{(1)}(x_0, x_0)\}^n}\ \frac{\prod_{j=1}^{n} G^{(1)}(x_j, x_0) \prod_{j=n+1}^{2n} G^{(1)}(x_0, x_j)}{\{G^{(1)}(x_0, x_0)\}^n}$$

If we introduce the function $\mathcal{E}(x)$ which is defined by Eq. (7.37), and make use of the normalized form of the correlation function, we may write the latter identity in the form

$$G^{(n)}(x_1 \cdots x_{2n}) = g^{(n)}(x_0 \cdots x_0) \prod_{j=1}^{n} \mathcal{E}^*(x_j) \prod_{j=n+1}^{2n} \mathcal{E}(x_j). \tag{8.7}$$

Now, as we have shown earlier, the functions $\mathcal{E}(x)$ can only depend on the choice of the arbitrary reference point x_0 through a constant phase factor. Since that phase factor cancels out of the product which occurs in Eq. (8.7), it follows that for fields with first order coherence the functions $g^{(n)}(x_0 \cdots x_0)$ are independent of x_0. In other words, the condition of first order coherence alone is sufficient to bring all of the higher order correlation functions into a factorized form, although not exactly the form, in general, which is required for higher order coherence. The difference is that Eq. (8.7) contains the constant factors $g^{(n)}(x_0 \ldots x_0)$ which should be unity if higher order coherence is to hold. Now the higher order coherence conditions (8.4) do require these coefficients to have unit absolute value for $n \leq M$. Then, since the $g^{(n)}(x_0 \cdots x_0)$ must be real and positive, they must be equal to one.

Hence the conditions (8.4) do indeed imply the factorization condition (8.5).

TREATMENT OF ARBITRARY POLARIZATIONS

From a mathematical standpoint, very little need be added to our earlier discussions in order to treat fields with arbitrary polarization properties rather than the fully polarized fields we have been discussing. All we need do, as we have already noted, in order to deal with the general tensor character of the correlation functions, is to think of every coordinate in the formulae we have derived as specifying a tensor index as well as a position and time.

Thus the relations (6.7) for n = 1 and (6.17), for example, may be generalized to read

$$\{G^{(1)}_{\mu\nu}(x_1, x_2)\}^* = G^{(1)}_{\nu\mu}(x_2, x_1) \tag{8.8}$$

and

$$|G^{(1)}_{\mu\nu}(x_1, x_2)|^2 \leq G^{(1)}_{\mu\mu}(x_1, x_1)\, G^{(1)}_{\nu\nu}(x_2, x_2). \tag{8.9}$$

It may be worth noting that all information about the state of polarization of the field is contained in the correlation tensor $G^{(1)}_{\mu\nu}(x, x)$. Let us denote this tensor by $\mathcal{G}_{\mu\nu}$. We see immediately that $\mathcal{G}_{\mu\nu}$ is a Hermitian matrix, $\mathcal{G}_{\mu\nu}{}^* = \mathcal{G}_{\nu\mu}$. If we substitute $A = \sum_{\nu=1}^{3} \lambda_\nu E^{(+)}_\nu(x)$ in the general inequality $\mathrm{Tr}\{\rho A^\dagger A\} \geq 0$ we find

$$\sum_{\mu,\nu=1}^{3} \lambda_\mu{}^* \lambda_\nu \mathcal{G}_{\mu\nu} \geq 0. \tag{8.10}$$

Thus $\mathcal{G}_{\mu\nu}$ is also positive definite. Because of its Hermitian character $\mathcal{G}_{\mu\nu}$ can be diagonalized, that is to say there exist three real and positive eigenvalues λ_p and three (generally complex) eigenvectors $\hat{e}^{(p)}$, such that

$$\mathcal{G} \cdot \hat{e}^{(p)*} = \lambda_p \hat{e}^{(p)*} \quad ; \quad \hat{e}^{(p)} \cdot \mathcal{G} = \lambda_p \hat{e}^{(p)}. \tag{8.11}$$

Note that both the λ_p and the $\hat{e}^{(p)}$ depend in general on the space-time point x, that occurs in the definition of $\mathcal{G}$.

The $\hat{e}^{(p)}$ are either found to be mutually orthogonal if the λ's have no degeneracy,

or they can be chosen orthogonal if the λ's are degenerate. Hence we may assume

$$\hat{e}^{(p)} \cdot \hat{e}^{(q)*} = \delta_{pq}. \tag{8.12}$$

Since the tensor product

$$\hat{e}^{(p)} \cdot \mathcal{G} \cdot \hat{e}^{(q)*} = \lambda_p \delta_{pq} \tag{8.13}$$

expresses the correlation of the field components in the directions of $\hat{e}^{(p)}$ and $\hat{e}^{(q)}$ there are three "directions" (i.e., complex directions) in which the field components are mutually uncorrelated. Any field may thus be regarded as a superposition of three orthogonally polarized fields whose amplitudes are (instantaneously) uncorrelated.

The eigenvalues $\lambda^{(p)}$ are the intensities corresponding to the three polarizations. The total intensity is given by

$$\mathrm{Tr}\, \mathcal{G} = \sum_p \lambda_p. \tag{8.14}$$

A set of normalized intensities can be defined as

$$I_p = \frac{\lambda_p}{\sum_{j=1}^{3} \lambda_j} \qquad (p = 1, 2, 3).$$

These numbers can be interpreted as specifying the degree of polarization of the field. In an isotropic radiation field we must have $I_p = 1/3$, $(p = 1, 2, 3)$. If the field is stationary i.e., $[\rho, H] = 0$ then $\mathcal{G}$ is time independent and the λ_p and I_p and $\hat{e}^{(p)}$ become fixed at any spatial position r.

If we are considering a beam with a single direction of propagation $\hat{k}$, then clearly $\hat{k} \cdot \mathcal{G} = \mathcal{G} \cdot \hat{k} = 0$ (since light is a transverse wave). Hence $\hat{k}$ is an eigenvector of $\mathcal{G}$ corresponding to the eigenvalue $\lambda = 0$. Then there are two remaining eigenvalues λ_p, $p = 1, 2$. The net polarization of the beam is usually defined as $|I_1 - I_2| = |\lambda_1 - \lambda_2|/(\lambda_1 + \lambda_2)$. The two polarizations $\hat{e}^{(p)}$ for $p = 1, 2$ clearly lie in the plane perpendicular to $\hat{k}$.

The higher order correlation tensors are defined by

$$G^{(n)}_{\mu_1 \cdots \mu_{2n}}(x_1 \cdots x_{2n}) = \mathrm{Tr}\{\rho E_{\mu_1}{}^{(-)}(x_1) \cdots E^{(-)}_{\mu_n}(x_n) E^{(+)}_{\mu_{n+1}}(x_{n+1}) \cdots E^{(+)}_{\mu_{2n}}(x_{2n})\}. \tag{8.15}$$

The coherence condition, Eq. (8.5), may evidently be restated for fields of arbitrary polarization by requiring that there exist a vector function $\mathcal{E}_\mu(x)$ such that

$$G_{\mu_1 \cdots \mu_{2n}}(x_1 \cdots x_{2n}) = \prod_{j=1}^{n} \mathcal{E}^*_{\mu_j}(x_j) \prod_{j=n+1}^{2n} \mathcal{E}_{\mu_j}(x_j) \tag{8.16}$$

for $n \leq M$.

As a last remark on polarizations we note that first order coherence implies full polarization of the field, i.e., if we have

$$G^{(1)}_{\mu\nu}(xx) = \mathcal{G}_{\mu\nu} = \mathcal{E}_\mu{}^*(x)\, \mathcal{E}_\nu(x), \tag{8.17}$$

then clearly the vector $\mathcal{E}_\mu(x)$ itself is an eigenvector. The corresponding intensity is $\sum_{\mu=1}^{3} |\mathcal{E}_\mu(x)|^2$, which is the full intensity of the field present.

COHERENT STATES OF THE FIELD - INTRODUCTION

Let us try to construct states in which the fields have full coherence, that is to say, states in which all the correlation functions $G^{(n)}$ factorize according to Eqs. (8.5) or (8.16). If there existed simultaneous eigenstates of the operators $E^{(+)}$ and $E^{(-)}$, such eigenstates would clearly bring about the desired factorization. However, since $E^{(+)}$ and $E^{(-)}$ do not commute (and have a commutator which is a c- number) it is clear that no such eigenstates exist. We may reduce our demand to a more plausible level by noting that in the correlation functions the field operators always occur in normal order. Therefore, it is sufficient to secure coherence if the state of the field is simply an eigenstate of $E^{(+)}$ in the restricted sense

$$E_{\mu}^{(+)}(x)\,|> \;=\; \mathcal{E}_{\mu}(x)\,|> \quad . \tag{8.18}$$

This is true because the adjoint relation is

$$<|\,E_{\mu}^{(-)}(x) \;=\; \mathcal{E}_{\mu}^{*}(x) <| \quad , \tag{8.19}$$

and together the two relations lead to the desired factorization of the correlation functions.

Since the operator $E^{(+)}$ is neither Hermitian nor normal (i.e., it does not commute with its Hermitian adjoint), there is no a priori reason why eigenstates of this form should exist. Indeed it is easily shown that the similar relation

$$<|\,E^{(+)}(x) \;=\; \mathcal{E}\,(x) <| \tag{8.20}$$

can have no normalizable solution at all. The simplest way to show that Eq. (8.18) has solutions is to construct them.

If any solution of Eq. (8.18) is to exist then it is clear that the function $\mathcal{E}_{\mu}(x)$ must satisfy the same wave equation and boundary conditions as the operator $E_{\mu}^{(+)}(x)$. The latter has the Fourier expansion

$$\begin{aligned} \mathbf{E}^{(+)}(\mathbf{r},t) &= -\frac{1}{c}\,\frac{\partial \mathbf{A}^{(+)}}{\partial t} \\ &= i\sum_{k}\left\{\frac{\hbar\omega_k}{2}\right\}^{1/2} a_k \mathbf{u}_k(\mathbf{r})\,e^{-i\omega_k t} \quad . \end{aligned} \tag{8.21}$$

Here the time independent operators a_k are described completely by means of their commutation relations

$$\begin{aligned} [a_k, a_{k'}] \;&=\; [a_k^{\dagger}, a_{k'}^{\dagger}] \;=\; 0 \\ [a_k, a_{k'}^{\dagger}] \;&=\; \delta_{kk'} \quad . \end{aligned} \tag{8.22}$$

For $\mathcal{E}(\mathbf{r},t)$ we must have a corresponding expansion

$$\mathcal{E}(\mathbf{r},t) \;=\; i\sum_{k}\left\{\frac{\hbar\omega_k}{2}\right\}^{1/2} \alpha_k \mathbf{u}_k(\mathbf{r})\,e^{-i\omega_k t} \quad , \tag{8.23}$$

where the coefficients α_k are a set of numbers which can take on arbitrary complex values.

Now if we substitute the expansions (8.21) and (8.23) in the equation which determines the eigenstates, we see that the coefficients of each mode function must separately be equal. Hence the eigenstate must satisfy the conditions

$$a_k\,|> \;=\; \alpha_k\,|> \tag{8.24}$$

for all modes k.

The coefficients α_k correspond in a simple way to the classical Fourier coefficients C_k which we introduced in the first lecture. More specifically if we compare Equations (1.8) and (8.23) we see that the correspondence is

$$C_k \;=\; i\left\{\frac{\hbar\omega_k}{2}\right\}^{1/2} \alpha_k \quad . \tag{8.25}$$

This relation shows that to describe classical fields we shall have to deal with parameters α_k of large modulus, i.e., if we let $\hbar \to 0$ then α_k increases as $\hbar^{-1/2}$.

To construct the desired eigenstate we can begin with the construction of a state $|\,\alpha_k>_k$ for the single mode k, such that

$$a_k\,|\alpha_k>_k \;=\; \alpha_k\,|\alpha_k>_k \quad . \tag{8.26}$$

The state for the entire system is then given by the direct product

$$|\,> \;=\; \prod_k |\,\alpha_k>_k \quad . \tag{8.27}$$

We shall call these states the coherent states. From the fact that they remain the same, up to a numerical factor, when we apply an annihilation operator a_k, it follows immediately that they cannot be eigenstates of the photon number operator.

The sense in which states of the type (8.27) are coherent includes, of course, optical coherence (they secure factorization of the first order coherence function). But it also includes a sense used in communication theory which we have mentioned earlier. There a coherent signal is a pure signal, one that has no noise. A classical signal of this type is ideally one with a precisely defined set of Fourier coefficients C_k. But this is exactly the kind of field we are talking about in the more general quantum mechanical context. Our precise specification of the Fourier coefficients α_k means, as we shall see, that we are as close as possible to having no noise in the signal. It can not mean, however, that there is no noise at all. Unpredictably fluctuating fields are present even in the vacuum. Our detectors detect individual photons, and photons tend to arrive randomly. Even when we specify the field as accurately as we can, we can only make predictions about the response of our counter in statistical terms; there will be some inevitable noise, and the coherent states of the field only tend to reduce that noise to a minimum.

REFERENCES

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Proc. Roy. Soc. (London) A242 300 (1957)
A243 291 (1957)

(2) G.A. Rebka and R.V. Pound Nature 180, 1035 (1957)

NOTE ON LECTURES IX - XI

Following the introduction of the coherent states, Lectures IX through XI presented the techniques for using them as a basis for the expansion of arbitrary states and operators, and for the representation of density operators in particular. Since the subject matter of these three lectures overlapped materially, if not precisely, the content of a recent paper by the lecturer, we are including a reprint of the paper itself at this point, rather than a repetition of its contents. The reader who has followed the lectures this far should have no difficulty in beginning the paper at Section III (Coherent States of a Single Mode), and following its presentation through Section IX (Density Operators for the Field). Following the reprint, the notes begin again with those of Lecture XII. That lecture resumes the story near the end of Section IX of the paper, which it is intended to amplify.

Reprint Errata

P. 2770	Eq. (3.23)	first line	Instead of 0 read $0>$
		second line	Instead of $e^{aa^\dagger}$ read $e^{a\,a^\dagger}$
P. 2783	Eq. (9.19)	Replace $\alpha_k^\dagger$ and α_k by $a_k^\dagger$ and a_k	

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Coherent and Incoherent States of the Radiation Field*

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Methods are developed for discussing the photon statistics of arbitrary radiation fields in fully quantum-mechanical terms. In order to keep the classical limit of quantum electrodynamics plainly in view, extensive use is made of the coherent states of the field. These states, which reduce the field correlation functions to factorized forms, are shown to offer a convenient basis for the description of fields of all types. Although they are not orthogonal to one another, the coherent states form a complete set. It is shown that any quantum state of the field may be expanded in terms of them in a unique way. Expansions are also developed for arbitrary operators in terms of products of the coherent state vectors. These expansions are discussed as a general method of representing the density operator for the field. A particular form is exhibited for the density operator which makes it possible to carry out many quantum-mechanical calculations by methods resembling those of classical theory. This representation permits clear insights into the essential distinction between the quantum and classical descriptions of the field. It leads, in addition, to a simple formulation of a superposition law for photon fields. Detailed discussions are given of the incoherent fields which are generated by superposing the outputs of many stationary sources. These fields are all shown to have intimately related properties, some of which have been known for the particular case of blackbody radiation.

I. INTRODUCTION

FEW problems of physics have received more attention in the past than those posed by the dual wave-particle properties of light. The story of the solution of these problems is a familiar one. It has culminated in the development of a remarkably versatile quantum theory of the electromagnetic field. Yet, for reasons which are partly mathematical and partly, perhaps, the accident of history, very little of the insight of quantum electrodynamics has been brought to bear on the problems of optics. The statistical properties of photon beams, for example, have been discussed to date almost exclusively in classical or semiclassical terms. Such discussions may indeed be informative, but they inevitably leave open serious questions of self-consistency, and risk overlooking quantum phenomena which have no classical analogs. The wave-particle duality, which should be central to any correct treatment of photon statistics, does not survive the transition to the classical limit. The need for a more consistent theory has led us to begin the development of a fully quantum-mechanical approach to the problems of photon statistics. We have quoted several of the results of this work in a recent note,[1] and shall devote much of the present paper to explaining the background of the material reported there.

Most of the mathematical development of quantum electrodynamics to date has been carried out through the use of a particular set of quantum states for the field. These are the stationary states of the noninteracting field, which corresponds to the presence of a precisely defined number of photons. The need to use these states has seemed almost axiomatic inasmuch as nearly all quantum electrodynamical calculations have been carried out by means of perturbation theory. It is characteristic of electrodynamical perturbation theory that in each successive order of approximation it describes processes which either increase or decrease the number of photons present by one. Calculations performed by such methods have only rarely been able to deal with more than a few photons at a time. The

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[1] R. J. Glauber, Phys. Rev. Letters 10, 84 (1963).

description of the light beams which occur in optics, on the other hand, may require that we deal with states in which the number of photons present is large and intrinsically uncertain. It has long been clear that the use of the usual set of photon states as a basis offers at best only an awkward way of approaching such problems.

We have found that the use of a rather different set of states, one which arises in a natural way in the discussion of correlation and coherence[2,3] properties of fields, offers much more penetrating insights into the role played by photons in the description of light beams. These states, which we have called coherent ones, are of a type that has long been used to illustrate the time-dependent behavior of harmonic oscillators. Since they lack the convenient property of forming an orthogonal set, very little attention has been paid them as a set of basis states for the description of fields. We shall show that these states, though not orthogonal, do form a complete set and that any state of the field may be represented simply and uniquely in terms of them. By suitably extending the methods used to express arbitrary states in terms of the coherent states, we may express arbitrary operators in terms of products of the corresponding state vectors. It is particularly convenient to express the density operator for the field in an expansion of this type. Such expansions have the property that whenever the field possesses a classical limit, they render that limit evident while at the same time preserving an intrinsically quantum-mechanical description of the field.

The earlier sections of the paper are devoted to a detailed introduction of the coherent states and a survey of some of their properties. We then undertake in Secs. IV and V the expansion of arbitrary states and operators in terms of the coherent states. Section VI is devoted to a discussion of the particular properties of density operators and the way these properties are represented in the new scheme. The application of the formalism to physical problems is begun in Sec. VII, where we introduce a particular form for the density operator which seems especially suited to the treatment of radiation by macroscopic sources. This form for the density operator leads to a particularly simple way of describing the superposition of radiation fields. A form of the density operator which corresponds to a very commonly occurring form of incoherence is then discussed in Sec. VIII and shown to be closely related to the density operator for blackbody radiation. In Sec. IX the results established earlier for the treatment of single modes of the radiation field are generalized to treat the entire field. The photon fields generated by arbitrary distributions of classical currents are shown to have an especially simple description in terms of coherent states. Finally, in Sec. X the methods of the preceding sections are illustrated in a discussion of certain forms of coherent and incoherent fields and of their spectra and correlation functions.

II. FIELD-THEORETICAL BACKGROUND

We have, in an earlier paper,[3] discussed the separation of the electric field operator $\mathbf{E}(\mathbf{r}t)$ into its positive-frequency part $\mathbf{E}^{(+)}(\mathbf{r}t)$ and its negative-frequency part $\mathbf{E}^{(-)}(\mathbf{r}t)$. These individual fields were then used to define a succession of correlation functions $G^{(n)}$, the simplest of which takes the form

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\mathrm{tr}\{\rho E_\mu^{(-)}(\mathbf{r}t)E_\nu^{(+)}(\mathbf{r}'t')\}, \quad (2.1)$$

where ρ is the density operator which describes the field and the symbol tr stands for the trace. We noted, in discussing these functions, that there exist quantum-mechanical states which are eigenstates of the positive- and negative-frequency parts of the fields in the senses indicated by the relations

$$E_\mu^{(+)}(\mathbf{r}t)|\ \rangle=\mathcal{E}_\mu(\mathbf{r}t)|\ \rangle, \quad (2.2)$$

$$\langle\ |E_\mu^{(-)}(\mathbf{r}t)=\mathcal{E}_\mu^*(\mathbf{r}t)\langle\ |, \quad (2.3)$$

in which the function $\mathcal{E}_\mu(\mathbf{r}t)$ plays the role of an eigenvalue. It is possible, as we shall note, to find eigenstates $|\ \rangle$ which correspond to arbitrary choices of the eigenvalue function $\mathcal{E}_\mu(\mathbf{r}t)$, provided they obey the Maxwell equations satisfied by the field operator $E_\mu(\mathbf{r}t)$ and contain only positive frequency terms in their Fourier resolutions.

The importance of the eigenstates defined by Eqs. (2.2) and (2.3) is indicated by the fact that they cause the correlation functions to factorize. If the field is in an eigenstate of this type we have $\rho=|\ \rangle\langle\ |$, and the first-order correlation function therefore reduces to

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\mathcal{E}_\mu^*(\mathbf{r}t)\mathcal{E}_\nu(\mathbf{r}'t'). \quad (2.4)$$

An analogous separation into a product of $2n$ factors takes place in the nth- order correlation function. The existence of such factorized forms for the correlation functions is the condition we have used to define fully coherent fields. The eigenstates $|\ \rangle$, which we have therefore called the coherent states, have many properties which it will be interesting to study in detail. For this purpose, it will be useful to introduce some of the more directly related elements of quantum electrodynamics.

The electric and magnetic field operators $\mathbf{E}(\mathbf{r}t)$ and $\mathbf{B}(\mathbf{r}t)$ may be derived from the operator $\mathbf{A}(\mathbf{r}t)$, which represents the vector potential, via the relations

$$\mathbf{E}=-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}, \quad \mathbf{B}=\nabla\times\mathbf{A}. \quad (2.5)$$

We shall find it convenient, in discussing the quantum states of the field, to describe the field by means of a discrete succession of dynamical variables rather than

[2] R. J. Glauber, in Proceedings of the Third International Conference on Quantum Electronics, Paris, France, 1963 (to be published).
[3] R. J. Glauber, Phys. Rev. 130, 2529 (1963).

a continuum of them. For this reason we assume that the field we are discussing is confined within a spatial volume of finite size, and expand the vector potential within that volume in an appropriate set of vector mode functions. The amplitudes associated with these oscillation modes then form a discrete set of variables whose dynamical behavior is easily discussed.

The most convenient choice of a set of mode functions, $\mathbf{u}_k(\mathbf{r})$, is usually determined by physical considerations which have little direct bearing on our present work. In particular, we need not specify the nature of the boundary conditions for the volume under study; they may be either the periodic boundary conditions which lead to traveling wave modes, or the conditions appropriate to reflecting surfaces which lead to standing waves. If the volume contains no refracting materials, the mode function $\mathbf{u}_k(\mathbf{r})$, which corresponds to frequency ω_k, may be taken to satisfy the wave equation

$$\nabla^2\mathbf{u}_k+\frac{\omega_k^2}{c^2}\mathbf{u}_k=0 \quad (2.6)$$

at interior points. More generally, whatever the form of the wave equation or the boundary conditions may be, we shall assume that the mode functions form a complete set which satisfies the orthonormality condition

$$\int\mathbf{u}_k^*(\mathbf{r})\cdot\mathbf{u}_l(\mathbf{r})d\mathbf{r}=\delta_{kl}, \quad (2.7)$$

and the transversality condition

$$\nabla\cdot\mathbf{u}_k(\mathbf{r})=0. \quad (2.8)$$

The plane-wave mode functions appropriate to a cubical volume of side L may be written as

$$\mathbf{u}_k(\mathbf{r})=L^{-3/2}\hat{\epsilon}^{(\lambda)}\exp(i\mathbf{k}\cdot\mathbf{r}), \quad (2.9)$$

where $\hat{\epsilon}^{(\lambda)}$ is a unit polarization vector. This example illustrates the way in which the mode index k may represent an abbreviation for several discrete variables, i.e., in this case the polarization index ($\lambda=1,2$) and the three Cartesian components of the propagation vector $\mathbf{k}$. The polarization vector $\hat{\epsilon}^{(\lambda)}$ is required to be perpendicular to $\mathbf{k}$ by the condition (2.8), and the permissible values of $\mathbf{k}$ are determined in a familiar way by means of periodic boundary conditions.

The expansion we shall use for the vector potential takes the form

$$\mathbf{A}(\mathbf{r}t)=c\sum_k\left(\frac{\hbar}{2\omega_k}\right)^{1/2}\times(a_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_kt}+a_k^\dagger\mathbf{u}_k^*(\mathbf{r})e^{i\omega_kt}), \quad (2.10)$$

in which the normalization factors have been chosen to render dimensionless the pair of complex-conjugate amplitudes a_k and $a_k^\dagger$. In the classical form of electromagnetic theory these Fourier amplitudes are complex numbers which may be chosen arbitrarily but remain constant in time when no charges or currents are present. In quantum electrodynamics, on the other hand, these amplitudes must be regarded as mutually adjoint operators. The amplitude operators, as we have defined them, will likewise remain constant when no field sources are active in the system studied.

The dynamical behavior of the field amplitudes is governed by the electromagnetic Hamiltonian which, in rationalized units, takes the form

$$H=\tfrac{1}{2}\int(\mathbf{E}^2+\mathbf{B}^2)d\mathbf{r}. \quad (2.11)$$

With the use of Eqs. (2.7,8) and of a suitable set of boundary conditions on the mode functions, the Hamiltonian may be reduced to the form

$$H=\tfrac{1}{2}\sum_k\hbar\omega_k(a_k^\dagger a_k+a_ka_k^\dagger). \quad (2.12)$$

This expression is the source of a well-known and extremely fruitful analogy between the mode amplitudes of the field and the coordinates of an assembly of one-dimensional harmonic oscillators. The quantum mechanical properties of the amplitude operators a_k and $a_k^\dagger$ may be described completely by adopting for them the commutation relations familiar from the example of independent harmonic oscillators:

$$[a_k,a_{k'}]=[a_k^\dagger,a_{k'}^\dagger]=0, \quad (2.13a)$$

$$[a_k,a_{k'}^\dagger]=\delta_{kk'}. \quad (2.13b)$$

Having thus separated the dynamical variables of the different modes, we are now free to discuss the quantum states of the modes independently of one another. Our knowledge of the state of each mode may be described by a state vector $|\ \rangle_k$ in a Hilbert space appropriate to that mode. The states of the entire field are then defined in the product space of the Hilbert spaces for all of the modes.

To discuss the quantum states of the individual modes we need only be familiar with the most elementary aspects of the treatment of a single harmonic oscillator. The Hamiltonian $\frac{1}{2}\hbar\omega_k(a_k^\dagger a_k+a_ka_k^\dagger)$ has eigenvalues $\hbar\omega_k(n_k+\frac{1}{2})$, where n_k is an integer ($n_k=0,1,2\cdots$). The state vector for the ground state of the oscillator will be written as $|\ \rangle_k$. It is defined by the condition

$$a_k|0\rangle_k=0. \quad (2.14)$$

The state vectors for the excited states of the oscillator may be obtained by applying integral powers of the operator $a_k^\dagger$ to $|0\rangle_k$. These states are written in normalized form as

$$|n_k\rangle_k=\frac{(a_k^\dagger)^{n_k}}{(n_k!)^{1/2}}|0\rangle_k, \quad (n_k=0,1,2\cdots). \quad (2.15)$$

The way in which the operators a_k and $a_k^\dagger$ act upon these states is indicated by the relations

$$a_k|n_k\rangle_k = n_k^{1/2}|n_k-1\rangle_k, \tag{2.16}$$

$$a_k^\dagger|n_k\rangle = (n_k+1)^{1/2}|n_k+1\rangle_k, \tag{2.17}$$

$$a_k^\dagger a_k|n_k\rangle = n_k|n_k\rangle. \tag{2.18}$$

With these preliminaries completed we are now ready to discuss the coherent states of the field in greater detail. The expansion (2.10) for the vector potential exhibits its positive frequency part as the sum containing the photon annihilation operators a_k and its negative frequency part as that involving the creation operators, $a_k^\dagger$. The positive frequency part of the electric field operator is thus given, according to (2.10), by

$$\mathbf{E}^{(+)}(rt) = i\sum_k (\tfrac{1}{2}\hbar\omega_k)^{1/2} a_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}. \tag{2.19}$$

The eigenvalue functions $\mathcal{E}(rt)$ defined by Eq. (2.2) must clearly satisfy the Maxwell equations, just as the operator $\mathbf{E}^{(+)}(rt)$ does. They therefore possess an expansion in normal modes similar to Eq. (2.19). In other words we may introduce a set of c-number Fourier coefficients α_k which permit us to write the eigenvalue function as

$$\mathcal{E}(rt) = i\sum_k (\tfrac{1}{2}\hbar\omega_k)^{1/2} \alpha_k \mathbf{u}_k(\mathbf{r}) e^{-i\omega_k t}. \tag{2.20}$$

Since the mode functions $\mathbf{u}_k(\mathbf{r})$ form an orthogonal set, it then follows that the eigenstate $|\ \rangle$ for the field obeys the infinite succession of relations

$$a_k|\ \rangle = \alpha_k|\ \rangle, \tag{2.21}$$

for all modes k. To find the states which satisfy these relations we seek states, $|\alpha_k\rangle_k$, of the individual modes which individually obey the relations

$$a_k|\alpha_k\rangle_k = \alpha_k|\alpha_k\rangle_k. \tag{2.22}$$

The coherent states $|\ \rangle$ of the field, considered as a whole, are then seen to be direct products of the individual states $|\alpha_k\rangle_k$,

$$|\ \rangle = \prod_k |\alpha_k\rangle_k. \tag{2.23}$$

III. COHERENT STATES OF A SINGLE MODE

The next few sections will be devoted to discussing the description of a single mode oscillator. We may therefore simplify the notation a bit by dropping the mode index k as a subscript to the state vector and to the amplitude parameters and operators. To find the oscillator state $|\alpha\rangle$ which satisfies

$$a|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.1}$$

we begin by taking the scalar product of both sides of the equation with the nth excited state, $\langle n|$. By using the Hermitian adjoint form of the relation (2.17), we find the recursion relation

$$(n+1)^{1/2}\langle n+1|\alpha\rangle - \alpha\langle n|\alpha\rangle \tag{3.2}$$

for the scalar products $\langle n|\alpha\rangle$. We immediately find from the recursion relation that

$$\langle n|\alpha\rangle = \frac{\alpha^n}{(n!)^{1/2}}\langle 0|\alpha\rangle. \tag{3.3}$$

These scalar products are the expansion coefficients of the state $|\alpha\rangle$ in terms of the complete orthonormal set $|n\rangle$ $(n=0, 1, \cdots)$. We thus have

$$|\alpha\rangle = \sum_n |n\rangle\langle n|\alpha\rangle$$

$$= \langle 0|\alpha\rangle \sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle. \tag{3.4}$$

The squared length of the vector $|\alpha\rangle$ is thus

$$\langle\alpha|\alpha\rangle = |\langle 0|\alpha\rangle|^2 \sum_n \frac{|\alpha|^{2n}}{n!}$$

$$= |\langle 0|\alpha\rangle|^2 e^{|\alpha|^2}. \tag{3.5}$$

If the state $|\alpha\rangle$ is normalized so that $\langle\alpha|\alpha\rangle = 1$ we may evidently define its phase by choosing

$$\langle 0|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}. \tag{3.6}$$

The coherent states of the oscillator therefore take the forms

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{\alpha^n}{(n!)^{1/2}}|n\rangle \tag{3.7}$$

and

$$\langle\alpha| = e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{(\alpha^*)^n}{(n!)^{1/2}}\langle n|. \tag{3.8}$$

These forms show that the average occupation number of the nth state is given by a Poisson distribution with mean value $|\alpha|^2$,

$$|\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}e^{-|\alpha|^2}. \tag{3.9}$$

They also show that the coherent state $|\alpha\rangle$ corresponding to $\alpha=0$ is the unique ground state of the oscillator, i.e., the state $|n\rangle$ for $n=0$.

An alternative approach to the coherent states will also prove quite useful in the work to follow. For this purpose we assume that there exists a unitary operator D which acts as a displacement operator upon the amplitudes $a^\dagger$ and a. We let D be a function of a complex parameter β, and require that it displace the amplitude operators according to the scheme

$$D^{-1}(\beta)aD(\beta) = a+\beta, \tag{3.10}$$

$$D^{-1}(\beta)a^\dagger D(\beta) = a^\dagger+\beta^*. \tag{3.11}$$

Then if $|\alpha\rangle$ obeys Eq. (3.1), it follows that $D^{-1}(\beta)|\alpha\rangle$ is an eigenstate of a corresponding to the eigenvalue $\alpha-\beta$,

$$aD^{-1}(\beta)|\alpha\rangle = (\alpha-\beta)D^{-1}(\beta)|\alpha\rangle. \tag{3.12}$$

In particular, if we choose $\beta=\alpha$, we find

$$aD^{-1}(\alpha)|\alpha\rangle = 0.$$

Since the ground state of the oscillator is uniquely defined by the relation (2.14), it follows that $D^{-1}(\alpha)|\alpha\rangle$ is just the ground state, $|0\rangle$. The coherent states, in other words, are just displaced forms of the ground state of the oscillator,

$$|\alpha\rangle = D(\alpha)|0\rangle. \tag{3.13}$$

To find an explicit form for the displacement operator $D(\alpha)$, we begin by considering infinitesimal displacements in the neighborhood of $D(0)=1$. For arbitrary displacements $d\alpha$, we see easily from the commutation rules (2.13) that $D(d\alpha)$ may be chosen to have the form

$$D(d\alpha) = 1+a^\dagger d\alpha - a d\alpha^*, \tag{3.14}$$

which holds to first order in $d\alpha$. To formulate a simple differential equation obeyed by the unknown operator we consider increments of α of the form $d\alpha = \alpha d\lambda$ where λ is a real parameter. Then if we assume the operators D to possess the group multiplication property

$$D(\alpha(\lambda+d\lambda)) = D(\alpha d\lambda)D(\alpha\lambda), \tag{3.15}$$

we find the differential equation

$$\frac{d}{d\lambda}D(\alpha\lambda) = (\alpha a^\dagger - \alpha^* a)D(\alpha\lambda), \tag{3.16}$$

whose solution, evaluated for $\lambda=1$, is the unitary operator

$$D(\alpha) = e^{\alpha a^\dagger - \alpha^* a}. \tag{3.17}$$

The coherent states $|\alpha\rangle$ may therefore be written in the form

$$|\alpha\rangle = e^{\alpha a^\dagger - \alpha^* a}|0\rangle \tag{3.18}$$

which is correctly normalized since $D(\alpha)$ is unitary.

It is interesting to discuss the relationship between the two forms we have derived for the coherent states. For this purpose we invoke a simple theorem on the multiplication of exponential functions of operators. If $\mathfrak{A}$ and $\mathfrak{B}$ are any two operators, whose commutator $[\mathfrak{A},\mathfrak{B}]$ commutes with each of them,

$$[[\mathfrak{A},\mathfrak{B}],\mathfrak{A}] = [[\mathfrak{A},\mathfrak{B}],\mathfrak{B}] = 0, \tag{3.19}$$

it may be shown[4] that

$$\exp(\mathfrak{A})\exp(\mathfrak{B}) = \exp\{\mathfrak{A}+\mathfrak{B}+\tfrac{1}{2}[\mathfrak{A},\mathfrak{B}]\}. \tag{3.20}$$

If we write $\mathfrak{A}=a^\dagger$ and $\mathfrak{B}=a$, this theorem permits us to resolve the exponential $D(\alpha)$ given by Eq. (3.17) into the product

$$D(\alpha) = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}e^{-\alpha^* a}. \tag{3.21}$$

Products of this type, which have been ordered so that the annihilation operators all stand to the right of the creation operators, will be said to be in normal form. Their convenience is indicated by the fact that the exponential $\exp[-\alpha^* a]$, when applied to the ground state $|0\rangle$, reduces in effect to unity, i.e., we have

$$e^{-\alpha^* a}|0\rangle = |0\rangle, \tag{3.22}$$

since the exponential may be expanded in series and the definition (2.14) of the ground state applied. It follows then that the coherent states may be written as

$$|\alpha\rangle = D(\alpha)|0\rangle$$

$$= e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^\dagger}|0\rangle \tag{3.23}$$

$$= e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{(\alpha a^\dagger)^n}{n!}|0\rangle. \tag{3.24}$$

Since the excited states of the oscillator are given by $|n\rangle = (n!)^{-1/2}(a^\dagger)^n|0\rangle$, we have once again derived the expression

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2}\sum_n \frac{\alpha^n}{n!}|n\rangle.$$

It may help in visualizing the coherent states if we discuss the form they take in coordinate space and in momentum space. We therefore introduce a pair of Hermitian operators q and p to represent, respectively, the coordinate of the mode oscillator and its momentum. These operators, which must satisfy the canonical commutation relation, $[q,p]=i\hbar$, may be defined for our purposes by the familiar expressions

$$q = (\hbar/2\omega)^{1/2}(a^\dagger+a), \tag{3.25a}$$

$$p = i(\hbar\omega/2)^{1/2}(a^\dagger-a). \tag{3.25b}$$

To find the expectation value of q and p in the coherent states we need only use Eq. (3.1), which defines these states, and its corresponding Hermitian adjoint form. We have then

$$\langle\alpha|q|\alpha\rangle = (2\hbar/\omega)^{1/2}\,\mathrm{Re}\,\alpha, \tag{3.26a}$$

$$\langle\alpha|p|\alpha\rangle = (2\hbar\omega)^{1/2}\,\mathrm{Im}\,\alpha, \tag{3.26b}$$

where $\mathrm{Re}\,\alpha$ and $\mathrm{Im}\,\alpha$ stand for the real and imaginary parts of α.

To find the wave functions for the coherent states we write the defining equation (3.1) in the form

$$(2\hbar\omega)^{-1/2}(\omega q + ip)|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.27}$$

and take the scalar product of both members with the conjugate state $\langle q'|$, which corresponds to the eigenvalue q' for q. Since the momentum may be represented by a derivative operator, i.e., $\langle q'|p = -i\hbar(d/dq')\langle q'|$, we find that the coordinate space wave function, $\langle q'|\alpha\rangle$,

[4] A. Messiah, *Quantum Mechanics* (North-Holland Publishing Company, Amsterdam, 1961), Vol. I, p. 442.

obeys the differential equation

$$\frac{d}{dq'}\langle q'|\alpha\rangle=-2\left(\frac{\omega}{2\hbar}\right)^{1/2}\left\{\left(\frac{\omega}{2\hbar}\right)^{1/2}q'-\alpha\right\}\langle q'|\alpha\rangle. \quad (3.28)$$

The equation may be integrated immediately to yield a solution for the wave function which, in normalized form, is

$$\langle q'|\alpha\rangle=(\omega/\pi\hbar)^{1/4}\exp\{-[(\omega/2\hbar)^{1/2}q'-\alpha]^2\}. \quad (3.29)$$

An analogous argument furnishes the momentum space wave function. If we take the scalar product of Eq. (3.27) with a momentum eigenstate $\langle p'|$, and use the relation $\langle p'|q=i\hbar(\partial/\partial p')\langle p'|$, we reach a differential equation whose normalized solution is

$$\langle p'|\alpha\rangle=(\pi\hbar\omega)^{-1/4}\exp\{-[(2\hbar\omega)^{-1/2}p'+i\alpha]^2\}. \quad (3.30)$$

Both of these wave functions are simply displaced forms of the ground-state wave function of the oscillator. The parameters $(\hbar/\omega)^{1/2}$ and $(\hbar\omega)^{1/2}$ correspond to the amplitudes of the zero-point fluctuations of the coordinate and momentum, respectively, for an oscillator of unit mass. The fact that the wave functions for the coherent states have this elementary structure should be no surprise in view of the way they are generated in Eq. (3.13), by means of displacements in the complex α plane.

The time-independent states $|\alpha\rangle$ which we have been describing are those characteristic of the Heisenberg picture of quantum mechanics. The Schrödinger picture, alternatively, would make use of the time-dependent states $\exp(-iHt/\hbar)|\alpha\rangle$. If we omit the zero-point energy $\frac{1}{2}\hbar\omega$ from the oscillator Hamiltonian and write $H=\hbar\omega a^\dagger a$, it is then clear from the expansion (3.7) for $|\alpha\rangle$ that the corresponding Schrödinger state takes the same form with α replaced by $\alpha e^{-i\omega t}$. We may thus write the Schrödinger state as $|\alpha e^{-i\omega t}\rangle$. With the substitution of $\alpha e^{-i\omega t}$ for α in Eqs. (3.26a) and (3.26b), we see that the expectation values of the coordinate and momentum carry out a simple harmonic motion with coordinate amplitude $(2\hbar/\omega)^{1/2}|\alpha|$. The same substitutions in the wave functions (3.29) and (3.30) show that the Gaussian probability densities characteristic of the ground state of the oscillator are simply carried back and forth in the same motion as the expectation values. Such wave packets are, of course, quite familiar; they were introduced to quantum mechanics at a very early stage by Schrödinger,[5] and have often been used to illustrate the way in which the behavior of the oscillator approaches the classical limit.

Another connection in which the wave packets (3.29) and (3.30) have been discussed in the past has to do with the particular way in which they localize the coordinate q' and the momentum p'. Wave packets can, of course, be found which localize either variable more sharply, but only at the expense of the localization of the other. There is a sense in which the wave packets (3.29) and (3.30) furnish a unique compromise; they minimize the product of the uncertainties of the variables q' and p'. If we represent expectation values by means of the angular brackets $\langle\ \rangle$ and define the variances

$$(\Delta q)^2=\langle q^2\rangle-\langle q\rangle^2, \quad (3.31a)$$

$$(\Delta p)^2=\langle p^2\rangle-\langle p\rangle^2, \quad (3.31b)$$

we find, for the wave functions (3.29) and (3.30), that the product of the variances is

$$(\Delta p)^2(\Delta q)^2=\tfrac{1}{4}\hbar^2.$$

According to the uncertainty principle, this is the minimum value such a product can have.[6] There thus exists a particular sense in which the description of an oscillator by means of the wave functions (3.29) and (3.30) represents as close an approach to classical localization as is possible.

The uses we shall make of the coherent states in quantum electrodynamics will not, in fact, require the explicit introduction of coordinate or momentum variables. We have reviewed the familiar representations of the coherent states in terms of these variables in the hope that they may be of some help in understanding the various applications of the states which we shall shortly undertake.

One property of the states $|\alpha\rangle$ which is made clear by the wave-function representations is that two such states are not, in general, orthogonal to one another. If we consider, for example, the wave functions $\langle q'|\alpha\rangle$ and $\langle q'|\alpha'\rangle$ for values of α' close to α, it is evident that the functions are similar in form and overlap one another appreciably. For values of α' quite different from α, however, the overlap is at most quite small. We may therefore expect that the scalar product $\langle\alpha|\alpha'\rangle$, which is unity for $\alpha'=\alpha$, will tend to decrease in absolute magnitude as α' and α recede from one another in the complex plane. The scalar product may, in fact, be calculated more simply than by using wave functions if we employ the representations (3.7) and (3.8). We then find

$$\langle\alpha|\beta\rangle=e^{-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}\sum_{n,m}\frac{(\alpha^*)^n\beta^m}{(n!m!)^{1/2}}\langle n|m\rangle,$$

which, in view of the orthonormality of the $|n\rangle$ states, reduces to

$$\langle\alpha|\beta\rangle=\exp\{\alpha^*\beta-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2\}. \quad (3.32)$$

The absolute magnitude of the scalar product is given by

$$|\langle\alpha|\beta\rangle|^2=\exp\{-|\alpha-\beta|^2\}, \quad (3.33)$$

which shows that the coherent states tend to become approximately orthogonal for values of α and β which are sufficiently different. The fact that these states are not even approximately orthogonal for $|\alpha-\beta|$ of order unity may be regarded as an expression of the overlap caused by the presence of the displaced zero-point fluctuations.

Since the coherent states do not form an orthogonal set, they appear to have received little attention as a possible system of basis vectors for the expansion of arbitrary states.[7] We shall show in the following section that such expansions can be carried out conveniently and uniquely and that they possess exceedingly useful properties. In later sections we shall, by generalizing the procedure to deal with bilinear combinations of states $|\alpha\rangle$ and $\langle\beta|$, develop analogous expansions for operators[1] as well.

IV. EXPANSION OF ARBITRARY STATES IN TERMS OF COHERENT STATES

While orthogonality is a convenient property for a set of basis states it is not a necessary one. The essential property of such a set is that it be complete. The set of coherent states $|\alpha\rangle$ for a mode oscillator can be shown without difficulty to form a complete set. To give a proof we need only demonstrate that the unit operator may be expressed as a suitable sum or an integral, over the complex α plane, of projection operators of the form $|\alpha\rangle\langle\alpha|$. In order to describe such integrals we introduce the differential element of area in the α plane

$$d^2\alpha=d(\mathrm{Re}\,\alpha)d(\mathrm{Im}\,\alpha) \quad (4.1)$$

(i.e., $d^2\alpha$ is real). If we write $\alpha=|\alpha|e^{i\vartheta}$, we may easily prove the integral identity

$$\int(\alpha^*)^n\alpha^m e^{-|\alpha|^2}d^2\alpha$$

$$=\int_0^\infty|\alpha|^{n+m+1}e^{-|\alpha|^2}d|\alpha|\int_0^{2\pi}e^{i(m-n)\vartheta}d\vartheta$$

$$=\pi n!\delta_{nm}, \quad (4.2)$$

in which the integration is carried out, as indicated, over the entire area of the complex plane. With the aid of this identity and the expansions (3.7,8) for the coherent states, we may immediately show

$$\int|\alpha\rangle\langle\alpha|d^2\alpha=\pi\sum_n|n\rangle\langle n|.$$

Since the n-quantum states are known to form a complete orthonormal set, the indicated sum over n is simply the unit operator. We have thus shown[1]

$$\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|d^2\alpha=1, \quad (4.3)$$

which is a completeness relation for the coherent states of precisely the type desired.

An arbitrary state of an oscillator must possess an expansion in terms of the n-quantum states of the form

$$|\ \rangle=\sum_n c_n|n\rangle,$$

$$=\sum_n c_n\frac{(a^\dagger)^n}{(n!)^{1/2}}|0\rangle, \quad (4.4)$$

where $\sum|c_n|^2=1$. The series which occurs in Eq. (4.4) may be used to define a function f of a complex variable z,

$$f(z)=\sum c_n\frac{z^n}{(n!)^{1/2}}. \quad (4.5)$$

It is clear from the normalization condition on the c_n that this series converges for all finite z, and thus represents a function which is analytic throughout the finite complex plane. We shall speak of the functions $f(z)$ for which $\sum|c_n|^2=1$ as the set of normalized entire functions. There is evidently a one-to-one correspondence which exists between such entire functions and the states of the oscillator. One way of approaching the description of the oscillator is to regard the functions $f(z)$ themselves as the elements of a Hilbert space. The properties of this space and of expansions carried out in it have been studied in some detail by Segal[8] and Bargmann.[9] The method we shall use for expanding arbitrary states in terms of the coherent states has been developed as a simple generalization of the usual method for carrying out changes of basis states in quantum mechanics. It is evidently equivalent, however, to one of the expansions stated by Bargmann.

If we designate the arbitrary state which corresponds to the function $f(z)$ by $|f\rangle$, then we may rewrite Eq. (4.4) as

$$|f\rangle=f(a^\dagger)|0\rangle. \quad (4.6)$$

To secure the expansion of $|f\rangle$ in terms of the states $|\alpha\rangle$, we multiply $|f\rangle$ by the representation (4.3) of the unit operator. We then find

$$|f\rangle=\frac{1}{\pi}\int|\alpha\rangle\langle\alpha|f(a^\dagger)|0\rangle d^2\alpha,$$

[5] E. Schrödinger, Naturwissenschaften 14, 664 (1926). For a more recent treatment see L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 67.

[6] W. Heisenberg, *The Physical Principles of the Quantum Theory* (University of Chicago Press, Chicago, 1930, reprinted by Dover Publications, Inc., New York, 1930), pp. 16–19.

[7] Uses of these states as generating functions for the n-quantum states have, however, been made by J. Schwinger, Phys. Rev. 91, 728 (1953).

[8] I. E. Segal, Illinois J. Math. 6, 520 (1962).

[9] V. Bargmann, Commun. Pure and Appl. Math. 14, 187 (1961); Proc. Natl. Acad. Sci. U. S. 48, 199 (1962).

which reduces, since $\langle\alpha|f(\alpha^\dagger)=\langle\alpha|f(\alpha^*)$, to

$$|f\rangle=\frac{1}{\pi}\int|\alpha\rangle f(\alpha^*)e^{-\frac{1}{2}|\alpha|^2}d^2\alpha, \tag{4.7}$$

which is an expansion of the desired type.

It is worth noting that the expansion (4.7) can easily be inverted to furnish an explicit form for the function $f(\alpha^*)$ which corresponds to any vector $|f\rangle$. For this purpose we take the scalar product of both sides of Eq. (4.7) with the coherent state $\langle\beta|$, and then, using Eq. (3.32), evaluate the scalar product $\langle\beta|\alpha\rangle$ to find

$$\langle\beta|f\rangle=\frac{1}{\pi}e^{-\frac{1}{2}|\beta|^2}\int e^{\beta^*\alpha-|\alpha|^2}f(\alpha^*)d^2\alpha. \tag{4.8}$$

Since $f(\alpha^*)$ may be expanded in a convergent power series we note the relation

$$\frac{1}{\pi}\int e^{\beta^*\alpha-|\alpha|^2}(\alpha^*)^n d^2\alpha=(\beta^*)^n, \tag{4.9}$$

from which we may derive the more general identity

$$\frac{1}{\pi}\int e^{\beta^*\alpha-|\alpha|^2}f(\alpha^*)d^2\alpha=f(\beta^*). \tag{4.10}$$

On substituting the latter identity in Eq. (4.8) we find

$$f(\beta^*)=e^{\frac{1}{2}|\beta|^2}\langle\beta|f\rangle. \tag{4.11}$$

There is thus a unique correspondence between functions $f(\alpha^*)$ which play the role of expansion amplitudes in Eq. (4.7) and the vectors $|f\rangle$ which describe the state of the oscillator.

An expansion analogous to Eq. (4.7) also exists for the adjoint state vectors. If we let $g(\alpha^*)$ be an entire function of α^* we may construct for the state $\langle g|$ the expansion

$$\langle g|=\frac{1}{\pi}\int[g(\beta^*)]^*\langle\beta|e^{-\frac{1}{2}|\beta|^2}d^2\beta. \tag{4.12}$$

The scalar product of the two states $\langle g|$ and $|f\rangle$ may then be expressed as

$$\langle g|f\rangle=\pi^{-2}\int[g(\beta^*)]^*f(\alpha^*)\exp\{\beta^*\alpha-|\alpha|^2-|\beta|^2\}d^2\alpha d^2\beta.$$

The identity (4.10) permits us to carry out the integration over the variable α to find

$$\langle g|f\rangle=\frac{1}{\pi}\int[g(\beta^*)]^*f(\beta^*)e^{-|\beta|^2}d^2\beta. \tag{4.13}$$

This expression for the scalar product of two vectors is, in essence, the starting point used by Bargmann in his discussion[10] of the Hilbert space of functions $f(z)$.

[10] Some of Bargmann's arguments are summarized by S. Schweber, J. Math. Phys. 3, 831 (1962), who has used them in connection with the formulation of quantum mechanics in terms of Feynman amplitudes. We are indebted to Dr. S. Bergmann for calling this reference to our attention.

It may be worth noting, for its mathematical interest, that the coherent states $|\alpha\rangle$ are not linearly independent of one another, as the members of a complete orthogonal set would be. Thus, for example, the expansion (4.7) may be used to express any given coherent state linearly in terms of all of the others, i.e., in view of Eqs. (4.11) and (3.32) we may write

$$|\alpha\rangle=\frac{1}{\pi}\int|\beta\rangle e^{\beta^*\alpha-\frac{1}{2}|\alpha|^2-\frac{1}{2}|\beta|^2}d^2\beta. \tag{4.14}$$

There exist many other types of linear dependence among the states $|\alpha\rangle$. We may, for example, note the identity

$$\int|\alpha\rangle\alpha^n e^{-\frac{1}{2}|\alpha|^2}d^2\alpha=0, \tag{4.15}$$

which holds for all integral $n>0$. It is clear from the latter result that if we admitted as expansion coefficients in Eq. (4.7) more general functions than $f(\alpha^*)$, say functions $F(\alpha,\alpha^*)$, there would be many additional ways of expanding any state in terms of coherent states. The constraint implicit in Eq. (4.7), that the expansion function must depend analytically upon the variable α^* is what renders the expansion unique. The virtue of an expansion scheme in which the coefficients are uniquely determined is evident. It becomes possible, by inverting the expansion as in Eq. (4.11), to construct an explicit solution for the expansion coefficient of any state, no matter what representation it was expressed in initially.

V. EXPANSION OF OPERATORS IN TERMS OF COHERENT STATE VECTORS

Our knowledge of the condition of an oscillator mode is rarely explicit enough in practice to permit the specification of its quantum state. Instead, we must describe it in terms of a mixture of states which is expressed by means of a density operator. The same reasons that lead us to express arbitrary states in terms of the coherent states, therefore, suggest that we develop an expansion for the density operator in terms of these states as well. We shall begin by considering in the present section a rather more general class of operators and then specialize to the case of the density operator in the section which follows.

A general quantum mechanical operator T may be expressed in terms of its matrix elements connecting states with fixed numbers of quanta as

$$T=\sum_{n,m}|n\rangle T_{nm}\langle m|, \tag{5.1}$$

$$=\sum T_{nm}(n!m!)^{-1/2}(a^\dagger)^n|0\rangle\langle 0|a^m. \tag{5.2}$$

If we use this expression for T to calculate the matrix element which connects the two coherent states $\langle\alpha|$ and $\langle\beta|$ we find

$$\langle\alpha|T|\beta\rangle=\sum_{n,m}T_{nm}(n!m!)^{-1/2}(\alpha^*)^n\beta^m\langle\alpha|0\rangle\langle 0|\beta\rangle. \tag{5.3}$$

It is evidently convenient to define a function $\mathcal{T}(\alpha^*,\beta)$ as

$$\mathcal{T}(\alpha^*,\beta)=\sum_{n,m}T_{nm}(n!m!)^{-1/2}(\alpha^*)^n\beta^m. \tag{5.4}$$

The operators which occur in quantum mechanics are often unbounded ones such as those of Eqs. (2.16)–(2.18). Those operators and the others we are apt to encounter have the property that the magnitudes of the matrix elements T_{nm} are dominated by an expression of the form Mn^jm^k for some fixed positive values of M, j, and k. It then follows that the double series (5.4) converges throughout the finite α^* and β planes and represents an entire function of both variables.

To secure the expansion of the operator T in terms of the coherent states, we may use the representation (4.3) of the unit operator to write

$$T=\frac{1}{\pi^2}\int|\alpha\rangle\langle\alpha|T|\beta\rangle\langle\beta|d^2\alpha d^2\beta, \tag{5.5}$$

$$=\frac{1}{\pi^2}\int|\alpha\rangle\mathcal{T}(\alpha^*,\beta)\langle\beta|\,|\alpha\rangle\langle 0|\beta\rangle d^2\alpha d^2\beta,$$

$$=\frac{1}{\pi^2}\int|\alpha\rangle\mathcal{T}(\alpha^*,\beta)\langle\beta|\exp\{-\tfrac{1}{2}|\alpha|^2-\tfrac{1}{2}|\beta|^2\}d^2\alpha d^2\beta. \tag{5.6}$$

The inversion of this expansion, or the solution for $\mathcal{T}(\alpha^*,\beta)$, is accomplished by the same method we used to invert Eq. (4.7) and secure the amplitude function (4.11). The result of the inversion is

$$\mathcal{T}(\alpha^*,\beta)=\langle\alpha|T|\beta\rangle\exp\{\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2\}. \tag{5.7}$$

We see, thus, that the expansion of operators, as well as of arbitrary quantum states, in terms of the coherent states is a unique one.

The law of operator multiplication is easily expressed in terms of the functions $\mathcal{T}$. If $T=T_1T_2$ and $\mathcal{T}_1$ and $\mathcal{T}_2$ are the functions appropriate to the latter two operators, we note that

$$\langle\alpha|T|\beta\rangle=\langle\alpha|T_1T_2|\beta\rangle$$

$$=\frac{1}{\pi}\int\langle\alpha|T_1|\gamma\rangle\langle\gamma|T_2|\beta\rangle d^2\gamma. \tag{5.8}$$

The function $\mathcal{T}$ which represents the product is therefore given by

$$\mathcal{T}(\alpha^*,\beta)=\frac{1}{\pi}\int\mathcal{T}_1(\alpha^*,\gamma)\mathcal{T}_2(\gamma^*,\beta)e^{-|\gamma|^2}d^2\gamma. \tag{5.9}$$

The expansion function for the operator $T^\dagger$, the Hermitian adjoint of T, is obtained by substituting T_{mn}^* for T_{nm} in Eq. (5.4). It is given by $[\mathcal{T}(\beta^*,\alpha)]^*$. If the operator T is Hermitian the function $\mathcal{T}$ must satisfy the identity

$$\mathcal{T}(\alpha^*,\beta)=[\mathcal{T}(\beta^*,\alpha)]^*, \tag{5.10}$$

since the expansions of T and $T^\dagger$ are unique.

The functions $\mathcal{T}(\alpha^*,\beta)$ which represent normal products of the operators $a^\dagger$ and a such as $(a^\dagger)^n a^m$ are immediately seen from Eqs. (5.7) and (3.32) to be

$$\mathcal{T}(\alpha^*,\beta)=(\alpha^*)^n\beta^m\exp[\alpha^*\beta]. \tag{5.11}$$

In particular, the unit operator corresponds to $n=m=0$.

It may be worth noting at this point that many of the foregoing formulas can be abbreviated somewhat by adopting a normalization different from the conventional one for the coherent states. If we introduce the symbol $\|\alpha\rangle$ for the states normalized in the new way and define these as

$$\|\alpha\rangle=|\alpha\rangle e^{\frac{1}{2}|\alpha|^2}, \tag{5.12}$$

then we may write the scalar product of two such states as $\langle\alpha\|\beta\rangle$. We see from Eq. (3.32) that this scalar product is

$$\langle\alpha\|\beta\rangle=\exp[\alpha^*\beta]. \tag{5.13}$$

We may next, following Bargmann,[9] introduce an element of measure $d\mu(\alpha)$ which is defined as

$$d\mu(\alpha)=\frac{1}{\pi}e^{-|\alpha|^2}d^2\alpha. \tag{5.14}$$

With these alterations, all of the Gaussian functions, and factors of π, in the preceding formulas become absorbed, as it were, into the notation. The Eqs. (5.6) and (5.7), for example, reduce to the briefer forms

$$T=\int\|\alpha\rangle\mathcal{T}(\alpha^*,\beta)\langle\beta\|d\mu(\alpha)d\mu(\beta) \tag{5.15}$$

and

$$\mathcal{T}(\alpha^*,\beta)=\langle\alpha\|T\|\beta\rangle. \tag{5.16}$$

A more significant property of the states $\|\alpha\rangle$ is that they are given by the expansion

$$\|\alpha\rangle=\sum_n\frac{\alpha^n}{(n!)^{1/2}}|n\rangle \tag{5.17}$$

and thus obey the relation

$$a^\dagger\|\alpha\rangle=\frac{\partial}{\partial\alpha}\|\alpha\rangle. \tag{5.18}$$

While the properties of the alternatively normalized states $\|\alpha\rangle$ are worth bearing in mind, we have chosen not to adopt this normalization in the present paper in order to retain the more conventional interpretation of

scalar products as probability amplitudes. The advantage afforded by the relation (5.18) is not a great one since all of the operators we shall have to deal with are either already in normally ordered form, or easily so ordered.

VI. GENERAL PROPERTIES OF THE DENSITY OPERATOR

The formalism we have developed in the two preceding sections has been intended to provide a background for the expression of the density operator of a mode in terms of the vectors that represent coherent states. Viewed in mathematical terms, the use of the coherent state vectors in this way leads to considerable simplification in the calculation of statistical averages. The fact that these states are eigenstates of the field operators $\mathbf{E}^{(\pm)}(rt)$ means that normally ordered products of the field operators, when they are to be averaged, may be replaced by the products of their eigenvalues, i.e., treated not as operators, but as numbers. The field correlation functions such as $G^{(1)}$ given by Eq. (2.1) are averages of just such operator products. Their evaluation may be carried out quite conveniently through use of the representations we shall discuss.

Any density operator ρ may, according to the methods of the preceding section, be represented in a unique way by means of a function of two complex variables, $R(\alpha^*,\beta)$, which is analytic throughout the finite α^* and β planes. The function R is given explicitly, by means of Eq. (5.7), as

$$R(\alpha^*,\beta)=\langle\alpha|\rho|\beta\rangle\exp[\tfrac{1}{2}|\alpha|^2+\tfrac{1}{2}|\beta|^2]. \quad (6.1)$$

If we happen to know the matrix representation of ρ in the basis formed by the n-quantum states, the function R is evidently given by

$$R(\alpha^*,\beta)=\sum_{n,m}\langle n|\rho|m\rangle(n!m!)^{-1/2}(\alpha^*)^n\beta^m. \quad (6.2)$$

If we do not know the matrix elements $\langle n|\rho|m\rangle$ they may be found quite simply from a knowledge of $R(\alpha^*,\beta)$. One method for finding them is to consider $R(\alpha^*,\beta)$ as a generating function and identify its Taylor series with the series (6.2). A second method is to note that if we multiply Eq. (6.2) by $\alpha^i(\beta^*)^j\exp[-(|\alpha|^2+|\beta|^2)]$ and integrate over the α and β planes, then all terms save that for $n=i$ and $m=j$ vanish in the sum on the right and we have

$$\langle i|\rho|j\rangle=\frac{1}{\pi^2}\int R(\alpha^*,\beta)(i!j!)^{-1/2}\alpha^i(\beta^*)^j e^{-(|\alpha|^2+|\beta|^2)}d^2\alpha d^2\beta. \quad (6.3)$$

Given the knowledge of $R(\alpha^*,\beta)$, we may write the density operator as

$$\rho=\frac{1}{\pi^2}\int|\alpha\rangle R(\alpha^*,\beta)\langle\beta|e^{-\frac{1}{2}(|\alpha|^2+|\beta|^2)}d^2\alpha d^2\beta. \quad (6.4)$$

The statistical average of an operator T is given by the trace of the product ρT. If we calculate this average by using the representation (6.4) for ρ we must note that the trace of the expression $|\alpha\rangle\langle\beta|T$, regarded as an operator, is the matrix element $\langle\beta|T|\alpha\rangle$. Then, if we express the matrix element in terms of the function $\mathcal{T}(\alpha^*,\beta)$ defined by Eq. (5.7) we find

$$\mathrm{tr}\{\rho T\}=\frac{1}{\pi^2}\int R(\alpha^*,\beta)\mathcal{T}(\beta^*,\alpha)e^{-|\alpha|^2-|\beta|^2}d^2\alpha d^2\beta. \quad (6.5)$$

If T is any operator of the form $(a^\dagger)^n a^m$, its representation $\mathcal{T}(\beta^*,\alpha)$ is given by Eq. (5.11). In particular for $n=m=0$, we have the unit operator $T=1$ which is represented by $\mathcal{T}(\beta^*,\alpha)=\exp[\beta^*\alpha]$. Hence, the trace of ρ itself, which must be normalized to unity, is

$$\mathrm{tr}\rho=1=\frac{1}{\pi^2}\int R(\alpha^*,\beta)\exp[\beta^*\alpha-|\alpha|^2-|\beta|^2]d^2\alpha d^2\beta.$$

Since $R(\alpha^*,\beta)$ is an entire function of α^*, we may use Eq. (4.10) to carry out the integration over the α plane. In this way we see that the normalization condition on R is

$$\frac{1}{\pi}\int R(\beta^*,\beta)e^{-|\beta|^2}d^2\beta=1. \quad (6.6)$$

The density operator is Hermitian and hence has real eigenvalues. These eigenvalues may be interpreted as probabilities and so must be positive numbers. Since ρ is thus a positive definite operator, its expectation value in any state, e.g., the state $|f\rangle$ defined by Eq. (4.6), must be non-negative,

$$\langle f|\rho|f\rangle\geq 0. \quad (6.7)$$

If, for example, we choose the state $|f\rangle$ to be a coherent state $|\alpha\rangle$ we find that the function R, which is given by Eq. (6.1), satisfies the inequality

$$R(\alpha^*,\alpha)\geq 0. \quad (6.8)$$

If we let the state $|f\rangle$ be specified as in Eq. (4.7) by an entire function $f(\alpha^*)$, then we find from the inequality (6.7) the more general condition for positive definiteness

$$\int[f(\alpha^*)]^*f(\beta^*)R(\alpha^*,\beta)e^{-|\alpha|^2-|\beta|^2}d^2\alpha d^2\beta\geq 0, \quad (6.9)$$

which must hold for all entire functions f.

In many types of physical experiments, particularly those dealing with fields which oscillate at extremely high frequencies, we cannot be said to have any *a priori* knowledge of the time-dependent parameters. The predictions we make in such circumstances are unchanged by displacements in time. They may be derived from a density operator which is stationary, that is, one which commutes with the Hamiltonian operator or, more simply, with $a^\dagger a$. The necessary and sufficient condition that a function $R(\alpha^*,\beta)$ correspond to a stationary density operator is that it depend only on the product of its two variables, $\alpha^*\beta$. There must, in other words, exist an analytic function $\mathcal{S}$ such that

$$R(\alpha^*,\beta)=\mathcal{S}(\alpha^*\beta). \quad (6.10)$$

That this condition is a sufficient one is clear from the invariance of R under the multiplication of both α and β by a phase factor, $e^{i\varphi}$. The condition may be derived as a necessary one directly from the vanishing of the commutator of ρ with $a^\dagger a$. An alternative and perhaps simpler way of seeing the result depends on noting that a stationary ρ can only be a function of the Hamiltonian for the mode, or of $a^\dagger a$. It is therefore diagonal in the basis formed by the n-quantum states, i.e., $\langle n|\rho|m\rangle=\delta_{nm}\langle n|\rho|n\rangle$. Examination of the series expansion (6.2) for R then shows that it then takes the form of Eq. (6.10).

VII. THE P REPRESENTATION OF THE DENSITY OPERATOR

In the preceding sections we have demonstrated the generality of the use of the coherent states as a basis. Not all fields require for their description density operators of quite so general a form. Indeed for a broad class of radiation fields which includes, as we shall see, virtually all of those studied in optics, it becomes possible to reduce the density operator to a considerably simpler form. This form is one which brings to light many similarities between quantum electrodynamical calculations and the corresponding classical ones. Its use offers deep insights into the reasons why some of the fundamental laws of optics, such as those for superposition of fields and calculation of the resulting intensities, are the same as in classical theory, even when very few quanta are involved. We shall continue, for the present, to limit consideration to a single mode of the field.

One type of oscillator state which interests us particularly is, of course, a coherent state. The density operator for a pure state $|\alpha\rangle$ is just the projection operator

$$\rho=|\alpha\rangle\langle\alpha|. \quad (7.1)$$

The unique representation of this operator as a function $R(\beta^*,\gamma)$ is easily shown, from Eq. (6.1), to be

$$R(\beta^*,\gamma)=\exp[\beta^*\alpha+\gamma\alpha^*-|\alpha|^2]. \quad (7.2)$$

Other functions $R(\beta^*,\gamma)$, which satisfy the analyticity requirements necessary for the representations of density operators, may be constructed by forming linear combinations of exponentials such as (7.2) for various values of the complex parameter α. The functions R, which we form in this way, represent statistical mixtures of the coherent states. The most general such function R may be written as

$$R(\beta^*,\gamma)=\int P(\alpha)\exp[\beta^*\alpha+\gamma\alpha^*-|\alpha|^2]d^2\alpha, \quad (7.3)$$

where $P(\alpha)$ is a weight function defined at all points of the complex α plane. Since $R(\beta^*,\gamma)$ must satisfy the Hermiticity condition, Eq. (5.10), we require that the weight function be real-valued, i.e., $[P(\alpha)]^*=P(\alpha)$. The function $P(\alpha)$ need not be subject to any regularity conditions, but its singularities must be integrable ones.[11] It is convenient to allow $P(\alpha)$ to have delta-function singularities so that we may think of a pure coherent state as represented by a special case of Eq. (7.3). A real-valued two-dimensional delta function which is suited to this purpose may be defined as

$$\delta^{(2)}(\alpha)=\delta(\mathrm{Re}\,\alpha)\delta(\mathrm{Im}\,\alpha). \quad (7.4)$$

The pure coherent state $|\beta\rangle$ is then evidently described by

$$P(\alpha)=\delta^{(2)}(\alpha-\beta), \quad (7.5)$$

and the ground state of the oscillator is specified by setting $\beta=0$.

The density operator ρ which corresponds to Eq. (7.3) is just a superposition of the projection operators (7.1),

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha. \quad (7.6)$$

It is the kind of operator we might naturally be led to if we were given knowledge that the oscillator is in a coherent state, but one which corresponds to an unknown eigenvalue α. The function $P(\alpha)$ might then be thought of as playing a role analogous to a probability density for the distribution of values of α over the complex plane.[12] Such an interpretation may, as we shall see, be justified at times. In general, however, it is not possible to interpret the function $P(\alpha)$ as a probability distribution in any precise way since the projection operators $|\alpha\rangle\langle\alpha|$ with which it is associated are not orthogonal to one another for different values of α. There is an approximate sense, as we have noted in connection with Eq. (3.33), in which two states $|\alpha\rangle$ and $|\alpha'\rangle$ may be said to become orthogonal to one another for $|\alpha-\alpha'|\gg 1$, i.e., when their wave packets (3.29) and those of the form (3.30) do not appreciably overlap. When the function $P(\alpha)$ tends to vary little over such large ranges of the parameter α, the nonorthogonality of the coherent states will make little difference, and $P(\alpha)$ will then be interpretable approximately as a probability density. The functions $P(\alpha)$

[11] If the singularities of $P(\alpha)$ are of types stronger than those of delta functions, e.g., derivatives of delta functions, the field represented will have no classical analog.

[12] The existence of this form for the density operator has also been observed by E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963). His note is discussed briefly at the end of Sec. X.

which vary this slowly will, in general, be associated with strong fields, ones which may be described approximately in classical terms.

We shall call the expression (7.6) for the density operator the P representation in order to distinguish it from the more general form based on the functions R discussed earlier. The normalization property of the density operator requires that $P(\alpha)$ obey the normalization condition

$$\mathrm{tr}\rho=\int P(\alpha)d^2\alpha=1. \tag{7.7}$$

It is interesting to examine the conditions that the positive definiteness of ρ places upon $P(\alpha)$. If we apply the condition (6.9) to the function $R(\beta^*,\gamma)$ given by Eq. (7.3) we find

$$\int[f(\beta^*)]^*f(\gamma^*)P(\alpha)\exp[\beta^*\alpha+\gamma\alpha^*-|\alpha|^2-|\beta|^2-|\gamma|^2]$$
$$\times d^2\alpha d^2\beta d^2\gamma\geq 0. \tag{7.8}$$

The γ integration may be carried out via Eq. (4.10) and the β integration by means of its complex conjugate. We then have the condition that

$$\int|f(\alpha^*)|^2P(\alpha)e^{-|\alpha|^2}d^2\alpha\geq 0 \tag{7.9}$$

must hold for all entire functions $f(\alpha^*)$. In particular, the choice $f(\alpha^*)=\exp[\beta\alpha^*-\frac{1}{2}|\beta|^2]$ leads to the simple condition

$$\int P(\alpha)e^{-|\alpha-\beta|^2}d^2\alpha\geq 0, \tag{7.10}$$

which must hold for all complex values of β. It corresponds to the requirement $\langle\beta|\rho|\beta\rangle\geq 0$. These conditions are immediately satisfied if $P(\alpha)$ is positive valued as it would be, were it a probability density. They are not strong enough, however, to exclude the possibility that $P(\alpha)$ takes on negative values over some suitably restricted regions of the plane.[13] This result serves to underscore the fact that the weight function $P(\alpha)$ cannot, in general, be interpreted as a probability density.[14]

If a density operator is specified by means of the P representation, its matrix elements connecting the n-quantum states are given by

$$\langle n|\rho|m\rangle=\int P(\alpha)\langle n|\alpha\rangle\langle\alpha|m\rangle d^2\alpha. \tag{7.11}$$

When Eqs. (3.3) and (3.6) are used to evaluate the scalar products in the integrand we find

$$\langle n|\rho|m\rangle=(n!m!)^{-1/2}\int P(\alpha)\alpha^n(\alpha^*)^me^{-|\alpha|^2}d^2\alpha. \tag{7.12}$$

This form for the density matrix indicates a fundamental property of the fields which are most naturally described by means of the P representation. If $P(\alpha)$ is a weight function with singularities no stronger than those of delta function type, it will, in general, possess nonvanishing complex moments of arbitrarily high order. [The unique exception is the choice $P(\alpha)=\delta^{(2)}(\alpha)$ which corresponds to the ground state of the mode.] It follows then that the diagonal matrix elements $\langle n|\rho|n\rangle$, which represent the probabilities for the presence of n photons in the mode, take on nonvanishing values for arbitrarily large n. There is thus no upper bound to the number of photons present when the function P is well behaved in the sense we have noted.[15]

Stationary density operators correspond in the P representation to functions $P(\alpha)$ which depend only on $|\alpha|$. This correspondence is made clear by Eq. (7.2) which shows that such $P(\alpha)$ lead to functions $R(\beta^*,\gamma)$ which are unaltered by a common phase change of β and γ. It is seen equally well through Eq. (7.12) which shows that $\langle n|\rho|m\rangle$ reduces to diagonal form when the weight function $P(\alpha)$ is circularly symmetric.

Some indication of the importance, in practical terms, of the P representation for the density operator can be found by considering the way in which photon fields produced by different sources become superposed. Since we are only discussing the behavior of one mode of the field for the present, we are only dealing with a fragment of the full problem, but all the modes may eventually be treated similarly. We shall illustrate the superposition law by assuming there are two different transient radiation sources coupled to the field mode and that they may be switched on and off separately. The first source will be assumed, when it is turned on alone at time t_1, to excite the mode from its ground state $|0\rangle$ to the coherent state $|\alpha_1\rangle$. If we assume that the source has ceased radiating by a time t_2, the state of the field remains $|\alpha_1\rangle$ for all later times. We may alternatively consider the case in which the first source remains inactive and the second one is switched on at

[13] An example of a weight function $P(\alpha)$ which takes on negative values but leads to a positive-definite density operator is given by the form

$$P(\alpha)=(1+\lambda)(\pi n)^{-1}\exp[-|\alpha|^2/n]-\lambda\delta^{(2)}(\alpha)$$

for $n>0$ and $0<\lambda<n^{-1}$. The matrix representation of the corresponding density operator, which is given by Eq. (7.12), is seen to be diagonal and to have only positive eigenvalues.

[14] A familiar example of a function which plays a role analogous to that of a probability density, but may take on negative values in quantum-mechanical contexts is the Wigner distribution function, E. P. Wigner, Phys. Rev. 40, 749 (1932).

[15] Density operators for fields in which the number of photons present possesses an upper bound N are represented by functions $R(\beta^*,\gamma)$ which are polynomials of Nth degree in β^* and in γ. It is evident from the behavior of such polynomials for large $|\beta|$ and $|\gamma|$ that any weight function $P(\alpha)$ which corresponds to $R(\beta^*,\gamma)$ through Eq. (7.2) would have to have singularities much stronger than those of a delta function. Such fields are probably represented more conveniently by means of the R function.

time t_2. The second source will then be assumed to bring the mode from its ground state to the coherent state $|\alpha_2\rangle$. We now ask what state the mode will be brought to if the two sources are allowed to act in succession, the first at t_1 and the second at t_2.

The answer for this simple case may be seen without performing any detailed calculations by making use of the unitary displacement operators described in Sec. III. The action of the first source is represented by the unitary operator $D(\alpha_1)$ which displaces the oscillator state from the ground state to the coherent state $|\alpha_1\rangle=D(\alpha_1)|0\rangle$. The action of the second source is evidently represented by the displacement operator $D(\alpha_2)$, so that when it is turned on after the first source, it brings the oscillator to the superposed state

$$|\ \rangle=D(\alpha_2)D(\alpha_1)|0\rangle. \tag{7.13}$$

Since the displacement operators are of the exponential form (3.17), their multiplication law is given by Eq. (3.20). We thus find

$$D(\alpha_2)D(\alpha_1)=D(\alpha_1+\alpha_2)\exp[\tfrac{1}{2}(\alpha_2\alpha_1^*-\alpha_2^*\alpha_1)]. \tag{7.14}$$

The exponential which has been separated from the D operators in this relation has a purely imaginary argument and, hence, corresponds to a phase factor. The superposed state, (7.13), in other words, is just the coherent state $|\alpha_1+\alpha_2\rangle$ multiplied by a phase factor. The phase factor has no influence upon the density operator for the superposed state, which is

$$\rho=|\alpha_1+\alpha_2\rangle\langle\alpha_1+\alpha_2|. \tag{7.15}$$

To vary the way in which the sources are turned on in the imaginary experiment we have described, e.g., to turn the two sources on at other times or in the reverse order, would only alter the final state through a phase factor and would thus lead to the same final density operator. The amplitudes of successive coherent excitations of the mode add as complex numbers in quantum theory, just as they do in classical theory.

Let us suppose next that the sources in the same experiment are somewhat less ideal and that, instead of exciting the mode to pure coherent states, they excite it to conditions described by mixtures of coherent states of the form (7.6). The first source acting alone, we assume, brings the field to a condition described by the density operator

$$\rho_1=\int P_1(\alpha_1)|\alpha_1\rangle\langle\alpha_1|d^2\alpha_1. \tag{7.16}$$

The condition produced by the second source, when it acts alone, is assumed to be represented by

$$\rho=\int P_2(\alpha_2)|\alpha_2\rangle\langle\alpha_2|d^2\alpha_2,$$
$$=\int P_2(\alpha_2)D(\alpha_2)|0\rangle\langle 0|D^{-1}(\alpha_2)d^2\alpha_2.$$

If the second source is turned on after the first, it brings the field to a condition described by the density operator

$$\rho=\int P_2(\alpha_2)D(\alpha_2)\rho_1D^{-1}(\alpha_2)d^2\alpha_2,$$
$$=\int P_2(\alpha_2)P_1(\alpha_1)|\alpha_1+\alpha_2\rangle\langle\alpha_1+\alpha_2|d^2\alpha_1d^2\alpha_2. \tag{7.17}$$

The latter density operator may be written in the general form

$$\rho=\int P(\alpha)|\alpha\rangle\langle\alpha|d^2\alpha,$$

if we define the weight function $P(\alpha)$ for the superposed excitations to be

$$P(\alpha)=\int\delta^{(2)}(\alpha-\alpha_1-\alpha_2)P_1(\alpha_1)P_2(\alpha_2)d^2\alpha_1d^2\alpha_2, \tag{7.18}$$
$$=\int P_1(\alpha-\alpha')P_2(\alpha')d^2\alpha'. \tag{7.19}$$

We see immediately from Eq. (7.18) that P is correctly normalized if P_1 and P_2 are. The simple convolution law for combining the weight functions is one of the unique features of the description of fields by means of the P representation. It is quite analogous to the law we would use in classical theory to describe the probability distribution of the sum of two uncertain Fourier amplitudes for a mode.

The convolution theorem can often be used to separate fields into component fields with simpler properties. Suppose we have a field described by a weight function $P(\alpha)$ which has a mean value of α given by

$$\bar{\alpha}=\int\alpha P(\alpha)d^2\alpha. \tag{7.20}$$

It is clear from Eq. (7.19) that any such field may be regarded as the sum of a pure coherent field which corresponds to the weight function $\delta^{(2)}(\alpha-\bar{\alpha})$ and an additional field represented by $P(\alpha+\bar{\alpha})$ for which the mean value of α vanishes. Fields with vanishing mean values of α will be referred to as unphased fields.

The use of the P representation of the density operator, where it is not too singular, leads to simplifications in the calculation of statistical averages which go somewhat beyond those discussed in the last section. Thus, for example, the statistical average of any normally ordered product of the creation and annihilation operators, such as $(a^\dagger)^na^m$, reduces to a simple average of $(\alpha^*)^n\alpha^m$ taken with respect to the weight

function $P(\alpha)$, i.e., we have

$$\mathrm{tr}\{\rho(a^\dagger)^n a^m\} = \int P(\alpha)\langle\alpha|(a^\dagger)^n a^m|\alpha\rangle d^2\alpha,$$

$$= \int P(\alpha)(\alpha^*)^n \alpha^m d^2\alpha. \qquad (7.21)$$

This identity means, in practice, that many quantum-mechanical calculations can be carried out by means which are analogous to those already familiar from classical theory.

The mean number of photons which are present in a mode is the most elementary measure of the intensity of its excitation. The operator which represents the number of photons present is seen from Eq. (2.18) to be $a^\dagger a$. The average photon number, written as $\langle n\rangle$, is therefore given by

$$\langle n\rangle = \mathrm{tr}\{\rho a^\dagger a\}. \qquad (7.22)$$

According to Eq. (7.21), with its two exponents set equal to unity, we have

$$\langle n\rangle = \int P(\alpha)|\alpha|^2 d^2\alpha, \qquad (7.23)$$

i.e., the average photon number is just the mean squared absolute value of the amplitude α. When two fields described by distributions P_1 and P_2 are superposed, the resulting intensities are found from rules of the form which have always been used in classical electromagnetic theory. For unphased fields the intensities add "incoherently"; for coherent states the amplitudes add "coherently."

The use of the P representation of the density operator in describing fields brings many of the results of quantum electrodynamics into forms similar to those of classical theory. While these similarities make applications of the correspondence principle particularly clear, they must not be interpreted as indicating that classical theory is any sort of adequate substitute for the quantum theory. The weight functions $P(\alpha)$ which occur in quantum theoretical applications are not accurately interpretable as probability distributions, nor are they derivable as a rule from classical treatments of the radiation sources. They depend upon Planck's constant, in general, in ways that are unfathomable by classical or semiclassical analysis.

Since a number of calculations having to do with photon statistics have been carried out in the past by essentially classical methods, it may be helpful to discuss the relation between the P representation and the classical theory a bit further. It is worth noting in particular that the definition we have given the amplitude α as an eigenvalue of the annihilation operator is an intrinsically quantum-mechanical one. If we wish to represent a given classical field amplitude for the mode as an eigenvalue, then we see from Eq. (2.20) that the appropriate value of α has a magnitude which is proportional to $\hbar^{-1/2}$. In the dimensionless terms in which α is defined, the classical description of the mode only applies to the region $|\alpha|\gg 1$ of the complex α plane, i.e., to amplitudes of oscillation which are large compared with the range of the zero-point fluctuations present in the wave packet (3.29) and (3.30). Classical theory can therefore, in principle, only furnish us with the grossest sort of information about the weight function $P(\alpha)$. When the weight function extends appreciably into the classical regions of the plane, classical theory can only be relied upon, crudely speaking, to tell us average values of the function $P(\alpha)$ over areas whose dimensions, $|\Delta\alpha|$, are of order unity or larger. From Eq. (7.10) we see that such average values will always be positive; in the classical limit they may always be interpreted as probabilities.

VIII. THE GAUSSIAN DENSITY OPERATOR

The Gaussian function is a venerable statistical distribution, familiar from countless occurrences in classical statistics. We shall indicate in this section that it has its place in quantum field theory as well, where it furnishes the natural description of the most commonly occurring type of incoherence.[1]

Let us assume that the field mode we are studying is coupled to a number of sources which are essentially similar but are statistically independent of one another in their behavior. Such sources might, in practice, simply be several hypothetical subdivisions of one large source. If we may represent the contribution of each source (numbered $j=1,\cdots N$) to the excitation of the mode by means of a weight function $p(\alpha_j)$, we may then construct the weight function $P(\alpha)$ which describes the superposed fields by means of the generalized form of the convolution theorem

$$P(\alpha) = \int \delta^{(2)}\Big(\alpha - \sum_{j=1}^{N}\alpha_j\Big)\prod_{j=1}^{N} p(\alpha_j)d^2\alpha_j. \qquad (8.1)$$

Since the weight functions which appear in this expression are all real valued, it is sometimes convenient to think of the amplitudes α in their arguments not as complex numbers, but as two-dimensional real vectors $\boldsymbol{\alpha}$ (i.e., $\boldsymbol{\alpha}_x = \mathrm{Re}\,\alpha$, $\boldsymbol{\alpha}_y = \mathrm{Im}\,\alpha$). Then if λ is an arbitrary complex number represented by the vector $\boldsymbol{\lambda}$, we may use a two-dimensional scalar product for the abbreviation

$$\mathrm{Re}\,\lambda\,\mathrm{Re}\,\alpha + \mathrm{Im}\,\lambda\,\mathrm{Im}\,\alpha = \boldsymbol{\alpha}\cdot\boldsymbol{\lambda}. \qquad (8.2)$$

Using this notation, we may define the two-dimensional Fourier transform of the weight function $p(\alpha)$ as

$$\xi(\boldsymbol{\lambda}) = \int \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})p(\alpha)d^2\alpha. \qquad (8.3)$$

The superposition law (8.1) then shows that the Fourier transform of the weight function $P(\alpha)$ is given by

$$\Xi(\boldsymbol{\lambda}) = \int \exp(i\boldsymbol{\lambda}\cdot\boldsymbol{\alpha})P(\alpha)d^2\alpha,$$

$$= [\xi(\boldsymbol{\lambda})]^N. \qquad (8.4)$$

If the individual sources are stationary ones their weight function $p(\alpha)$ depends only on $|\alpha|$. The transform $\xi(\boldsymbol{\lambda})$ may then be approximated for small values of $|\boldsymbol{\lambda}|$ by

$$\xi(\boldsymbol{\lambda}) = 1 - \tfrac{1}{4}\boldsymbol{\lambda}^2\int |\alpha|^2 p(\alpha)d^2\alpha,$$

$$= 1 - \tfrac{1}{4}\boldsymbol{\lambda}^2\langle|\alpha|^2\rangle. \qquad (8.5)$$

For values of $|\boldsymbol{\lambda}|$ which are smaller still (i.e., $|\boldsymbol{\lambda}|^2 < N^{-1/2}\langle|\alpha|^2\rangle^{-1}$), the transform Ξ for the superposed field may be approximated by

$$\Xi(\boldsymbol{\lambda}) \approx \exp\{-\tfrac{1}{4}\boldsymbol{\lambda}^2 N\langle|\alpha|^2\rangle\}. \qquad (8.6)$$

Since the weight function $p(\alpha)$ may take on negative values it is necessary at this point to verify that the second moment $\langle|\alpha|^2\rangle$ is positive. That it is indeed positive is indicated by Eqs. (7.22) and (7.23) which show that $\langle|\alpha|^2\rangle$ is the mean number of photons which would be radiated by each source in the absence of the others. For large values of N the transform $\Xi(\boldsymbol{\lambda})$ therefore decreases rapidly as $|\boldsymbol{\lambda}|$ increases. Since the function becomes vanishingly small for $|\boldsymbol{\lambda}|$ lying outside the range of approximation noted earlier, we may use (8.6) more generally as an asymptotic approximation to $\Xi(\boldsymbol{\lambda})$ for large N. When we calculate the transform of this asymptotic expression for $\Xi(\boldsymbol{\lambda})$ we find

$$P(\alpha) = (2\pi)^{-2}\int \exp(-i\boldsymbol{\alpha}\cdot\boldsymbol{\lambda})\Xi(\boldsymbol{\lambda})d^2\boldsymbol{\lambda},$$

$$= \frac{1}{\pi N\langle|\alpha|^2\rangle}\exp(-\boldsymbol{\alpha}^2/N\langle|\alpha|^2\rangle). \qquad (8.7)$$

The mean value of $|\alpha|^2$ for such a weight function is evidently $N\langle|\alpha|^2\rangle$, but by the general theorem expressed in Eq. (7.23), this mean value is just the average of the total number of quanta present in the mode. If we write the latter average as $\langle n\rangle$, and resume the use of the complex notation for the variable α, the weight function (8.7) may be written as

$$P(\alpha) = \frac{1}{\pi\langle n\rangle}e^{-|\alpha|^2/\langle n\rangle}. \qquad (8.8)$$

The weight function $P(\alpha)$ is positive everywhere and takes the same form as the probability distribution for the total displacement which results from a random walk in the complex plane. However, because the coherent states $|\alpha\rangle$ are not an orthogonal set, $P(\alpha)$ can only be accurately interpreted as a probability distribution for $\langle n\rangle \gg 1$. We may note that it is not ultimately necessary, in order to derive Eq. (8.8), to assume that the weight functions corresponding to the individual sources are all the same. All that is required to carry out the proof is that the moments of the individual functions be of comparable magnitudes. The mean squared value of $|\alpha|$ is then given more generally by $\sum_j \langle|\alpha_j|^2\rangle$, rather than the value in Eq. (8.7), but this value is still the mean number of quanta in the mode, as indicated in Eq. (8.8).

It should be clear from the conditions of the derivation that the Gaussian distribution $P(\alpha)$ for the excitation of a mode possesses extremely wide applicability. The random or chaotic sort of excitation it describes is presumably characteristic of most of the familiar types of noncoherent macroscopic light sources, such as gas discharges, incandesant radiators, etc.

The Gaussian density operator

$$\rho = \frac{1}{\pi\langle n\rangle}\int e^{-|\alpha|^2/\langle n\rangle}|\alpha\rangle\langle\alpha|d^2\alpha \qquad (8.9)$$

may be seen to take on a very simple form as well in the basis which specifies the photon numbers. To find this form we substitute in Eq. (8.9) the expansions (3.7) and (3.8) for the coherent states and note the identity

$$\pi^{-1}(l!m!)^{-1/2}\int \exp[-C|\alpha|^2]\alpha^l(\alpha^*)^m d^2\alpha = \delta_{lm}C^{-(m+1)},$$

which holds for $C>0$. If we write $C = (1+\langle n\rangle)/\langle n\rangle$ we then find

$$\rho = \frac{1}{1+\langle n\rangle}\sum_m \left\{\frac{\langle n\rangle}{1+\langle n\rangle}\right\}^m |m\rangle\langle m|. \qquad (8.10)$$

In other words, the number of quanta in the mode is distributed according to the powers of the parameter $\langle n\rangle/(1+\langle n\rangle)$. The Planck distribution for blackbody radiation furnishes an illustration of a density operator which has long been known to take the form of Eq. (8.10). The thermal excitation which leads to the blackbody distribution is an ideal example of the random type we have described earlier, and so it should not be surprising that this distribution is one of the class we have derived. It is worth noting, in particular, that while the Planck distribution is characteristic of thermal equilibrium, no such limitation is implicit in the general form of the density operator (8.9). It will apply whenever the excitation has an appropriately random quality, no matter how far the radiator is from thermal equilibrium.

The Gaussian distribution function $\exp[-|\alpha|^2/\langle n\rangle]$ is phrased in terms which are explicitly quantum mechanical. In the limit which would represent a classical field both $|\alpha|^2$ and the average quantum number $\langle n\rangle$ become infinite as $\hbar^{-1}$, but their quotient, which is the argument of the Gaussian function, remains

well defined. The form which the distribution takes in the classical limit is a familiar one. Historically, one of the origins of the random walk problem is to be found in the discussion of a classical harmonic oscillator which is subject to random excitations.[16] Such oscillators have complex amplitudes which are described under quite general conditions by a Gaussian distribution. If we were armed with this knowledge, and lacked the quantum-mechanical analysis given earlier, we might be tempted to assume that a Gaussian distribution derived in this way from classical theory can describe the photon distribution. To demonstrate the fallacy of this view we must examine more closely the nature of the parameter $\langle n\rangle$ which is, after all, the only physical constant involved in the distribution. We may take, as a simple illustration, the case of thermal excitation corresponding to temperature T. Then the mean photon number is given by $\langle n\rangle=[\exp(\hbar\omega/\kappa T)-1]^{-1}$, where κ is Boltzmann's constant, and the distribution $P(\alpha)$ takes the form

$$P(\alpha)=\frac{1}{\pi}[e^{\hbar\omega/\kappa T}-1]\exp[-(e^{\hbar\omega/\kappa T}-1)|\alpha|^2]. \quad (8.11)$$

To reach the classical analog of this distribution we would assume that the classical field energy in the mode, $H=\frac{1}{2}\int(\mathbf{E}^2+\mathbf{B}^2)d\mathbf{r}$, is distributed with a probability proportional to $\exp[-H/\kappa T]$. The distribution for the amplitude α that results is

$$P_{cl}(\alpha)=(\hbar\omega/\pi\kappa T)\exp[-\hbar\omega|\alpha|^2/\kappa T], \quad (8.12)$$

which is seen to be a first approximation in powers of $\hbar$ to the correct distribution. (Again, we must remember that the quantity $\hbar|\alpha|^2$ is to be construed as a classical parameter.) The distribution $P_{cl}(\alpha)$ only extends into the classical region of the plane, $|\alpha|\gg 1$, for low-frequency modes, that is, only for $(\hbar\omega/\kappa T)\ll 1$ are the modes sufficiently excited to be accurately described by classical theory. For higher frequencies the two distributions differ greatly in nature even though both are Gaussian. The classical distribution retains much too large a radius in the α plane as $\hbar\omega$ increases beyond κT, rather than narrowing extremely rapidly as the correct distribution does.[17] That error, in fact, epitomizes the ultraviolet catastrophe of the classical radiation theory. The example we have discussed is, of course, an elementary one, but it should serve to illustrate some of the points noted in the preceding section regarding the limitations of the classical distribution function.

The expression for the thermal density operator of an oscillator in terms of coherent quantum states appears to offer new and instructive approaches to many familiar problems. It permits us, for example, to derive the thermal averages of exponential functions of the operators a and $a^\dagger$ in an elementary way. The thermal average of the operator $D(\beta)$ defined by Eq. (3.17) is an illustration. It is given by

$$\mathrm{tr}\{\rho D(\beta)\}=\frac{1}{\pi\langle n\rangle}\int e^{-|\alpha|^2/\langle n\rangle}\langle\alpha|D(\beta)|\alpha\rangle d^2\alpha. \quad (8.13)$$

The expectation value in the integrand is, in this case

$$\begin{aligned}\langle\alpha|D(\beta)|\alpha\rangle&=\langle 0|D^{-1}(\alpha)D(\beta)D(\alpha)|0\rangle,\\&=\exp[\beta\alpha^*-\beta^*\alpha]\langle 0|D(\beta)|0\rangle,\\&=\exp[\beta\alpha^*-\beta^*\alpha]\langle 0|\beta\rangle,\\&=\exp[\beta\alpha^*-\beta^*\alpha-\tfrac{1}{2}|\beta|^2], \quad (8.14)\end{aligned}$$

where the properties of $D(\alpha)$ as a displacement operator have been used in the intermediate steps. When the integration indicated in Eq. (8.13) is carried out, we find

$$\mathrm{tr}\{\rho D(\beta)\}=\exp[-|\beta|^2(\langle n\rangle+\tfrac{1}{2})], \quad (8.15)$$

which is a frequently used corollary of Bloch's theorem on the distribution function of an oscillator coordinate.[18]

IX. DENSITY OPERATORS FOR THE FIELD

The developments introduced in Secs. III–VIII have all concerned the description of the quantum state of a single mode of the electromagnetic field. We may describe the field as a whole by constructing analogous methods to deal with all its modes at once. For this purpose we introduce a basic set of coherent states for the entire field and write them as

$$|\{\alpha_k\}\rangle\equiv\prod_k|\alpha_k\rangle_k, \quad (9.1)$$

where the notation $\{\alpha_k\}$, which will be used in several other connections, stands for the set of all amplitudes α_k. It is clear then, from the arguments of Sec. IV, that any state of the field determines uniquely a function $f(\{\alpha_k^*\})$ which is an entire function of each of the variables α_k^*. If the Hilbert space vector which represents the state is known and designated as $|f\rangle$, the function f is given by

$$f(\{\alpha_k^*\})=\langle\{\alpha_k\}|f\rangle\exp(\tfrac{1}{2}\textstyle\sum_k|\alpha_k|^2), \quad (9.2)$$

which is the direct generalization of Eq. (4.11). The expansion for the state $|f\rangle$ in terms of coherent states is then

$$|f\rangle=\int|\{\alpha_k\}\rangle f(\{\alpha_k^*\})\prod_k\pi^{-1}e^{-|\alpha_k|^2}d^2\alpha_k, \quad (9.3)$$

which generalizes Eq. (4.7).

All of the operators which occur in field theory possess expansions in terms of the vectors $|\{\alpha_k\}\rangle$ and their adjoints. To construct such representations is simply a matter of generalizing the formulas of Sec. V to deal with an infinite set of amplitude variables. We therefore proceed directly to a discussion of the density operator. For any density operator ρ we may define a function $R(\{\alpha_k^*\},\{\beta_k\})$ which is an entire function of each of the variables α_k^* and β_k for all modes k. This function, as may be seen from Eq. (6.1), is given by

$$\begin{aligned}R(\{\alpha_k^*\},\{\beta_k\})&=\langle\{\alpha_k\}|\rho|\{\beta_k\}\rangle\\&\times\exp[\tfrac{1}{2}\textstyle\sum(|\alpha_k|^2+|\beta_k|^2)]. \quad (9.4)\end{aligned}$$

The corresponding representation of the density operator is

$$\begin{aligned}\rho=\int|\{\alpha_k\}\rangle R(\{\alpha_k^*\},\{\beta_k\})\langle\{\beta_k\}|\prod_k\pi^{-2}\\\times e^{-\frac{1}{2}(|\alpha_k|^2+|\beta_k|^2)}d^2\alpha_k d^2\beta_k. \quad (9.5)\end{aligned}$$

If the set of integers $\{n_k\}$ is used to specify the familiar stationary states which have n_k photons in the kth mode, we may regard R as a generating function for the matrix elements of ρ connecting these states, i.e., as a generalization of Eq. (6.2) we have

$$\begin{aligned}R(\{\alpha_k^*\},\{\beta_k\})=\sum_{\{n_k\},\{m_k\}}\langle\{n_k\}|\rho|\{m_k\}\rangle\\\times\prod_k(n_k!m_k!)^{-1/2}(\alpha_k^*)^{n_k}\beta_k^{m_k}. \quad (9.6)\end{aligned}$$

The matrix elements of ρ in the stationary basis are then given by

$$\begin{aligned}&\langle\{n_k\}|\rho|\{m_k\}\rangle\\&=\int R(\{\alpha_k^*\},\{\beta_k\})\prod_k\pi^{-2}(n_k!m_k!)^{-1/2}\alpha_k^{n_k}(\beta_k^*)^{m_k}\\&\times e^{-(|\alpha_k|^2+|\beta_k|^2)}d^2\alpha_k d^2\beta_k. \quad (9.7)\end{aligned}$$

The normalization condition on R is clearly

$$\int R(\{\beta_k^*\},\{\beta_k\})\prod_k\pi^{-1}e^{-|\beta_k|^2}d^2\beta_k=1. \quad (9.8)$$

The positive definiteness condition, Eq. (6.9), may also be generalized in an evident way to deal with the full set of amplitude variables.

It may help as a simple illustration of the foregoing formulae to consider the representation of a single-photon wave packet. The state which is empty of all photons is the one for which the amplitudes α_k all vanish. If we write that state as $|\mathrm{vac}\rangle$, then we may write the most general one-photon state as $\sum_k q(k)a_k^\dagger|\mathrm{vac}\rangle$, where the function $q(k)$ plays the role of a packet amplitude. The function f which represents this state is then

$$f(\{\alpha_k^*\})=\sum_k q(k)\alpha_k^*, \quad (9.9)$$

and the corresponding function R which determines the density operator is

$$R(\{\alpha_k^*\},\{\beta_k\})=\sum_k q(k)\alpha_k^*\sum_{k'}q^*(k')\beta_{k'}. \quad (9.10)$$

The normalization condition (9.8) corresponds to the requirement $\sum|q(k)|^2=1$. Since the state we have considered is a pure one, the function R factorizes into the product of two functions, one having the form of f and the other of its complex conjugate. If the packet amplitudes $q(k)$ were in some degree unpredictable, as they usually are, the packet could no longer be represented by a pure state. The function R would then be an average taken over the distribution of the amplitudes $q(k)$ and hence would lose its factorizable form in general. Whenever an upper bound exists for the number of photons present, i.e., the number of photons is required to be less than or equal to some integer N, we will find that R is a polynomial of at most Nth degree in the variables $\{\alpha_k^*\}$ and of the same degree in the $\{\beta_k\}$.

There will, of course, exist many types of excitation for which the photon numbers are unbounded. Among these are the ones which are more conveniently described by means of a generalized P distribution, i.e., the excitations for which there exists a reasonably well-behaved real-valued function $P(\{\alpha_k\})$ such that

$$\begin{aligned}R(\{\beta_k^*\},\{\gamma_k\})=\int P(\{\alpha_k\})\\\times\exp\Big[\sum_k(\beta_k\alpha_k+\gamma_k\alpha_k^*-|\alpha_k|^2)\Big]\prod_k d^2\alpha_k. \quad (9.11)\end{aligned}$$

When R possesses a representation of this type the density operator (9.5) may be reduced by means of Eq. (4.14) and its complex conjugate to the simple form

$$\rho=\int P(\{\alpha_k\})|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k, \quad (9.12)$$

which is the many-mode form of the P representation given by Eq. (7.6). The function P must satisfy the positive definiteness condition

$$\int|f(\{\alpha_k^*\})|^2P(\{\alpha_k\})\prod_k e^{-|\alpha_k|^2}d^2\alpha_k\geq 0 \quad (9.13)$$

for all possible choices of entire functions $f(\{\alpha_k^*\})$. The matrix elements of the density operator in the representation based on the n-photon states are

$$\begin{aligned}\langle\{n_k\}|\rho|\{m_k\}\rangle=\int P(\{\alpha_k\})\\\times\prod_k(n_k!m_k!)^{-1/2}\alpha_k^{n_k}(\alpha_k^*)^{m_k}e^{-|\alpha_k|^2}d^2\alpha_k. \quad (9.14)\end{aligned}$$

[16] Lord Rayleigh, *The Theory of Sound*, (MacMillan and Company Ltd., London, 1894), 2nd ed., Vol. I, p. 35; *Scientific Papers* (Cambridge University Press, Cambridge, England, 1899–1920), Vol. I, p. 491, Vol. IV, p. 370.

[17] For frequencies in the middle of the visible spectrum and temperatures under 3000°K the quantum mechanical distribution (8.11) will have ara dius which corresponds to $|\alpha|^2\ll 10^{-4}$, i.e., the distribution is far from classical in nature. Comparable radii characterize the distributions for nonthermal incoherent sources.

[18] F. Bloch, Z. Physik 74, 295 (1932).

Stationary density operators, i.e., ones which commute with the Hamiltonian correspond to functions $P(\{\alpha_k\})$ which depend on the amplitude variables only through their magnitudes $\{|\alpha_k|\}$.

The superposition of two fields is described by forming the convolution integral of their distribution functions, much as in the case of a single mode. Thus, if two fields, described by $P_1(\{\beta_k\})$ and $P_2(\{\gamma_k\})$, respectively, are superposed, the resulting field has a distribution function

$$P(\{\alpha_k\})=\int\prod_k \delta^{(2)}(\alpha_k-\beta_k-\gamma_k) \times P_1(\{\beta_k\})P_2(\{\gamma_k\})\prod_k d^2\beta_k d^2\gamma_k. \quad (9.15)$$

For fields which are represented by means of the density operator (9.12) all of the averages of normally ordered operator products can be calculated by means of formulas which, as in the case of a single mode, greatly resemble those of classical theory. Thus, the parameters $\{\alpha_k\}$ play much the same role in these calculations as the random Fourier amplitudes of the field do in the familiar classical theory of microwave noise.[19] Furthermore, the weight function $P(\{\alpha_k\})$ plays a role similar to that of the probability distribution for the Fourier amplitudes. Although this resemblance is extremely convenient in calculations, and offers immediate insight into the application of the correspondence principle, we must not lose sight of the fact that the function $P(\{\alpha_k\})$ is, in general, an explicitly quantum-mechanical structure. It may assume negative values, and is not accurately interpretable as a probability distribution except in the classical limit of strongly excited or low frequency fields.

In the foregoing discussions we have freely assumed that the density operator which describes the field is known and that it may, therefore, be expressed either in the representation of Eq. (9.5) or in the P representation of Eq. (9.12). For certain types of incoherent sources which we have discussed in Sec. VIII and will mention again in Sec. X, the explicit construction of these density operators is not at all difficult. But to find accurate density operators for other types of sources, including the recently developed coherent ones, will require a good deal of physical insight. The general problem of treating quantum mechanically the interaction of a many-atom source both with the radiation field and with an excitation mechanism of some sort promises to be a complicated one. It will have to be approached, no doubt, through greatly simplified models.

Since very little is known about the density operator for radiation fields, some insight may be gained by examining the form it takes on in one of the few completely soluble problems of quantum electrodynamics. We shall study the photon field radiated by an electric current distribution which is essentially classical in nature, one that does not suffer any noticeable reaction from the process of radiation. We may then represent the radiating current by a prescribed vector function of space and time $\mathbf{j}(\mathbf{r},t)$. The Hamiltonian which describes the coupling of the quantized electromagnetic field to the current distribution takes the form

$$H_1(t)=-\frac{1}{c}\int \mathbf{j}(\mathbf{r},t)\cdot\mathbf{A}(\mathbf{r},t)d\mathbf{r}. \quad (9.16)$$

The introduction of an explicitly time-dependent interaction of this type means that the state vector for the field, $|\ \rangle$, which previously was fixed (corresponding to the Heisenberg picture) will begin to change with time in accordance with the Schrödinger equation

$$i\hbar\frac{\partial}{\partial t}|\ \rangle=H_1(t)|\ \rangle, \quad (9.17)$$

which is the one appropriate to the interaction representation. The solution of this equation is easily found.[20] If we assume that the initial state of the field at time $t=-\infty$ is one empty of all photons, then the state of the field at time t may be written in the form

$$|t\rangle=\exp\left\{\frac{i}{\hbar c}\int_{-\infty}^{t}dt'\int\mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')d\mathbf{r}+i\varphi(t)\right\}|\text{vac}\rangle. \quad (9.18)$$

The function $\varphi(t)$ which occurs in the exponent is a real-valued c-number phase function. It is easily evaluated, but cancels out of the product $|t\rangle\langle t|$ and so has no bearing on the construction of the density operator. The exponential operator which occurs in Eq. (9.18) may be expressed quite simply in terms of the displacement operators we discussed in Sec. III. For this purpose we define a displacement operator D_k for the kth mode as

$$D_k(\beta_k)=\exp[\beta_k a_k^\dagger-\beta_k^* a_k]. \quad (9.19)$$

Then it is clear from the expansion (2.10) for the vector potential that we may write

$$\exp\left\{\frac{i}{\hbar c}\int_{-\infty}^{t}dt'\int\mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')d\mathbf{r}\right\}=\prod_k D_k[\alpha_k(t)], \quad (9.20)$$

where the time-dependent amplitudes $\alpha_k(t)$ are given by

$$\alpha_k(t)=\frac{i}{(2\hbar\omega)^{1/2}}\int_{-\infty}^{t}dt'\int d\mathbf{r}\,\mathbf{u}_k^*(\mathbf{r})\cdot\mathbf{j}(\mathbf{r},t')e^{i\omega t'}. \quad (9.21)$$

The density operator at time t may therefore be written as

$$|t\rangle\langle t|=\prod_k D_k[\alpha_k(t)]|\text{vac}\rangle\langle\text{vac}|\prod_k D_k^{-1}[\alpha_k(t)] \quad (9.22)$$

$$=|\{\alpha_k(t)\}\rangle\langle\{\alpha_k(t)\}|. \quad (9.23)$$

The radiation by any prescribed current distribution, in other words, always leads to a pure coherent state.

It is only a slight generalization of the model we have just considered to imagine that the current distribution $\mathbf{j}(\mathbf{r},t)$ is not wholly predictable. In that case the amplitudes $\alpha_k(t)$ defined by Eq. (9.21) become random variables which possess collectively a probability distribution function which we may write as $p(\{\alpha_k\},t)$. The density operator for the field radiated by such a random current then becomes

$$\rho(t)=\int p(\{\alpha_k\},t)|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k. \quad (9.24)$$

We see that the density operator for a field radiated by a random current which suffers no recoil in the radiation process always takes the form of the P representation of Eq. (9.12). The weight function in this case does admit interpretation as a probability distribution, but it has a classical structure associated directly with the properties of the radiating current rather than with particular (nonorthogonal) states of the field. The assumption we have made in defining the model, that the current suffers negligible reaction, is a strong one but is fairly well fulfilled in radiating systems operated at radio or microwave frequencies. The fields produced by such systems should be accurately described by density operators of the form (9.24).

X. CORRELATION AND COHERENCE PROPERTIES OF THE FIELD

Any eigenvalue function $\mathcal{E}(rt)$ which satisfies the appropriate field equations and contains only positive frequency terms determines a set of mode amplitudes $\{\alpha_k\}$ uniquely through the expansion (2.20). This set of mode amplitudes then determines a coherent state of the field, $|\{\alpha_k\}\rangle$, such that

$$\mathbf{E}^{(+)}(rt)|\{\alpha_k\}\rangle=\mathcal{E}(rt)|\{\alpha_k\}\rangle. \quad (10.1)$$

To discuss the general form which the field correlation functions take in such states it is convenient to abbreviate a set of coordinates $(\mathbf{r}_j,t_j)$ by a single symbol x_j. The nth-order correlation function is then defined as[3]

$$G_{\mu_1\cdots\mu_{2n}}^{(n)}(x_1\cdots x_{2n})=\text{tr}\{\rho E_{\mu_1}^{(-)}(x_1)\cdots \times E_{\mu_n}^{(-)}(x_n)E_{\mu_{n+1}}^{(+)}(x_{n+1})\cdots E_{\mu_{2n}}^{(+)}(x_{2n})\}. \quad (10.2)$$

The density operator for the coherent state defined by Eq. (10.1) is the projection operator

$$\rho=|\{\alpha_k\}\rangle\langle\{\alpha_k\}|. \quad (10.3)$$

For this operator it follows from Eq. (10.1) and its Hermitian adjoint that the correlation functions reduce to the factorized form

$$G_{\mu_1\cdots\mu_{2n}}^{(n)}(x_1\cdots x_{2n})=\prod_{j=1}^{n}\mathcal{E}_{\mu_j}^*(x_j)\prod_{k=n+1}^{2n}\mathcal{E}_{\mu_k}(x_k). \quad (10.4)$$

In other words, the field which corresponds to the state $|\{\alpha_k\}\rangle$ satisfies the conditions for full coherence according to the definition[3] given earlier.

It is worth noting that the state $|\{\alpha_k\}\rangle$ is not the only one which leads to the set of correlation functions (10.4). Indeed, let us consider a state which corresponds not to the amplitudes $\{\alpha_k\}$, but to a set $\{e^{i\varphi}\alpha_k\}$ which differs by a common phase factor (i.e., φ is real and independent of k). Then the corresponding eigenvalue function becomes $e^{i\varphi}\mathcal{E}(rt)$, but such a change leaves the correlation functions (10.4) unaltered. It is clear from this invariance property of the correlation functions that certain mixtures of the coherent states also lead to the same set of functions. Thus, if $|\{\alpha_k\}\rangle$ is the state defined by Eq. (10.1), and $\mathcal{L}(\varphi)$ is any real-valued function of φ normalized in the sense

$$\int_0^{2\pi}\mathcal{L}(\varphi)d\varphi=1, \quad (10.5)$$

we see that the density operator

$$\rho=\int_0^{2\pi}\mathcal{L}(\varphi)|\{e^{i\varphi}\alpha_k\}\rangle\langle\{e^{i\varphi}\alpha_k\}|d\varphi \quad (10.6)$$

leads for all choices of $\mathcal{L}(\varphi)$ to the set of correlation functions (10.4). Such a density operator is, of course, a special case of the general form (9.12), one which corresponds to an over-all uncertainty in the phase of the $\{\alpha_k\}$. The particular choice $\mathcal{L}(\varphi)=(2\pi)^{-1}$, which corresponds to complete ignorance of the phase, represents the usual state of our knowledge about high-frequency fields. We have been careful, therefore, to define coherence in terms of a set of correlation functions which are independent of the over-all phase.

Since nonstationary fields of many sorts can be represented by means of eigenvalue functions, it becomes a simple matter to construct corresponding quantum states. As an illustration we may consider the example of an amplitude-modulated plane wave. For this purpose we make use of the particular set of mode functions defined by Eq. (2.9). Then if the carrier wave has frequency ω and the modulation is periodic and has frequency $\zeta\omega$ where $0<\zeta<1$, we may write an appropriate eigenvalue function as

$$\mathcal{E}(rt)=i\left(\frac{\hbar\omega}{2L^3}\right)^{1/2}\mathcal{E}^{(k)}\alpha_k \times\{1+M\cos[\zeta(\mathbf{k}\cdot\mathbf{r}-\omega t)-\delta]\}e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}. \quad (10.7)$$

[19] J. Lawson and G. E. Uhlenbeck, *Threshold Noise Signals* (McGraw Hill Book Company, Inc., New York, 1950), pp. 33–56.

[20] R. J. Glauber, Phys. Rev. 84, 395 (1951).

When this expression is expanded in plane-wave modes it has only three nonvanishing amplitude coefficients. These are α_k itself and the two sideband amplitudes

$$\alpha_{k(1-\xi)}=\tfrac{1}{2}M(1-\xi)^{-1/2}e^{i\delta}\alpha_k,$$
$$\alpha_{k(1+\xi)}=\tfrac{1}{2}M(1+\xi)^{-1/2}e^{-i\delta}\alpha_k. \quad (10.8)$$

The coherent state which corresponds to the modulated wave may be constructed immediately from the knowledge of these amplitudes. In practice, of course, we will not often know the phase of α_k, and so the wave should be represented not by a single coherent state, but by a mixture of the form (10.6). Representations of other forms of modulated waves may be constructed similarly.

Incoherent fields, or the broad class of fields for which the correlation functions do not factorize, must be described by means of density operators which are more general in their structure than those of Eqs. (10.3) or (10.6). To illustrate the form taken by the correlation functions for such cases we may suppose the field to be described by the P representation of the density operator. Then the first-order correlation function is given by

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\int P(\{\alpha_k\})\sum_{k,k'}\tfrac{1}{2}\hbar(\omega\omega')^{1/2}u_{k\mu}^*(\mathbf{r})u_{k'\nu}(\mathbf{r}')\times\alpha_k^*\alpha_{k'}e^{i(\omega t-\omega' t')}\prod_l d^2\alpha_l. \quad (10.9)$$

Fields for which the P representation is inconveniently singular may, as we have noted earlier, always be described by means of analytic functions $R(\{\alpha_k^*\},\{\beta_k\})$ and corresponding density operators of the form (9.5). When that form of density operator is used to evaluate the first-order correlation function we find

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\int R(\{\alpha_k^*\},\{\beta_k\})\sum_{k',k''}\tfrac{1}{2}\hbar(\omega'\omega'')^{1/2}\times u_{k'\mu}^*(\mathbf{r})u_{k''\nu}(\mathbf{r}')\beta_{k'}^*\alpha_{k''}e^{i(\omega' t-\omega'' t')}\times\prod_l e^{\beta_l\alpha_l^*}d\mu(\alpha_l)d\mu(\beta_l), \quad (10.10)$$

where the differentials $d\mu(\alpha_l)$ and $d\mu(\beta_l)$ are those defined by Eq. (5.14). The higher order correlation functions are given by integrals analogous to (10.9) and (10.10). Their integrands contain polynomials of the $2n$th degree in the amplitude variables α_k and β_k^* in place of the quadratic forms which occur in the first-order functions.

The energy spectrum of a radiation field is easily derived from a knowledge of its first-order correlation function. If we return for a moment to the expansion (2.19) for the positive-frequency field operator, and write the negative-frequency field as its Hermitian adjoint, we see that these operators obey the identity

$$2\int \mathbf{E}^{(-)}(\mathbf{r}t)\cdot\mathbf{E}^{(+)}(\mathbf{r}t')d\mathbf{r}=\sum_k\hbar\omega a_k^\dagger a_k\exp[i\omega(t-t')]. \quad (10.11)$$

If we take the statistical average of both sides of this equation we may write the resulting relation as

$$\sum_\mu\int G_{\mu\mu}^{(1)}(\mathbf{r}t,\mathbf{r}t')d\mathbf{r}=\tfrac{1}{2}\sum_k\hbar\omega\langle n_k\rangle\exp[i\omega(t-t')], \quad (10.12)$$

where $\langle n_k\rangle$ is the average number of photons in the kth mode. The Fourier representation of the volume integral of $\sum_\mu G_{\mu\mu}^{(1)}$ therefore identifies the energy spectrum $\hbar\omega\langle n_k\rangle$ quite generally.

For fields which may be represented by stationary density operators, it becomes still simpler to extract the energy spectrum from the correlation function. For such fields the weight function $P(\{\alpha_k\})$ depends only on the absolute values of the α_k, so that we have

$$\int P(\{\alpha_k\})\alpha_{k'}^*\alpha_{k''}\prod_l d^2\alpha_l=\langle|\alpha_{k'}|^2\rangle\delta_{k'k''}=\langle n_{k'}\rangle\delta_{k'k''}. \quad (10.13)$$

By using Eq. (10.9) to evaluate the correlation function, and specializing to the case of plane-wave modes, we then find

$$\sum_\mu G_{\mu\mu}^{(1)}(\mathbf{r}t,\mathbf{r}t')=\tfrac{1}{2}L^{-3}\sum_{\mathbf{k},\lambda}\hbar\omega\langle n_{\mathbf{k},\lambda}\rangle e^{i\omega(t-t')}, \quad (10.14)$$

in which we have explicitly indicated the role of the polarization index λ. If the volume which contains the field is sufficiently large in comparison to the wavelengths of the excited modes, the sum over the modes in Eq. (10.14) may be expressed as an integral over $\mathbf{k}$ space $[\sum_k\to\int L^3(2\pi)^{-3}d\mathbf{k}]$. By defining an energy spectrum for the quanta present (i.e., an energy per unit interval of ω) as

$$w(\omega)=(2\pi)^{-3}\hbar k^3\sum_\lambda\int\langle n_{\mathbf{k},\lambda}\rangle d\Omega_{\mathbf{k}}, \quad (10.15)$$

where $d\Omega_k$ is an element of solid angle in $\mathbf{k}$ space, we may then rewrite Eq. (10.14) in the form

$$\sum_\mu G_{\mu\mu}^{(1)}(\mathbf{r}t,\mathbf{r}t')=\tfrac{1}{2}\int_0^\infty w(\omega)e^{i\omega(t-t')}d\omega. \quad (10.16)$$

With the understanding that $w(\omega)=0$ for $\omega<0$, we may extend the integral over ω from $-\infty$ to ∞. It is then clear that the relation (10.16) may be inverted to express the energy spectrum as the Fourier transform of the time-dependent correlation function,

$$w(\omega)=\frac{1}{\pi}\int_{-\infty}^\infty\sum_\mu G_{\mu\mu}^{(1)}(\mathbf{r}0,\mathbf{r}t)e^{i\omega t}dt. \quad (10.17)$$

A pair of relations analogous to Eqs. (10.16) and (10.17), and together called the Wiener-Khintchine theorem, has long been of use in the classical theory of random fields.[21] The relations we have derived are, in a sense, the natural quantum mechanical generalization of the Wiener-Khintchine theorem. All we have assumed is that the field is describable by a stationary form of the P representation of the density operator. The proof need not, in fact, rest upon the use of the P representation since we can construct a corresponding statement in terms of the more general representation (9.5).

Stationary fields, according to Eq. (6.10), are represented by entire functions $R=\mathfrak{F}(\{\alpha_k^*\beta_k\})$, i.e., functions which depend only on the set of products $\alpha_k^*\beta_k$. For such fields, then, the integral over the α and β planes which is required in Eq. (10.10) takes the form

$$\langle\beta_{k'}^*\alpha_{k''}\rangle=\int\mathfrak{F}(\{\alpha_k^*\beta_k\})\beta_{k'}^*\alpha_{k''}\prod_l e^{\beta_l\alpha_l^*}d\mu(\alpha_l)d\mu(\beta_l). \quad (10.18)$$

Since the range of integration of each of the α and β variables covers the entire complex plane, this integral cannot be altered if we change the signs of any of the variables. If, however, we replace the particular variables $\alpha_{k''}$ and $\beta_{k''}$ by $-\alpha_{k''}$ and $-\beta_{k''}$ the integral is seen to reverse in sign, unless we have

$$\langle\beta_{k'}^*\alpha_{k''}\rangle=\delta_{k'k''}\langle\beta_{k'}^*\alpha_{k'}\rangle. \quad (10.19)$$

The average $\langle\beta_k^*\alpha_k\rangle$, we may note from Eqs. (5.11) and (6.5), is just the mean number of quanta in the kth mode,

$$\langle\beta_k^*\alpha_k\rangle=\mathrm{tr}\{\rho a_k^\dagger a_k\}=\langle n_k\rangle. \quad (10.20)$$

We have thus shown that the general expression (10.10) for the first-order correlation function always satisfies Eq. (10.14) when the field is described by a stationary density operator. The derivation of the equations relating the energy spectrum to the time-dependent correlation function then proceeds as before.

The simplest and most universal example of an incoherent field is the type generated by superposing the outputs of stationary sources. We have shown in some detail in Sec. VIII that as the number of sources which contribute to the excitation of a single mode increases, the density operator for the mode takes on a Gaussian form in the P representation. It is not difficult to derive an analogous result for the case of sources which excite many modes at once. We shall suppose that the sources ($j=1\cdots N$) are essentially identical, and that their contributions to the excitation are described by a weight function $p(\{\alpha_{jk}\})$. The weight function $P(\{\alpha_k\})$ for the superposed fields is then given by the convolution theorem as

$$P(\{\alpha_k\})=\int\prod_k\delta^{(2)}\Big(\alpha_k-\sum_{j=1}^N\alpha_{jk}\Big)\prod_{j=1}^N p(\{\alpha_{jk}\})\prod_k d^2\alpha_{jk}. \quad (10.21)$$

Since the individual sources are assumed to be stationary, the function $p(\{\alpha_{jk}\})$ will only depend on the variables α_{jk} through their absolute magnitudes, $|\alpha_{jk}|$.

The derivation which leads from Eq. (10.21) to a Gaussian asymptotic form for $P(\{\alpha_k\})$ is so closely parallel to that of Eqs. (8.1)–(8.8) that there is no need to write it out in detail. The argument makes use of second-order moments of the function p which may, with the same type of vector notation used previously, be written as

$$\langle\alpha_k\alpha_{k'}\rangle=\int\alpha_k\alpha_{k'}p(\{\alpha_k\})\prod_l d^2\alpha_l. \quad (10.22)$$

The stationary character of the function p implies that such moments vanish for $k\neq k'$. With this observation, we may retrace our earlier steps to show that the many-dimensional Fourier transform of P takes the form of a product of Gaussians, one for each mode and each similar in form to that of Eq. (8.6). It then follows immediately that the weight function P for the field as a whole is given by a product of Gaussian factors each of the form of Eq. (8.8). We thus have

$$P(\{\alpha_k\})=\prod_k\frac{1}{\pi\langle n_k\rangle}e^{-|\alpha_k|^2/\langle n_k\rangle}, \quad (10.23)$$

where $\langle n_k\rangle$ is the average number of photons present in the kth mode when the fields are fully superposed. One of the striking features of this weight function is its factorized form. It is interesting to remember, therefore, that no assumption of factorizability has been made regarding the weight functions p which describe the individual sources. These sources may, indeed, be ones for which the various mode amplitudes are strongly coupled in magnitude. It is the stationary property of the sources which leads, because of the vanishing of the moments (10.22) for $k\neq k'$, to the factorized form for the weight function (10.23).

The density operator which corresponds to the Gaussian weight function (10.23) evidently describes an ideally random sort of excitation of the field modes. We may reasonably surmise that it applies, at least as a good approximation, to all of the familiar sorts of incoherent sources in laboratory use. It is clear, in particular, from the arguments of Sec. VII that the Gaussian weight function describes thermal sources

[21] The Wiener-Khintchine theorem is usually expressed in terms of cosine transforms since it deals with a real-valued correlation function for the classical field $\mathbf{E}$, rather than a complex one for the fields $\mathbf{E}^{(\pm)}$. The complex correlation functions are considerably more convenient to use for quantum mechanical purposes, as is shown in Ref. 3.

correctly. The substitution of the Planck distribution $\langle n_k\rangle=[\exp(\hbar\omega_k/\kappa T)-1]^{-1}$ into Eq. (10.23) leads to the density operator for the entire thermal radiation field. To the extent that the Gaussian weight function (10.23) may describe radiation by a great variety of incoherent sources there will be certain deep-seated similarities in the photon fields generated by all of them. One may, for example, think of these sources all as resembling thermal ones and differing from them only in the spectral distributions of their outputs. As a way of illustrating these similarities we might imagine passing blackbody radiation through a filter which is designed to give the spectral distribution of the emerging light a particular line profile. We may choose this artificial line profile to be the same as that of some true emission line radiated, say, by a discharge tube. We then ask whether measurements carried out upon the photon field can distinguish the true emission-line source from the artificial one. If the radiation by the discharge tube is described, as we presume, by a Gaussian weight function, it is clear that the two sources will be indistinguishable from the standpoint of any photon counting experiments. They are equivalent sorts of narrow-band, quantum-mechanical noise generators.

It is a simple matter to find the correlation functions for the incoherent fields[9] described by the Gaussian weight function (10.23). If we substitute this weight function into the expansion (10.9) for the first-order correlation function we find

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\tfrac{1}{2}\sum_{k}\hbar\omega u_{k\mu}^{*}(\mathbf{r})u_{k\nu}(\mathbf{r}')\langle n_k\rangle e^{i\omega(t-t')}. \quad (10.24)$$

When the mode functions $\mathbf{u}_k(\mathbf{r})$ are the plane waves of Eq. (2.9), and the volume of the system is sufficiently large, we may write the correlation function as the integral

$$G_{\mu\nu}^{(1)}(\mathbf{r}t,\mathbf{r}'t')=\frac{\hbar c}{2(2\pi)^3}\int\sum_{\lambda}e_{\mu}^{(\lambda)*}e_{\nu}^{(\lambda)}\langle n_{k,\lambda}\rangle k \times\exp\{-i[\mathbf{k}\cdot(\mathbf{r}-\mathbf{r}')-\omega(t-t')]\}d\mathbf{k}, \quad (10.25)$$

in which the index λ again labels polarizations. To find the second-order correlation function defined by Eq. (10.2) we may write it likewise as an expansion in terms of mode functions. The only new moments of the weight function which we need to know are those given by $\langle|\alpha_k|^4\rangle=2\langle|\alpha_k|^2\rangle^2=2\langle n_k\rangle^2$. We then find that the second-order correlation function may be expressed in terms of the first-order function as

$$G_{\mu_1\mu_2\mu_3\mu_4}^{(2)}(x_1x_2x_3x_4)=G_{\mu_1\mu_3}^{(1)}(x_1,x_3)G_{\mu_2\mu_4}^{(1)}(x_2,x_4) +G_{\mu_1\mu_4}^{(1)}(x_1,x_4)G_{\mu_2\mu_3}^{(1)}(x_2,x_3). \quad (10.26)$$

It is easily shown that all of the higher order correlation functions as well reduce to sums of products of the first-order function. The nth-order correlation function may be written as

$$G_{\mu_1\cdots\mu_{2n}}^{(n)}(x_1\cdots x_n,x_{n+1}\cdots x_{2n})=\sum_{\sigma}\prod_{j=1}^{n}G_{\mu_j\nu_j}^{(1)}(x_j,y_j), \quad (10.27)$$

where the indices ν_j and the coordinates y_j for $j=1\cdots n$ are a permutation of the two sets $\mu_{n+1}\cdots\mu_{2n}$ and $x_{n+1}\cdots x_{2n}$, and the sum is carried out over all of the $n!$ permutations. One of the family resemblances which links all fields represented by the weight function (10.23) is that their properties may be fully described through knowledge of the first-order correlation function.

The fields which have traditionally been called coherent ones in optical terminology are easily described in terms of the first-order correlation function given by Eq. (10.25). Since the light in such fields is accurately collimated and nearly monochromatic, the mean occupation number $\langle n_{k,\lambda}\rangle$ vanishes outside a small volume of $\mathbf{k}$-space. The criterion for accurate coherence is ordinarily that the dimensions of this volume be extremely small in comparison to the magnitude of $\mathbf{k}$. It is easily verified, if the field is fully polarized, and the two points $(\mathbf{r},t)$ and $(\mathbf{r}',t')$ are not too distantly separated, that the correlation function (10.25) falls approximately into the factorized form of Eq. (2.4). That is to say, fields of the type we have described approximately fulfill the condition for first-order coherence.[9] It is easily seen, however, from the structure of the higher order correlation functions that these fields can never have second or higher order coherence. In fact, if we evaluate the function $G^{(n)}$ given by Eq. (10.27) for the particular case in which all of the coordinates are set equal, $x_1=\cdots=x_{2n}=x$, and all of the indices as well, $\mu_1=\cdots=\mu_{2n}=\mu$, we find the result

$$G_{\mu\cdots\mu}^{(n)}(x\cdots x,x\cdots x)=n![G_{\mu\mu}^{(1)}(x,x)]^n. \quad (10.28)$$

The presence of the coefficient $n!$ in this expression is incompatible with the factorization condition (10.4) for the correlation functions of order n greater than one. The absence of second or higher order coherence is thus a general feature of stationary fields described by the Gaussian weight function (10.23). There exists, in other words, a fundamental sense in which these fields remain incoherent no matter how monochromatic or accurately collimated they are. We need hardly add that other types of field such as those generated by radio transmitters or masers may possess arbitrarily high orders of coherence.

During the completion of the present paper a note by Sudarshan[10] has appeared which deals with some of the problems of photon statistics that have been treated here.[11] Sudarshan has observed the existence of what we have called the P representation of the density operator and has stated its connection with the representation based on the n-quantum states. To that extent, his work agrees with ours in Secs. VII and IX. He has, however, made a number of statements which appear to attach an altogether different interpretation to the P representation. In particular, he regards its existence as demonstrating the "complete equivalence" of the classical and quantum mechanical approaches to photon statistics. He states further that there is a "one-to-one correspondence" between the weight functions P and the probability distributions for the field amplitudes of classical theory.

The relation between the P representation and classical theory has already been discussed at some length in Secs. VII–IX. We have shown there that the weight function $P(\alpha)$ is, in general, an intrinsically quantum-mechanical structure and not derivable from classical arguments. In the limit $\hbar\to 0$, which corresponds to large amplitudes of excitation for the modes, the weight functions $P(\alpha)$ may approach classical probability functions as asymptotic forms. Since infinitely many quantum states of the field may approach the same asymptotic form, it is clear that the correspondence between the weight functions $P(\alpha)$ and classical probability distributions is not at all one-to-one.

[11] In an accompanying note, L. Mandel and E. Wolf [Phys. Rev. Letters 10, 276 (1963)] warmly defend the classical approach to photon problems. Some of the possibilities and fundamental limitations of this approach should be evident from our earlier work. We may mention that the "implication" they draw from Ref. 1 and disagree with cannot be validly inferred from any reading of that paper.

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Note on Numbering of Equations:

In the lectures which follow, references to equations in the preceding reprint are indicated by a capital R followed by the equation number.

Lecture XII RADIATION BY A PREDETERMINED CHARGE-CURRENT DISTRIBUTION

Not many problems of quantum electrodynamics are in any sense exactly soluble. But there does exist one simple, completely soluble problem which has considerable physical meaning. That is the problem of finding the photon field radiated by an electric current distribution which is essentially classical in nature. By "classical" in this case we mean that we may represent the current by a prescribed vector function of space and time, $\mathbf{j}(\mathbf{r}t)$.

Such a model clearly can not represent the process of radiation by an individual atom, since the atomic current is affected by radiation recoil in essentially unpredictable ways. The model may, however, be an excellent approximation for dealing with radiation by aggregates of atoms which are large enough to show statistically predictable behavior for the total current vector. Note that in saying this we are not at all ignoring the reaction of the radiation process back upon the current. All we require is that whatever the reaction is, it be predictable at least in principle (as the radiation resistance of an antenna is, for example). It seems likely that this model, when allowance is made for statistical uncertainties in the current distribution, will accurately account for the photon fields generated by most macroscopic sources.

The interaction Hamiltonian which describes the coupling of the quantized electromagnetic field to the current distribution takes the form

$$\mathcal{H}_I(t) = \frac{1}{c}\int \mathbf{j}(\mathbf{r}, t)\cdot \mathbf{A}(\mathbf{r}, t)\, d\mathbf{r}. \qquad (12.1)$$

The state vector of the field changes with time in the interaction representation, obeying the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|t> = \mathcal{H}_I|t> . \qquad (12.2)$$

Now let us, as an abbreviation, introduce the operator $B(t)$ which is defined as

$$B(t) = \frac{1}{\hbar c}\int \mathbf{j}(\mathbf{r}, t)\cdot \mathbf{A}(\mathbf{r}, t)\, d\mathbf{r}. \qquad (12.3)$$

The operator $B(t)$ is simply a linear combination of values of the vector potential, and hence obeys a commutation relation of the same general type as the vector potential. In general $[B(t), B(t')]$ will be different from zero, but it is always an ordinary number.

Now the Schrödinger equation, Eq. (12.2), can be rewritten as

$$\frac{d}{dt}|t> = B(t)|t> . \qquad (12.4)$$

Because of the operator character of $B(t)$ the solution of this equation is not

$$\exp\left\{\int_{t_0}^{t} B(t')\,dt'\right\}\ |\ t_0> \qquad (12.5)$$

as it would be if $B(t)$ were an ordinary number. However because of the simple commutation relation obeyed by the B's this expression will turn out not to be quite as wrong as we might perhaps expect.

We know that the state $|t>$ at time t can be expressed by means of a unitary operator, $U(t, t_0)$, applied to the state $|t_0>$ as time t_0, i.e.,

$$|t> = U(t, t_0)|t_0> . \qquad (12.6)$$

The equations which determine $U(t, t_0)$ are evidently

$$\frac{d}{dt}U(t, t_0) = B(t)\,U(t, t_0) \qquad (12.7)$$

and the initial condition $U(t_0, t_0) = 1$.

In order to solve for the operator U let us begin by dividing the time interval $(t_0 - t)$ into sub-intervals of length Δt extending between the times $t_j = t_0 + j\Delta t$, where j is an integral. We may then reach the solution of Eq. (12.7) through a simple limiting process. We assume that the operator $B(t)$ is constant in value during each of the sub-intervals of time and allow its value to change at the times t_j. A rather fanciful picture of this variation is shown in the "graph" of the operator B versus time given by Fig. 11.

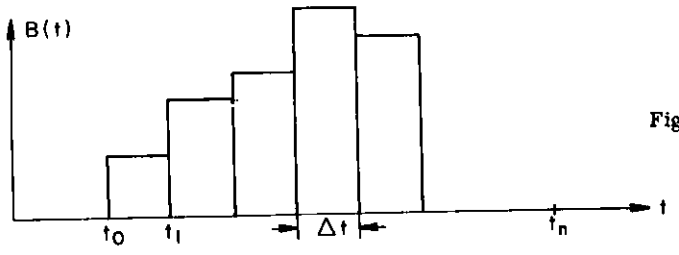

Figure 11

Since the operator B is constant in each of the sub-intervals, we can easily integrate the differential equation (12.7) for the individual sub-intervals. If $B(t)$ takes on the value B_j in the interval from t_{j-1} to t_j then we evidently have

$$U(t_j, t_{j-1}) = e^{B_j \Delta t} . \qquad (12.8)$$

Hence the transformation operator which corresponds to a succession of sub-intervals must be

$$U(t_n, t_0) = e^{B_n \Delta t} e^{B_{n-1}\Delta t} \cdots e^{B_1 \Delta t} . \qquad (12.9)$$

Now we can use the familiar theorem for multiplication of exponentials, Eq. (R 3.20) to evaluate the product. For n = 2, for example we have

$$U(t_2, t_0) = e^{B_2 \Delta t} e^{B_1 \Delta t} = \exp\left\{(B_1 + B_2)\Delta t + \frac{1}{2}[B_2, B_1](\Delta t)^2\right\}. \qquad (12.10)$$

The repetition of similar multiplications clearly leads to

$$U(t_n, t_0) = \exp\left\{\sum_{j=1}^{n} B_j \Delta t + \frac{1}{2}\sum_{j>k}[B_j, B_k](\Delta t)^2\right\} , \qquad (12.11)$$

which is an exact solution as long as $B(t)$ has the discontinuous time variation we have assumed.

We may consider the case in which the operator $B(t)$ varies continuously with time to be the limit in which $\Delta t \to 0$, i.e., we assume $t_n = t$ remains fixed and let $n \to \infty$. In that limit Eq. (12.12) becomes the general solution

$$U(t, t_0) = \exp\left\{\int_{t_0}^{t} B(t')dt' + \frac{1}{2}\int_{t_0}^{t}dt' \int_{t_0}^{t'}dt'' [B(t'), B(t'')]\right\} . \qquad (12.12)$$

If we compare this solution with the expression (12.5), which was reached by naively ignoring the operator character of B(t), we may see that the difference lies only in the addition of the term

$$\frac{1}{2}\int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \; [B(t'), B(t'')] \tag{12.13}$$

to the exponent. The commutator in this integral is an ordinary number and, in fact, a purely imaginary one. Hence the solution (12.12) only differs from (12.5) by a time dependent phase factor. If we let $i\varphi(t)$ represent the integral (12.14), then we may write the transformation operator as

$$\begin{aligned} U(t,t_0) &= \exp\left\{\int_{t_0}^{t} B(t')\,dt' + i\varphi(t)\right\} \\ &= \exp\left\{\frac{i}{\hbar c}\int_{t_0}^{t} \mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')\,dt'\,d\mathbf{r} + i\varphi(t)\right\}. \end{aligned} \tag{12.14}$$

Although the phase function $\varphi(t)$ is not altogether lacking in physical interest, (there is information contained in it, for example, on the interaction energy of the current and field) it does not have any influence on the calculation of density operators for the field, i.e., if the density operator has the initial value $\rho(t_0)$, then its value at time t is

$$\rho(t) = U(t,t_0)\,\rho(t_0)\,U^{\dagger}(t,t_0), \tag{12.15}$$

and we see immediately that the phase factor cancels.

If in particular the initial state is the vacuum state

$$|\,t_0\rangle = |\,\mathrm{vac}\rangle, \tag{12.16}$$

then at time t we have

$$e^{-i\varphi(t)}\,|t\rangle = \exp\left\{\frac{1}{\hbar c}\int_{t_0}^{t} \mathbf{j}(\mathbf{r},t')\cdot\mathbf{A}(\mathbf{r},t')\,d\mathbf{r}\,dt'\right\}|\,\mathrm{vac}\rangle. \tag{12.17}$$

Now if we introduce the expansion of the operator A in normal modes, Eq. (R 2.10), we see that the unitary operator which is applied to the vacuum state on the right side of Eq. (12.17) is simply a product of displacement operators which take the form

$$D_k(\alpha_k) = \exp\left[\alpha_k a_k^{\dagger} - \alpha_k^{*} a_k\right]. \tag{12.18}$$

More precisely, if we define the set of time-dependent amplitudes

$$\alpha_k(t) = \frac{i}{(2\hbar\omega_k)^{1/2}}\int_{t_0}^{t} \mathbf{j}(\mathbf{r},t')\cdot\mathbf{u}_k^{*}(\mathbf{r},t')\,e^{-i\omega_k t'}\,d\mathbf{r}\,dt', \tag{12.19}$$

then Eq. (12.17) may be rewritten as

$$e^{-i\varphi(t)}\,|t\rangle = \prod_k D_k(\alpha_k(t))\,|\,\mathrm{vac}\rangle. \tag{12.20}$$

It is clear from this result that a prescribed current distribution, radiating into the vacuum, always brings the field to a coherent state

$$e^{-i\varphi(t)}\,|t\rangle = |\{\alpha_k(t)\}\rangle. \tag{12.21}$$

More generally, if the field is initially in an arbitrary coherent state its state remains coherent under the influence of the current distribution.

The solution to the radiation problem we have found takes accurate account of the quantum mechanical properties of the field. It is related, however, in a simple way to the solution of the corresponding classical problem. The amplitudes $\alpha_k(t)$ are simply related to the time-dependent mode amplitudes for the classically radiated field through Eq. (8.22).

The density operator at time t which corresponds to the coherent state (12.21) is simply

$$\rho(t) = |\{\alpha_k(t)\}\rangle\langle\{\alpha_k(t)\}|, \tag{12.22}$$

which may be written in the P-representation as

$$\rho(t) = \int P(\{\beta_k\})\,|\{\beta_k\}\rangle\langle\{\beta_k\}|\prod_k d^2\beta_k, \tag{12.23}$$

by making use of the P-function

$$P(\{\beta_k\}) = \prod_k \delta^{(2)}(\beta_k - \alpha_k(t)). \tag{12.24}$$

The calculations we have carried out have dealt with a predetermined current distribution, i.e., one which behaves in a way which is in principle predictable. But in practice, of course, we may lack the information necessary to make such predictions and may have to resort to a statistical description of the behavior of the current. In that case, since we do not know the current $\mathbf{j}(\mathbf{r}, t)$ at any given time, it becomes impossible to make an exact specification of the set of amplitudes $\alpha_k(t)$ through Eq. (12.19). The best we can do is to state that the coefficients α_k have a certain probability distribution $p(\{\alpha_k\}, t)$ at time t whose dispersion corresponds to whatever randomness is present. Then it is clear that the density operator can be written in the form

$$\rho = \int p(\{\alpha_k\},t)\,|\{\alpha_k\}\rangle\langle\{\alpha_k\}|\prod_k d^2\alpha_k, \tag{12.25}$$

which is a fairly general form for the P-representation, but one in which the function P is obviously always positive.

Density operators having the general form of Eq. (12.25) with $p(\{\alpha_k\}, t)$ positive may arise from a variety of sources (e.g., thermal radiators, discharge tubes, etc.). Hence it is interesting to note that our arguments indicate that we can always construct for these cases some sort of random classical current distribution which will lead to the same field, i.e., the same density operator.

Lecture XIII PHASE-SPACE DISTRIBUTIONS FOR THE FIELD

In classical mechanics we can specify the state of a system by giving the instantaneous values of all coordinates and momenta. The evolution of the system then follows uniquely from the equations of motion. It can be visualized by considering the n coordinates and n momenta of the system as the coordinates of a point in a 2n-dimensional space, the phase space. The point which represents a system in this space moves along a uniquely determined trajectory. This picture is easily adapted to the uses of classical statistical mechanics. There, since we are characteristically uncertain of the initial coordinates and momenta of the system, we can speak only of probability distributions $P_{cl}(p_1' \cdots p_n', q_1', \cdots, q_n')$ for these variables. Instead of following the motion of a single point through the phase space, we must follow the motion of a whole "cloud" of them representing an ensemble of similarly prepared systems. The expectation value of any function of the p_i' and q_i' can then be calculated by means of an integral, involving the probability P_{cl} as

a weight function.

There has been, since the earliest days of quantum mechanics, a prevailing temptation to use the same sort of phase space picture for the description of quantum mechanical uncertainties. We shall not attempt to discuss these representations here in much generality since our interests are confined to the electromagnetic field. From a dynamical standpoint, the oscillations of each mode of the field are those of a harmonic oscillator. It will be quite sufficient, for the present discussion, to confine our attention to a single mode. In that case, the classical phase space has only two dimensions, corresponding to the variables p' and q'. The phase point for a mode with energy E moves classically along the ellipse $p'^2 + \omega^2 q'^2 = 2E$. (The mass parameter is set equal to unity.)

A coherent state of the mode will exist corresponding to any complex eigenvalue we specify for the operator

$$a = (2\hbar\omega)^{-\frac{1}{2}} (\omega q + ip) . \tag{13.1}$$

The amplitude α corresponding to the state $|\alpha>$ may be written as

$$\alpha = (2\hbar\omega)^{-\frac{1}{2}} (\omega q' + ip') , \tag{13.2}$$

where q' and p' are real numbers. Now we have shown in Section III of the reprinted paper that the state $|\alpha>$ may be described by a wave packet which has minimum uncertainty and the mean coordinate q' and the mean momentum p'.

Furthermore if we use the Schrödinger picture and follow the motion of the state with time, we know that the state remains coherent at all times, and that its time-dependent amplitude is simply $\alpha e^{-i\omega t}$. The motion of the amplitude vector in the complex α-plane takes place on the circle $|\alpha|$ = const. which simply represents an ellipse of the type noted earlier in the p', q' plane.

It is clear that the complex α-plane is simply a species of two-dimensional phase space. One therefore inevitably feels a great temptation to think of the coherent state wave packets in terms of probability "clouds" whose centers move on circular paths. Such an image, however, is an intrinsically classical one. In quantum mechanics the observables p and q are not simultaneously measureable (with more than limited accuracy), and therefore a certain lack of meaning, or at best an arbitrariness of meaning characterizes any attempt to speak of a joint probability distribution for the variables p' and q'. We can, of course, speak of the distribution of either variable in precisely defined terms, but these are alternative descriptions of the oscillator rather than a way of dealing with p' and q' simultaneously.

The P-representation of the density operator, which we introduced in the reprinted paper, can often be regarded as defining something at least comparable to a phase space distribution. The complex α-plane on which the P-function is defined, is indeed a species of phase space. Furthermore as we have noted in the paper, the P-function has a number of properties in common with probability distributions. However, as we have also seen, the function may take on negative values, and behave in singular ways which are altogether unlike those of a probability density. There is nothing inconsistent about such strange behavior because the function is not accessible to measurement as a joint probability distribution.

From the standpoint of similarity to classical theory, the function $P(\alpha)$ is simply one of a class of functions which possess, by definition, some of the properties of a phase space distribution and then inevitably lack others. We will discuss some other examples of such functions, which are perhaps best called quasi-probability densities, later in the lecture, and show their relation to the P-representation. First, however, let us turn to the question of how generally applicable the P-representation is.

THE P-REPRESENTATION AND THE MOMENT PROBLEM

Although it is clear, from the examples given in the reprinted paper, that the P-representation of the density operator is capable of representing a fairly broad variety of fields, no effort was made there to characterize that class of fields. Sudarshan has stated in a brief note,[1] however, that a "diagonal" representation of the density operator in terms of the coherent states may be used to represent an arbitrary field. He has given an explicit construction for the weight function of such a representation as an infinite sum of arbitrarily high-order derivatives of a delta function. He has said that, as a consequence of this construction, "the description of statistical states of a quantum mechanical system ... is completely equivalent to the description in terms of classical probability distributions."

The way in which Sudarshan's construction for the function $P(\alpha)$ may be reached is as follows: we consider the matrix elements of the density operator in the n-quantum state representation as known and note that, according to Eq. (R.7.12), these matrix elements are the complex moments,

$$< n|\rho|m > = (n!\, m!)^{-\frac{1}{2}} \int P(\alpha)\ (\alpha^*)^m\, \alpha^n d^2\alpha$$

of the weight function $P(\alpha)$. We then consider this sequence of equations for all n and m to define a species of two-dimensional moment problem, i.e., we seek a function $P(\alpha)$ which has the correct matrix of moments. The general problem when stated thus becomes a notoriously difficult one, and one which need not, for arbitrary matrix elements $<n|\rho|m>$, have a reasonable solution of any sort. Sudarshan's solution corresponds to taking advantage of some remarkable properties of the delta function and its derivatives which are perhaps most easily illustrated in a one-dimensional context.

Let us suppose that we are given the problem of finding a function f(x) on the interval $-\infty < x < \infty$ which has a specified set of moments M_n, i.e., we have

$$\int_{-\infty}^{\infty} f(x)\ x^n\, dx = M_n \qquad n = 0,\ 1,\ 2,\ \cdots\ . \tag{13.3}$$

If we write the j-th derivative of the delta function as

$$\delta^{(j)}(x) = \frac{d^j}{dx^j}\, \delta(x) , \tag{13.4}$$

then we observe that its moments are given by

$$\int_{-\infty}^{\infty} x^k \delta^{(j)}(x)\ dx = (-1)^j\, j!\ \delta_{jk} . \tag{13.5}$$

In other words, each derivative of the delta function has one and only one non-vanishing moment. It would seem then that we can construct a "solution" of the general moment problem simply by writing

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\ M_n \delta^{(n)}(x) . \tag{13.6}$$

The test of such a "solution" is ultimately whether or not it means anything.

Mathematicians have long noted that the delta function and its derivatives are not, strictly speaking, functions at all. More recently they have provided us with the theory of distributions (or generalized functions) as a means of dealing with these structures in more meaningful and rigorous terms.

Equations (13.4) and (13.5) assume a well-defined meaning in terms of distribution theory, but the theory shows that there is in general no useful meaning which can be attached to an infinite sum such as Eq. (13.6).

The "solution" exhibited by Sudarshan for the two dimensional moment problem takes the explicit form

$$P_s(\alpha) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(n!\,m!)^{\frac{1}{2}}}{(n+m)!} < n|\rho|m > \frac{1}{2\pi|\alpha|} e^{|\alpha|^2 - i(n-m)\theta} \{(-\frac{\partial}{\partial|\alpha|})^{n+m} \delta(|\alpha|)\}, \tag{13.7}$$

where we have written $\alpha = |\alpha| e^{i\theta}$. Recently Holliday and Sage[2] have shown, by considering a simple example explicitly, that this expression cannot be construed as a generalized function of any sort. The example was that of the thermal density operator, and for it they showed that when the series (13.7) is multiplied by an extremeley well behaved test function (which vanishes outside a circle of finite radius in the α-plane), and the product is then integrated, the integral diverges. More recently, Cahill[3] has shown that whenever there is no upper bound to the number of quanta present, the series (13.7) will fail to be interpretable as a distribution (or a generalized function).

While these results indicate that Sudarshan's proposed representation is not, in general, meaningful, they leave open the larger question of the generality of the P-representation. They allow the possibility, in other words, that there might exist other constructions of the P-representation which are meaningful for all states of the field. Recently, however, D. Kastler and the lecturer[4] have demonstrated that the P-representation lacks the generality necessary to represent all states. They have shown in particular that there exist quantum states of the field for which it is not possible to find functions $P(\alpha)$ which are distributions. That means that all general results derived by using the P-representation must be qualified by the assumption that the representation exists.

A POSITIVE-DEFINITE "PHASE SPACE DENSITY"

We will now consider some other examples of quasiprobability functions, with different types of behavior and different degrees of usefulness. The first of these is the diagonal element, $< \alpha|\rho|\alpha >$, of the density operator. It is clear that $< \alpha|\rho|\alpha>$ is non-negative and that it is a well-defined function of α for all ρ. It is therefore a good deal closer to being a phase space density in its behavior than is $P(\alpha)$.

From the general expression for $R(\alpha^*, \beta)$ given by Eq. (R 6.1),

$$R(\alpha^*, \beta) = <\alpha|\rho|\beta> \exp\{\frac{1}{2}(|\alpha|^2 + |\beta|^2)\},$$

we have

$$<\alpha|\rho|\alpha> = R(\alpha^*, \alpha) e^{-|\alpha|^2}. \tag{13.8}$$

Hence, according to Eq. (R 6.6), the normalization condition on $< \alpha|\rho|\alpha >$ is

$$\frac{1}{\pi}\int <\alpha|\rho|\alpha> d^2\alpha = \frac{1}{\pi}\int R(\alpha^*, \alpha) e^{-|\alpha|^2} d^2\alpha = 1. \tag{13.9}$$

If the P-representation exists for the density operator ρ and has a weight function $P(\beta)$, we clearly have

$$\begin{aligned} <\alpha|\rho|\alpha> &= \int P(\beta) |<\alpha|\beta>|^2 d^2\beta \\ &= \int P(\beta) e^{-|\alpha-\beta|^2} d^2\beta. \end{aligned} \tag{13.10}$$

The function we are considering is simply a Gaussian convolution of the P-function.

We can use the function $<\alpha|\rho|\alpha>$ to calculate averages of products of operators which are in antinormal order in much the same way as products in normal order are averaged by means of the P-representation. Let us consider, for example, the average

$$Tr\{\rho J(a) K(a^\dagger)\},$$

where J and K can be any functions of the annihilation and creation operators, respectively.

We can write this average as

$$Tr\{K(a^\dagger)\rho J(a)\} = \frac{1}{\pi}\int Tr\{|\alpha><\alpha| K(a^\dagger)\rho J(a)\} d^2\alpha$$

$$= \frac{1}{\pi}\int d^2\alpha <\alpha|K(a^\dagger)\rho J(a)|\alpha> = \frac{1}{\pi}\int <\alpha|\rho|\alpha> K(\alpha^*) J(\alpha) d^2\alpha. \tag{13.11}$$

Unfortunately we are not too often interested in evaluating the expectation values of antinormally ordered products of field operators. When the full set of modes of the field is considered such expectation values tend to contain divergent contributions from the vacuum fluctuations.

The function $<\alpha|\rho|\alpha>$ takes an interesting form for the n-th excited state of the oscillator. For these states we have

$$\rho_n = |n><n| = \frac{1}{n!}(a^\dagger)^n |0><0| a^n, \tag{13.12}$$

and therefore the result

$$<\alpha|\rho|\alpha> = \frac{1}{n!} |<\alpha|n>|^2 = \frac{|\alpha|^{2n}}{n!} e^{-|\alpha|^2}. \tag{13.13}$$

This is an extremely well-behaved function, especially when we compare it with the analogous expression in the P-representation, which contains the 2n-th derivative of a delta function. The function $x^n e^{-x}$ has a maximum at $x = n$ and is quite sharply peaked there for large values of n. If we want to express the result (13.13) as a distribution in phase space we can substitute the expression (13.2) for α and write

$$<\alpha|\rho|\alpha> = \frac{1}{n!} \frac{(p'^2 + \omega^2 q'^2)^n}{(2\hbar\omega)^n} \exp\{-\frac{p'^2 + \omega^2 q'^2}{2\hbar\omega}\}. \tag{13.14}$$

This function evidently has its maximum value on the ellipse $(1/2)(p'^2 + \omega^2 q'^2) = n\hbar\omega$, that is to say on the classical orbit in phase space. It drops to zero on either side of the classical orbit while remaining positive everywhere.

Another example for which we can easily illustrate this "phase space density" is that of the Gaussian density operator. For that case we have

$$\begin{aligned} R(\alpha^*, \beta) &= \int P(\gamma) \exp\{\alpha^*\gamma + \beta\gamma^* - |\gamma|^2\} d^2\gamma \\ &= \frac{1}{\pi<n>}\int \exp\{-\frac{|\gamma|^2}{<n>} + \alpha^*\gamma + \beta\gamma^* - |\gamma|^2\} d^2\gamma \\ &= \frac{1}{\pi<n>}\int \exp\{-|\gamma|^2 \frac{1+<n>}{<n>} + \alpha^*\gamma + \beta\gamma^*\} d^2\gamma. \end{aligned} \tag{13.15}$$

We can now make the substitution $\xi = \gamma\{\frac{1+<n>}{<n>}\}^{\frac{1}{2}}$, which reduces the integral to the standard form

$$R(\alpha^*, \beta) = \frac{1}{\pi(1+\langle n\rangle)} \int \exp\{-|\xi|^2 + [\frac{\langle n\rangle}{1+\langle n\rangle}]^{\frac{1}{2}}(\alpha^*\xi + \beta\xi^*)\}\, d^2\xi$$

$$= \frac{1}{\pi(1+\langle n\rangle)} \exp\{\frac{\langle n\rangle}{1+\langle n\rangle}\, \alpha^*\beta\}. \qquad (13.16)$$

Hence we find

$$\langle \alpha|\rho|\alpha\rangle = R(\alpha^*, \alpha)\, e^{-|\alpha|^2}$$

$$= \frac{1}{\pi(1+\langle n\rangle)} \exp\{-\frac{|\alpha|^2}{1+\langle n\rangle}\}. \qquad (13.17)$$

If $\langle n\rangle$ goes to zero this expression becomes the Gaussian function $(1/\pi)\exp(-|\alpha|^2)$. In the same case the weight function $P(\alpha)$ would be a delta function at the origin. If $\langle n\rangle$ goes to infinity we have

$$\langle \alpha|\rho|\alpha\rangle \cong \frac{1}{\pi\langle n\rangle}\, e^{-\frac{|\alpha|^2}{\langle n\rangle}} \cong P(\alpha). \qquad (13.18)$$

In this limit $\langle \alpha|\rho|\alpha\rangle$ becomes equal to the P-distribution. That is so because the limit of large $\langle n\rangle$ is just the classical limit. There $P(\alpha)$ does indeed become interpretable as a classical phase space density, and the distinction between normally and antinormally ordered operators also vanishes, as a consequence of the correspondence principle.

WIGNER'S "PHASE SPACE DENSITY"

The Wigner distribution can be considered as the grandfather of all our quasi-probability functions. It exists and is well-behaved for all quantum states but seems to take on negative values without hesitation. We shall follow the approach used by Moyal[5] to define the Wigner distribution.

We begin by discussing a species of characteristic function which is defined as

$$X(\mu, \nu) = \langle e^{i(\mu p + \nu q)}\rangle, \qquad (13.19)$$

where p and q are operators. By using our theorem for the decompositon of exponentials, Eq. (R 3.20), we may write this expression as

$$X(\mu, \nu) = \mathrm{Tr}\{\rho\, e^{\frac{i\mu p}{2}}\, e^{i\nu q}\, e^{\frac{i\mu p}{2}}\} \qquad (13.20)$$

If we restrict consideration to a pure state, use the coordinate representation, and recall the interpretation of exponential functions of the momentum as coordinate displacement operators, we may rewrite Eq. (13.20) as

$$X(\mu, \nu) = \int \psi^*(q'' - \frac{\mu\hbar}{2})\, e^{i\nu q''}\, \psi(q'' + \frac{\mu\hbar}{2})\, dq'', \qquad (13.21)$$

where $\psi(q')$ is the wave function of the pure state. The Wigner function is then the Fourier transform of this characteristic function

$$W(p', q') \equiv \frac{1}{(2\pi)^2} \int \exp\{-i(\mu p' + \nu q')\} X(\mu, \nu)\, d\mu\, d\nu$$

$$= \frac{1}{(2\pi)^2} \int \exp\{-i(\mu p' + \nu q')\} \int \psi^*(q'' - \frac{\mu\hbar}{2})\, e^{i\nu q''} \times$$

$$\psi(q'' + \frac{\mu\hbar}{2})\, dq''\, d\mu\, d\nu$$

$$= \frac{1}{2\pi} \int e^{-i\mu p'} \int \psi^*(q'' - \frac{\mu\hbar}{2})\, \delta(q' - q'')\, \psi(q'' + \frac{\mu\hbar}{2})\, dq''\, d\mu$$

$$= \frac{1}{2\pi} \int \psi^*(q' - \frac{\mu\hbar}{2})\, e^{-i\mu p'}\, \psi(q' + \frac{\mu\hbar}{2})\, d\mu. \qquad (13.22)$$

If we substitue $y = -\mu\hbar$ in the latter expression we derive the form of the distribution originally stated by Wigner,

$$W(p', q') = \frac{1}{2\pi\hbar} \int \psi^*(q' + \frac{1}{2}y)\, e^{\frac{ip'y}{\hbar}}\, \psi(q' - \frac{1}{2}y)\, dy. \qquad (13.23)$$

It is obvious that whenever we have a wave function we can derive a Wigner distribution from it. Thus the distribution always exists, but it is not necessarily positive. When we have a mixture of states we must of course take a suitably weighted average of (13.23) over all the states which occur.
The normalization condition for $W(q', p')$ is

$$\int W(q', p')\, dp'\, dq' = \int \delta(\mu)\, \delta(\nu)\, X(\mu, \nu)\, d\mu\, d\nu$$

$$= X(0, 0)$$

$$= 1. \qquad (13.24)$$

To compare the Wigner distribution with the others we have discussed, it is useful to express it in terms of the creation and annihilation operators $a^\dagger$ and a. Then if we define a complex Fourier transform variable

$$\lambda = -\mu\{\frac{\hbar\omega}{2}\}^{\frac{1}{2}} + i\nu[\frac{\hbar}{2\omega}]^{\frac{1}{2}}. \qquad (13.25)$$

we may write the operator which occurs in the exponent of the characteristic function as

$$-i(\mu p + \nu q) = \lambda a^\dagger - \lambda^* a, \qquad (13.26)$$

and the characteristic function itself becomes

$$X(\mu, \nu) = \langle e^{\lambda a^\dagger - \lambda^* a}\rangle$$

$$= \mathrm{Tr}\{\rho e^{\lambda a^\dagger} e^{-\lambda^* a}\} e^{-\frac{1}{2}|\lambda|^2}$$

$$= \mathrm{Tr}\{\rho e^{-\lambda^* a} e^{\lambda a^\dagger}\} e^{\frac{1}{2}|\lambda|^2}. \qquad (13.27)$$

We can now use the normally ordered form to express the Wigner function in terms of the P-representation. If we assume that the density operator possesses a P-representation, the characteristic function is given by

$$X(\mu, \nu) = \int P(\beta) \langle \beta|\, e^{\lambda a^\dagger}\, e^{-\lambda^* a}\, |\beta\rangle e^{-\frac{1}{2}|\lambda|^2}\, d^2\beta$$

$$= \int P(\beta)\, \exp\{\lambda\beta^* - \lambda^*\beta - \frac{1}{2}|\lambda|^2\} d^2\beta. \qquad (13.28)$$

In calculating the Fourier transform of X, i.e., the Wigner function, it is convenient to use a linear combination of α and α^* in the exponent rather than a combination of the classical variables q' and p'. We therefore write

$$i(\mu p' + \nu q') = \lambda\alpha^* - \lambda^*\alpha \qquad (13.29)$$

and

$$d\mu\, d\nu = \frac{2}{\hbar}\, d^2\lambda \quad . \tag{13.30}$$

Then the Fourier transform becomes

$$W(q', p') = \frac{1}{(2\pi)^2}\int \mathrm{Tr}\{\rho e^{\lambda(a^\dagger - \alpha^*)}\, e^{-\lambda^*(a-\alpha)}\}\, e^{-\frac{1}{2}|\lambda|^2}\, \frac{2}{\hbar}\, d^2\lambda$$

$$= \frac{1}{2\pi\hbar}\int P(\beta) \exp\{\lambda(\beta^* - \alpha^*) - \lambda^*(\beta - \alpha) - \frac{1}{2}|\lambda|^2\}\, d^2\lambda\, d^2\beta \,. \tag{13.31}$$

We can reduce this integral to a standard form by the substitution $\xi = \frac{\lambda}{\sqrt{2}}$ which leads to

$$W(q', p') = \frac{1}{\pi\hbar}\int P(\beta)\, \exp\{\sqrt{2}\,\xi(\beta^* - \alpha^*) - \sqrt{2}\,\xi^*(\beta - \alpha) - |\xi|^2\} d^2\xi\, d^2\beta$$

$$= \frac{1}{\pi\hbar}\int P(\beta)\, \exp\{-2|\beta - \alpha|^2\}\, d^2\beta. \tag{13.32}$$

It is sometimes convenient to think of the Wigner function more directly as a function of the complex variable α, and to change its normalization accordingly. We therefore recall that

$$d^2\alpha = \frac{1}{\{2\hbar\omega\}^{\frac{1}{2}}}\left\{\frac{\omega}{2\hbar}\right\}^{\frac{1}{2}} dp'\, dq' = \frac{dp'\, dq'}{2\hbar} \tag{13.33}$$

and define the function

$$W(\alpha) = 2\hbar\, W(p', q') \tag{13.34}$$

so that

$$\int W(\alpha)\, d^2\alpha = 1 \quad . \tag{13.35}$$

The Wigner function of complex argument is then given by

$$W(\alpha) = \frac{2}{\pi}\int P(\beta)\, e^{-2|\beta - \alpha|^2} d^2\beta \quad . \tag{13.36}$$

When we compare this expression with the one derived in the preceding section,

$$\langle \alpha|\rho|\alpha\rangle = \int P(\beta)\, e^{-|\beta - \alpha|^2}\, d^2\beta, \tag{13.37}$$

We see that both of these expressions are simply Gaussian convolutions of the P-distribution (when the latter exists). The quality which the Wigner distribution shares with the P-distribution, of becoming negative in places, would seem to be due to the fact that the averaging process expressed by Eq. (13.36) takes place over a radius which is $(\sqrt{2})^{-1}$ times smaller than that expressed by Eq. (13.37).

As an example, let us evaluate the Wigner distribution for a field described by a Gaussian density operator. For this case we have, according to Eq. (13.36),

$$W(\alpha) = \frac{2}{\pi^2\langle n\rangle}\int \exp\left\{-\frac{|\beta|^2}{\langle n\rangle} - 2|\beta - \alpha|^2\right\} d^2\beta$$

$$= \frac{2\, e^{-2|\alpha|^2}}{\pi^2\langle n\rangle}\int \exp\left\{-|\beta|^2\left(2 + \frac{1}{\langle n\rangle}\right) + 2(\beta^*\alpha + \alpha^*\beta)\right\} d^2\beta \quad . \tag{13.38}$$

We now use the substitution $\gamma = \left\{\frac{1 + 2\langle n\rangle}{\langle n\rangle}\right\}^{\frac{1}{2}}\beta$ to reduce the integral to the standard form

$$W(\alpha) = \frac{2\, e^{-2|\alpha|^2}}{\pi^2(2\langle n\rangle + 1)}\int \exp\left\{-|\gamma|^2 + 2\left\{\frac{\langle n\rangle}{2\langle n\rangle + 1}\right\}^{\frac{1}{2}}(\gamma^*\alpha + \alpha^*\gamma)\right\} d^2\gamma$$

$$= \frac{2}{\pi^2(2\langle n\rangle + 1)}\int \exp\left\{-\left|\gamma - 2\left\{\frac{\langle n\rangle}{2\langle n\rangle + 1}\right\}^{\frac{1}{2}}\alpha\right|^2\right\} d^2\gamma \times$$

$$\exp\left\{\left[\frac{4\langle n\rangle}{2\langle n\rangle + 1} - 2\right]|\alpha|^2\right\} . \tag{13.39}$$

The latter integral leads immediately to the result

$$W(\alpha) = \frac{2}{\pi(2\langle n\rangle + 1)}\, \exp\left\{-\frac{2}{2\langle n\rangle + 1}|\alpha|^2\right\} \quad . \tag{13.40}$$

Thus, the Wigner distribution also has the Gaussian shape. We consider again the two limiting cases $\langle n\rangle = 0$, for which

$$W(\alpha) = \frac{2}{\pi}\, e^{-2|\alpha|^2} \,, \tag{13.41}$$

and $\langle n\rangle \to \infty$, for which

$$W(\alpha) \cong \frac{1}{\pi\langle n\rangle}\, e^{-\frac{|\alpha|^2}{\langle n\rangle}} = P(\alpha) \quad . \tag{13.42}$$

The latter result is the one we anticipate for the correspondence limit.

The simple Gaussian form given by Eq. (13.40) may be used to derive the complete set of Wigner distributions for the n-quantum states. This is possible because the function (13.40) may be regarded as a generating function for the Wigner distributions. Let us consider, for a moment, the general case of a density operator which may be written in the form

$$\rho = (1 - x)\sum_{n=0}^{\infty} x^n\, |n\rangle\langle n|, \tag{13.43}$$

where x is an arbitrary parameter. If we let $W_n(\alpha)$ be the Wigner function for the n-th quantum state, then, as a consequence of the linearity of W in ρ, we must have

$$W(\alpha) = (1 - x)\sum_{n=0}^{\infty} x^n\, W_n(\alpha) \quad . \tag{13.44}$$

Now if we make the identification $x = \langle n\rangle/(1 + \langle n\rangle)$, it becomes clear from Eq. (R 8.10) that ρ given by Eq. (13.43) is simply the Gaussian density operator. We can therefore write Eq. (13.40) alternatively in terms of the variable x as

$$W(\alpha) = \frac{2(1 - x)}{\pi(1 + x)}\, \exp\left\{-2\left(\frac{1 - x}{1 + x}\right)|\alpha|^2\right\}$$

$$= \frac{2(1 - x)}{\pi(1 + x)}\, \exp\left\{\frac{x}{1 + x}\, 4|\alpha|^2\right\} \exp\{-2|\alpha|^2\}. \tag{13.45}$$

This rather complicated exponential is just the generating function for the Laguerre polynomials L_n. In more familiar notation the generating function reads as

$$\frac{\exp\{-\frac{\rho u}{1-u}\}}{1-u} = \sum_{n=0}^{\infty} L_n(\rho)\frac{u^n}{n!} \quad . \tag{13.46}$$

Hence Eq. (13.45) yields the expansion

$$W(\alpha) = (1-x)\frac{2}{\pi}\sum_{n=0}^{\infty} x^n \frac{(-1)^n}{n!} L_n(4|\alpha|^2) e^{-2|\alpha|^2} \quad . \tag{13.47}$$

The Wigner function for the n-th excited state of the oscillator may thus be identified as

$$W_n(\alpha) = \frac{2}{\pi}\frac{(-1)^n}{n!} L_n(4|\alpha|^2)\ e^{-2|\alpha|^2} . \tag{13.48}$$

These functions have quite a wiggly behavior in the complex phase plane. The n-th function has nodes on n concentric circles.

For the first two states we have, more explicitly,

$$W_0(\alpha) = \frac{2}{\pi} e^{-2|\alpha|^2} = \frac{2}{\pi}\exp\{-\frac{p'^2+\omega^2 q'^2}{2\hbar\omega}\} \tag{13.48}$$

$$W_1(\alpha) = \frac{2}{\pi}(4|\alpha|^2 - 1)\ e^{-2|\alpha|^2} \quad . \tag{13.49}$$

The function $W_1(\alpha)$ is sketched in Fig. 12.

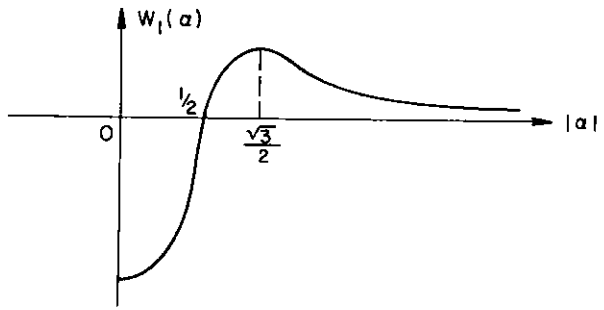

Figure 12

Its maximum lies at the radius $\alpha = \sqrt{3/2}$.

Each of the functions we have considered (the P-function, the function, $\langle\alpha|\rho|\alpha\rangle$, and the Wigner distribution) has its particular advantages. It should, however, be clear from the preceding discussions that we can construct numerous other such functions, each with virtues of its own. An element of arbitrariness underlies all such discussions of phase space distributions.

Note added in proof: In a recent preprint, Klauder, McKenna, and Currie confirm the conclusion that no useful weight function P need exist for arbitrary density operators. To minimize this difficulty they express matrix elements of the density operator through a limiting procedure involving an infinite sequence of operators expressed as P-representations. This procedure, however, does not preserve the most useful property of the P-representation, the reduction of statistical averages to simple integrals over the complex α-plane.

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Lecture XIV CORRELATION FUNCTIONS AND QUASIPROBABILITY DISTRIBUTIONS

In this lecture and in the ones which follow we shall begin to discuss applications of our formalism in somewhat more concrete terms. As a first step in that direction it will be useful to amplify several of the points which are stated rather briefly in the last section of the reprinted paper.

Let us suppose that the electromagnetic field is in a pure coherent state which we denote by $|\{\alpha_k\}>$. Then $|\{\alpha_k\}>$ is an eigenstate of the operator $E^{(+)}$,

$$E^{(+)}(rt)|\{\alpha_k\}> = \mathcal{E}(rt\{\alpha\})|\{\alpha_k\}>. \tag{14.1}$$

and the corresponding eigenvalue function $\mathcal{E}$, is a linear form in the variables $\{\alpha_k\}$, i.e., we have

$$\mathcal{E}(rt\{\alpha_k\}) = i\sum_k \left(\frac{\hbar\omega_k}{2}\right)^{\frac{1}{2}} u_k(r) e^{-i\omega_k t}\alpha_k . \tag{14.2}$$

The corresponding field is fully coherent since the correlation functions for all orders n fall into the factorized form

$$G^{(n)}_{\mu_1\cdots\mu_{2n}}(x_1\ldots x_{2n}) = \prod_{j=1}^{n}\mathcal{E}^*_{\mu_j}(x_j\{\alpha_k\})\prod_{j=n+1}^{2n}\mathcal{E}_{\mu_j}(x_j\{\alpha_k\}). \tag{14.3}$$

We have already noted that the term "coherence" is used frequently in the discussion of quantum mechanical problems of all sorts. Since the term is usually meant to imply that interference phenomena can take place, many of its uses are to be found in discussions of pure quantum mechanical states. Pure states, however, by no means exhaust the possibilities of securing interference. For most quantum mechanical systems there exist certain statistical mixtures of states which preserve essentially the same interference phenomena as are found for pure states. It is easy to exhibit these mixtures for the case of the electromagnetic field and to show that they may correspond to fields which are fully coherent in the sense of Eq. (14.3).

Instead of considering the field which corresponds to the set of amplitudes $\{\alpha_k\}$, let us consider the field corresponding to a set $\{\alpha_k'\}$ which we obtain by multiplying each of the coefficients α_k by a phase factor, $e^{i\phi}$, which is the same for all modes. If we have

$$\alpha_k' = e^{i\phi}\alpha_k, \tag{14.4}$$

then, since the eigenvalue function, $\mathcal{E}$, is linear, we must have

$$\mathcal{E}_\mu(r\,t\{\alpha_k'\}) = e^{i\phi}\mathcal{E}_\mu(rt\{\alpha_k\}). \tag{14.5}$$

Because the phase factors cancel when we construct the correlation functions, it is clear that the altered state of the field leads to the same set of correlation functions (14.3) as the original state. This invariance property, which is implicit in our definition of the correlation functions, means that we secure the same correlation functions not only for pure states corresponding to different values of the phase ϕ, but for arbitrary mixtures of such states as well.

Let us suppose that $\mathcal{L}(\phi)$ is a function which satisfies the normalization condition

$$\int_0^{2\pi}\mathcal{L}(\phi)\,d\phi = 1. \tag{14.6}$$

Then we may construct a density operator

$$\rho = \int_0^{2\pi} \mathcal{L}(\phi) |\{\alpha_k e^{i\phi}\}\rangle\langle\{\alpha_k e^{i\phi}\}| d\phi, \tag{14.7}$$

which represents mixtures of states with different values of the overall phase ϕ. (Note that $\mathcal{L}(\phi)$ must also statisfy a positive definiteness condition analogous to Eq. (R 7.9).) All such mixtures, i.e. all choices of $\mathcal{L}(\phi)$, lead to precisely the set of correlation functions (14.3); hence all such mixed states correspond to fully coherent fields.

It is most important, from a practical standpoint, that our definitions permit these mixed states to correspond to coherent fields. Our a priori knowledge of the state of high frequency fields usually contains no information about the overall phase ϕ. An ensemble of experiments performed with such fields must then be described by using a density operator of the form (14.7) with the special choice,

$$\mathcal{L}(\phi) = \frac{1}{2\pi}, \tag{14.8}$$

which represents our total ignorance of the phase. The indefinite character of this phase does not influence any of the interference intensities we have discussed thus far. It must therefore have no bearing on the coherence properties of a field. Our definition of coherence would hardly be very useful physically if it did not allow the appropriate mixed states as well as pure ones to be coherent.

FIRST ORDER CORRELATION FUNCTIONS FOR STATIONARY FIELDS

Virtually all of the famous experiments of optics may be described in terms of the first order correlation function for stationary light beams. Let us begin the evaluation of such a correlation function by using the normal mode expansion for the field operators to write it in the form

$$G^{(1)}_{\mu\nu}(rt, r't') = \frac{1}{2} \sum_{k,k'} \hbar(\omega_k \omega_{k'})^{\frac{1}{2}} \mathrm{Tr}\{\rho\, a_k^\dagger a_{k'}\} \times u_{k\mu}^*(r)\, u_{k'\nu}(r')\, e^{i(\omega_k t - \omega_{k'} t')}. \tag{14.9}$$

To evaluate the statistical averages $\mathrm{Tr}\{\rho a^\dagger_k a_{k'}\}$ we first note that these will always vanish when the modes k and k' are non-degenerate. We may prove that they vanish in this case by recalling that for stationary fields ρ commutes with the field Hamiltonian $\mathcal{H}_0$. Thus we have, for example,

$$\rho = e^{-\frac{i}{\hbar}\mathcal{H}_0 t}\, \rho\, e^{\frac{i}{\hbar}\mathcal{H}_0 t} \tag{14.10}$$

for all values of the parameter t. If we substitute the latter form for the operator into the expression for the desired trace we find

$$\mathrm{Tr}\{\rho a^\dagger_k a_{k'}\} = \mathrm{Tr}\{\rho\, e^{\frac{i}{\hbar}\mathcal{H}_0 t} a_k^\dagger a_{k'} e^{-\frac{i}{\hbar}\mathcal{H}_0 t}\} = \mathrm{Tr}\{\rho a_k^\dagger a_{k'}\} e^{i(\omega_k - \omega_{k'})t}. \tag{14.11}$$

Since the trace is independent of the parameter t, it must vanish whenever $\omega_k \neq \omega_{k'}$.

For the case of two different but degenerate modes, k and k', on the other hand, the quantity $\mathrm{Tr}(\rho\, a_k^\dagger a_{k'})$ need not vanish. More generally, if there are N degenerate modes the corresponding averages $\mathrm{Tr}(\rho\, a_k^\dagger a_{k'})$ can be regarded as forming the elements of an N × N Hermitian matrix which is not, in general, diagonal. It is always possible to diagonalize this matrix, however, by means of a

linear transformation which amounts simply to a redefinition of the set of degenerate mode functions. For any stationary state of the field represented by a density operator ρ, in other words, there will exist some particular choice of mode functions $u_k(r)$ such that the matrix reduces to diagonal form, i.e. we have

$$\mathrm{Tr}(\rho\, a_k^\dagger a_{k'}) = \langle n_k \rangle \delta_{kk'}, \tag{14.12}$$

where $\langle n_k \rangle$ is the mean occupation number of the k-th mode.

The convenience of working with particular choices of degenerate mode functions is easily illustrated by means of the polarization properties of light beams. For any plane wave state of a beam there are two degenerate polarization modes which are othogonal. If we were to choose a pair of plane polarization states as a basis, and were to describe a circularly polarized beam, for example, the quantities $\mathrm{Tr}(\rho a_k^\dagger a_{k'})$ would form a 2 × 2 matrix with four non-vanishing components. It is no surprise then that a more convenient choice of mode functions for that case consists of the two orthogonal circular polarizations. That choice reduces the matrix to one with only a single non-vanishing component.

Let us now return to our calculation of the first order correlation function for stationary fields. We see from Eqs. (14.12) and (14.9) that with a suitable choice of basis functions it is always possible to write the correlation function as an expansion of the form

$$G^{(1)}_{\mu\nu}(rt, r't') = \frac{1}{2} \sum_k \hbar\omega_k \langle n_k \rangle u^*_{k\mu}(r) u_{k\nu}(r')\, e^{i\omega_k(t-t')}, \tag{14.13}$$

which is determined simply by the set of average occupation numbers $\langle n_k \rangle$. An expansion of this type which is often useful is based on the set of plane wave modes of a large cubical volume of side L. These modes, whose functions $u_k(r)$ are given by Eq. (R 2.9), are so densely distributed in the space of the propagation vector k, when the volume of the system is large, that the sum over the states required in Eq. (14.13) may be replaced by the integral $(L/2\pi)^3 \int dk \ldots$. The expansion of the correlation function is then

$$G^{(1)}(rt, r't') = \frac{\hbar c}{2(2\pi)^3} \int \sum_{\lambda=1,2} \hat{e}_\mu^{(\lambda)*} \hat{e}_\nu^{(\lambda)} \langle n_{k,\lambda} \rangle k \times \exp\{-i[k\cdot(r-r') - \omega_k(t-t')]\}\, dk, \tag{14.14}$$

where λ is an index which labels the polarizations associated with propagation vector.

Let us suppose that the field consists of a well collimated light beam which is nearly monochromatic and is fully polarized. Then the mean occupation number $\langle n_{k,\lambda} \rangle$ will only take on non-vanishing values within a very small cell of k-space and, say, for $\lambda = 1$. Under these circumstances, if the magnitudes of $|r - r'|$ and $c|t - t'|$ remain small in comparison to the reciprocal dimensions of the volume in which $\langle n_{k,\lambda} \rangle$ differs from zero, it becomes possible to approximate the integral in Eq. (14.14) by neglecting the variation of the exponential in the integrand. If k_0 and ω_0 are the mean propagation vector and frequency of the beam we have

$$G^{(1)}_{\mu\nu}(rt, r't') \approx \frac{\hbar c}{2(2\pi)^3} N \hat{e}_\mu^{(1)*} \hat{e}_\nu^{(1)}\, e^{-i[k_0\cdot(r-r') - \omega_0(t-t')]}, \tag{14.15}$$

where

$$N = \int \langle n_{k,1} \rangle dk. \tag{14.16}$$

The light beam we have described is of just the sort most often used in interference experiments. It is also the kind most often referred to as "coherent" in the traditional terminology of optics. Now it is evident that by defining the field

$$\mathcal{E}(r,t) = \left\{\frac{\hbar c}{2(2\pi)^3}\right\} N^{\frac{1}{2}} \hat{e}^{(1)} e^{i(k_0 \cdot r - \omega_0 t)} \tag{14.17}$$

we may write the expression (14.15) for the correlation function in the factorized form

$$G^{(1)}_{\mu\nu}(rt, r't') \approx \mathcal{E}_\mu^*(rt)\ \mathcal{E}_\nu(r't'). \tag{14.18}$$

Hence the field in question does indeed satisfy the condition for first order coherence. It is worth emphasizing, however, that the factorization in Eq. (14.18) is an approximate one which tends to be most accurate for points r', t' near r, t. The imperfect collimation and monochromaticity of the beam define finite ranges of the variables r - r' and t - t', i.e. coherence distances and a coherence time, within which the factorization condition is obeyed. These ranges can, in principle, be made arbitrarily large by improving the quality of the beam.

This example illustrates the sense in which the coherence conditions must usually be regarded as idealizations. Given the practical sorts of field sources at our disposal, we cannot expect that the field correlations they generate will obey the coherence conditions over infinite ranges of the coordinate variables (even though in the case of laser fields these conditions may be known to hold over tens of thousands of miles).

CORRELATION FUNCTIONS FOR CHAOTIC FIELDS

A particularly important class of stationary fields, which arises whenever the source is essentially chaotic in nature, is one in which the weight function in the P-representation is a product of Gaussian factors, one for each mode. The density operator is then specified by

$$P(\{\alpha_k\}) = \prod_k \frac{1}{\pi \langle n_k \rangle} e^{-\frac{|\alpha_k|^2}{\langle n_k \rangle}}, \tag{14.19}$$

and it follows that all of the statistical properties of the field are determined by the set of average occupation numbers $\langle n_k \rangle$. The knowledge of this same set of numbers, on the other hand, is equivalent, according to Eq. (14.13), to specifying the first order correlation function for the field. There thus exists a fundamental sense in which the first order correlation function furnishes all the information we need for the description of fields specified by Gaussian weight functions. We may demonstrate this simplifying property more explicitly by showing that all of the higher order correlation functions for such fields can be expressed as sums of products of first order correlation functions.

In order to prove this theorem we shall construct a species of generating functional for the set of all correlation functions of the field. The essential tool for doing this is the operation of functional differentiation. If $F[\zeta(x)]$ is a functional of $\zeta(x)$, i.e. a function of the set of values of $\zeta(x)$ for all x, then we define its functional derivative with respect to $\zeta(x_0)$ to be

$$\frac{\delta F}{\delta \zeta(x_0)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{F[\zeta(x) + \epsilon \delta^{(4)}(x - x_0)] - F[\zeta(x)]\right\}, \tag{14.20}$$

where $\delta^{(4)}$ is a four-dimensional (space-time) delta function. As an illustration, if we apply this definition to an integral operator of the form

$$F = \int \zeta(x)\ E^{(-)}(x)\, d^4x, \tag{14.21}$$

we find

$$\frac{\delta F}{\delta \zeta(x_0)} = \int \delta^{(4)}(x - x_0)\ E^{(-)}(x)\ d^4x = E^{(-)}(x_0). \tag{14.22}$$

Now, let us define the generating functional

$$\Xi[\zeta(x), \eta(x)] = \mathrm{Tr}\left\{\rho\, e^{\int \zeta(x) E^{(-)}(x) d^4x}\ e^{\int \eta(x') E^{(+)}(x') d^4x'}\right\}, \tag{14.23}$$

which depends upon two independent functions $\zeta(x)$ and $\eta(x)$ and is the trace of a normally ordered product. Then we easily see that the functional derivatives of this expression, evaluated for $\zeta(x) = \eta(x) = 0$, are the correlation functions of the field; i.e. we have

$$\frac{\delta^2}{\delta \zeta(x_1)\, \delta \eta(x_2)} \Xi \Big|_{\zeta = \eta = 0} = \mathrm{Tr}\left\{\rho\ E^{(-)}(x_1)\ E^{(+)}(x_2)\right\} = G^{(1)}(x_1, x_2), \tag{14.24}$$

and more generally

$$\frac{\delta^{2n}}{\delta \zeta(x_1) \cdots \delta \zeta(x_n)\, \delta \eta(x_{n+1}) \cdots \delta \eta(x_{2n})} \Xi \Big|_{\zeta = \eta = 0} = G^{(n)}(x_1 \cdots x_n, x_{n+1} \cdots x). \tag{14.25}$$

(The tensor indices which have been suppressed in these expressions may be restored by considering each coordinate x to specify a component index as well as a position and a time, e.g. the function $\zeta(x)$ is actually a set of four functions $\zeta_\mu(r, t)$ for $\mu = 1, \cdots 4$, etc.)

It is convenient, at this point, to introduce the abbreviation

$$e(x, k) = i\left\{\frac{\hbar \omega_k}{2}\right\}^{\frac{1}{2}} u_k(r)\ e^{-i\omega_k t}, \tag{14.26}$$

which permits us to write the expansion of the operator $E^{(+)}$ in terms of the mode functions as

$$E^{(+)}(x) = \sum_k e(x, k)\ a_k. \tag{14.27}$$

Then when we use the P-representation for the density operator with the Gaussian weight function (14.19), the generating functional (14.23) may be written as

$$\Xi = \int e^{-\sum_k \frac{|\alpha_k|^2}{\langle n_k \rangle}}\ e^{\sum_k \int \zeta(x) e^*(x,k) \alpha_k^* d^4x}\ e^{\sum_k \int \eta(x') e(x',k) \alpha_k d^4x'}\ \prod \frac{d^2 \alpha_k}{\pi \langle n_k \rangle}. \tag{14.28}$$

This multiple integral factors into a product of integrals, one for each mode k. If we introduce the pair of complex parameters

$$\beta_k = \int \zeta(x)\ e^*(x, k)\ d^4x$$

$$\gamma_k = \int \eta(x')\ e(x', k)\ d^4x', \tag{14.29}$$

the integral factor for the k-th mode takes the familiar form

$$\int \exp\left\{-\frac{|\alpha_k|^2}{\langle n_k\rangle} + \beta_k\,\alpha_k^* + \gamma_k\,\alpha_k\right\}\frac{d^2\alpha_k}{\pi\langle n_k\rangle} = \exp\{\beta_k\,\gamma_k\langle n_k\rangle\}\,. \qquad (14.30)$$

Hence the generating functional is given by the product

$$\Xi = \prod_k \exp\{\beta_k\,\gamma_k\langle n_k\rangle\}$$
$$= \exp\Big\{\int \zeta(x)\sum_k e^*(x,\,k)\,e(x',\,k)\,\langle n_k\rangle\,\eta(x')\,d^4x\,d^4x'\Big\}. \qquad (14.31)$$

Now, according to Eqs. (14.13) and (14.26), the first order correlation function for the field is given by the expansion

$$G^{(1)}(x,\,x') = \sum_k e^*(x,\,k)\;e(x',\,k)\;\;\langle n_k\rangle, \qquad (14.32)$$

which is just the sum which occurs in the exponential function of Eq. (14.31). Hence the generating functional for the correlation functions of all orders may be expressed in terms of the first order correlation function as

$$\Xi[\zeta(x),\,\eta(x)] = \exp\Big\{\int \zeta(x)\,G^{(1)}(x,\,x')\,\eta(x')\;d^4x\,d^4x'\Big\}. \qquad (14.33)$$

We may now derive explicit expressions for the higher order correlation functions by evaluating the appropriate functional derivatives. In particular the n-th derivative with respect to ζ may be written as

$$\frac{\delta^n}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)}\,\Xi = \Big\{\prod_{j=1}^{n}\int G^{(1)}(x_j,\,x')\,\eta(x')\,d^4x'\Big\}\,\Xi\;. \qquad (14.34)$$

To evaluate the n-th order correlation function we must next differentiate n times with respect to the function η. Since $\zeta(x)$ is finally to be set equal to zero it is easy to see that all of the terms which come from differentiating the factor Ξ on the right side of Eq. (14.34) with respect to η will finally vanish. Hence we have simply

$$\frac{\delta^{2n}}{\delta\zeta(x_1)\cdots\delta\zeta(x_n)\,\delta\eta(x_{n+1})\cdots\delta\eta(x_{2n})}\,\Xi\Big|_{\substack{\zeta=0\\ \eta=0}}$$

$$= \frac{\delta^n}{\delta\eta(x_{n+1})\cdots\delta\eta(x_{2n})}\;\prod_{j=1}^{n}\int G^{(1)}(x_j,\,x')\,\eta(x')\,d^4x'$$

$$= \sum_P\prod_{j=1}^{n} G^{(1)}(x_j,\,x_{P(n+j)})\,; \qquad (14.35)$$

i.e., the derivative is a sum taken over the n! possible ways of permuting the set of coordinates $x_{n+1},\cdots x_{2n}$. Since the derivative we have evaluated, according to Eq. (14.25), is the n-th order correlation function, we have finally

$$G^{(n)}(x_1\cdots x_n,\;x_{n+1}\cdots x_{2n}) = \sum_P\prod_{j=1}^{n} G^{(1)}(x_j,\;x_{P(n+j)})\,. \qquad (14.36)$$

The n-th order correlation function for Gaussian fields is just a symmetrical sum of products of first order correlation functions.

To illustrate this result for the second order correlation function we may write

$$G^{(2)}(x_1x_2,\;x_3x_4) = G^{(1)}(x_1x_3)\,G^{(1)}(x_2x_4) + G^{(1)}(x_1x_4)\,G^{(1)}(x_2x_3)\,. \qquad (14.37)$$

Now if the field in question possesses first order coherence, we may write the first order correlation function in the factorized form of Eq. (7.15). The two terms of Eq. (14.37) are then equal and we find

$$G^{(2)}(x_1x_2,\;x_3x_4) = 2\,\mathcal{E}^*(x_1)\,\mathcal{E}^*(x_2)\,\mathcal{E}(x_3)\,\mathcal{E}(x_4)\,. \qquad (14.38)$$

The second order correlation function factorizes, but because of the presence of the factor of 2, it does so in a way which precludes the possibility that the field has second or higher order coherence. The n-th order correlation function for such fields in evidently given by

$$G^{(n)}(x_1\cdots x_{2n}) = n!\;\prod_{j=1}^{n}\mathcal{E}^*(x_j)\;\prod_{j=n+1}^{2n}\mathcal{E}(x_j)\,. \qquad (14.39)$$

QUASIPROBABILITY DISTRIBUTION FOR THE FIELD AMPLITUDE

Whenever the density operator for the field may be specified by means of the P-representation the function $P(\{\alpha_k\})$ plays a role analogous to that of a probability density for the individual mode amplitudes α_k. Of course when we make measurements upon a light beam, we are typically measuring not the individual amplitudes α_k, but the average values of various functions of the complex field strength eigenvalue, $\mathcal{E}(rt)$, which is a particular linear sum of the mode amplitudes,

$$\mathcal{E}(x,\{\alpha_k\}) = \sum_k e(x,\,k)\;\alpha_k. \qquad (14.40)$$

To describe the fullest variety of such measurements which we can make at a single space-time point $x = (r,\,t)$, it is convenient to derive from $P(\{\alpha\})$ a species of reduced quasiprobability distribution for the complex field amplitude $\mathcal{E}(x,\{\alpha_k\})$. This distribution function for the field amplitude will be quite useful in discussing the origin of the photon correlation effect discovered by Hanbury Brown, and Twiss.

To illustrate the kinds of averages we frequently want to discuss, let us note that the average intensity of the field at the point x is

$$G^{(1)}(x,\,x) = \int P(\{\alpha_k\})\;\Big|\,\mathcal{E}(x\{\alpha_k\})\,\Big|^2\;\prod_k d^2\alpha_k, \qquad (14.41)$$

and the average coincidence rate for the limiting case in which the two counters are placed at the same point and are sensitive at the same time is

$$G^{(1)}(x\,x,\,x\,x) = \int P(\{\alpha_k\})\;\Big|\,\mathcal{E}(x\{\alpha_k\})\,\Big|^4\;\prod_k d^2\alpha_k. \qquad (14.42)$$

These are examples of a general class of averages which take the form

$$\int P(\{\alpha_k\})\;F(\mathcal{E}(x\{\alpha_k\}))\;\prod_k d^2\alpha_k \qquad (14.43)$$

for suitably determined functions F. It is convenient now to separate the multidimensional integration over the complex amplitude parameters α_k into two steps, the integration over the subspace of the α_k-parameters in which the linear combination

$$\mathcal{E}(x\{\alpha_k\}) = \sum e(x,k)\,\alpha_k$$

remains constant, and then the further integration over the values this sum may take on. The first of these integrations is accomplished by defining the function

$$W(\mathcal{E},\,x) = \int P(\{\alpha_k\})\;\delta^{(2)}\Big(\mathcal{E} - \sum_k e(x,\,k)\;\alpha_k\Big)\;\prod_k d^2\alpha_k. \qquad (14.44)$$

We may then write the complete integral (14.43) in the form

$$\int P(\{\alpha_k\})\, F(\mathcal{E}(x\{\alpha_k\}))\prod_k d^2\alpha_k = \iint P(\{\alpha_k\})\,\delta^{(2)}\Big(\mathcal{E} - \sum_k e(x,k)\alpha_k\Big)\times F(\mathcal{E})\prod_k d^2\alpha_k d^2\mathcal{E}$$

$$= \int W(\mathcal{E}, x)\, F(\mathcal{E})\, d^2\mathcal{E}, \qquad (14.45)$$

where $d^2\mathcal{E} = d(\mathrm{Re}\,\mathcal{E})\, d(\mathrm{Im}\,\mathcal{E})$ is a real element of area in the complex field amplitude plane. The function $W(\mathcal{E}, x)$ defined by Eq. (14.44) evidently plays a role analogous to that of a probability distribution for the complex field amplitude at the space-time point x. Of course, since the function P from which it is derived is only a quasiprobability distribution, and is subject to all the restrictions mentioned in the last lecture, the same limitations will apply to the physical interpretation of the function $W(\mathcal{E}, x)$. It too can take on negative values, for example.

The function W furnishes a particularly simple description of fields which consist of many independently excited modes. Since the total field amplitude $\mathcal{E}$ is then the sum of a large number of independently distributed complex amplitudes proportional to the α_k, the distribution of the amplitude $\mathcal{E}$ will correspond to that of the endpoint of a many-step random walk in the complex plane. This distribution tends to take on a Gaussian form when the number of contributing modes is large, no matter how the mode amplitudes may be distributed individually. From a mathematical standpoint this argument differs hardly at all from the discussion of the central limit theorem given in Section VIII of the reprinted paper; i.e., the starting point, Eq. (14.44), becomes similar in structure to Eq. (R 8.1) when the function $P(\{\alpha_k\})$ is assumed to factorize into a product $\prod_k P_k(\alpha_k)$. As a slight generalization of the discussion given there we may let the individual mode excitations be non-stationary in character and have mean amplitudes

$$\int P_k(\alpha_k)\, \alpha_k\, d^2\alpha_k = \langle \alpha_k \rangle . \qquad (14.46)$$

Then by applying the central limit theorem, we find

$$W(\mathcal{E}, x) = \frac{1}{\pi \Sigma_k |e(x,k)|^2 \{\langle |\alpha_k|^2\rangle - |\langle \alpha_k\rangle|^2\}} \times \exp\left\{ - \frac{|\mathcal{E} - \Sigma_k\, e(x,k)\langle \alpha_k\rangle|^2}{\Sigma_k |e(x,k)|^2 \{\langle |\alpha_k|^2\rangle - |\langle \alpha_k\rangle|^2\}} \right\} . \qquad (14.47)$$

If the mean amplitudes $\langle \alpha_k \rangle$ vanish, as they do for example in the case of stationary fields, we have

$$W(\mathcal{E}, x) = \frac{1}{\pi \Sigma_k |e(x,k)|^2 \langle n_k\rangle} \exp\left\{ - \frac{|\mathcal{E}|^2}{\Sigma_k |e(x,k)|^2 \langle n_k\rangle} \right\}$$

$$= \frac{1}{\pi\, G^{(1)}(x,x)}\, e^{-\frac{|\mathcal{E}|^2}{G^{(1)}(x,x)}} . \qquad (14.48)$$

To illustrate the use of this expression for $W(\mathcal{E}, x)$, let us calculate the n-th order correlation function with all arguments equal. By letting $F(\mathcal{E}) = |\mathcal{E}|^{2n}$ in Eq. (14.45) we find

$$G^{(n)}(x \cdots x) = \int W(\mathcal{E}, x)\, |\mathcal{E}|^{2n}\, d^2\mathcal{E} . \qquad (14.49)$$

For the Gaussian form of W given by Eq. (14.48), the latter integral is simply

$$G^{(n)}(x \cdots x) = n!\{G^{(1)}(x, x)\}^n . \qquad (14.50)$$

An important class of fields which obey the separability conditions we have assumed in deriving these results is that specified by the Gaussian density operators discussed earlier. For these fields, in fact, Eq. (14.50) follows directly from Eq. (14.36). But since we have not had to assume that the functions $P_k(\alpha_k)$ are individually Gaussian in form to derive Eqs. (14.48) and (14.50), these results evidently hold true for a considerably broader variety of field excitations.

A sketch of the Gaussian distribution function $W(\mathcal{E}, x)$ is given in Fig. 13. Since this function plays a role akin to that of a probability distribution for the complex field amplitude $\mathcal{E}$, it is evident that the absolute magnitude of the field undergoes a considerable amount of fluctuation. Thus, while the most probable value of the field amplitude is $\mathcal{E} = 0$, the amplitude will occasionally stray out

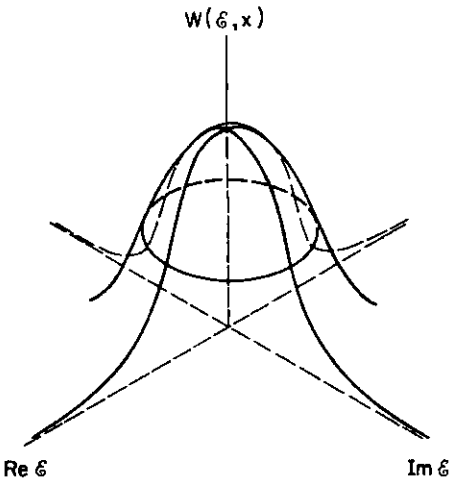

Figure 13

into the regions of the complex plane which represent the "tail" of the Gaussian and correspond to arbitrarily strong fields. The relation (14.50) between values of the correlation functions may also be stated as the relation

$$\langle |\mathcal{E}|^{2n} \rangle = n!\, \{\langle |\mathcal{E}|^2 \rangle\}^n \qquad (14.51)$$

between average moments $\langle |\mathcal{E}|^j \rangle$ of the function W. The extremely rapid increase with n of the ratio $\langle |\mathcal{E}|^{2n} \rangle / \{\langle |\mathcal{E}|^2 \rangle\}^n$, which the Gaussian distribution shows, is due to its "long-tailed" character.

Although the Gaussian form for the function $W(\mathcal{E}, x)$ will presumably apply almost universally to the radiation from natural or essentially chaotic sources, altogether different distributions may be required to describe the radiation from certain man-made sources. In fact the avoidance of fields which have the extremely random or noisy character of the Gaussian form of $W(\mathcal{E}, x)$ has been one of the major goals of radio-frequency technology. One of its earliest accomplishments was the development of oscillators which generate fields of extremely stable

modulus, e.g. broadcast carrier waves. These oscillators are non-linear devices and the contributions of the various mode amplitudes to the total field are not at all independently distributed as in the Gaussian case. For a stationary field generated by such an oscillator we might find the function $W(\mathcal{E}, x)$ to assume a form similar to that shown in Fig. (14); i.e., the modulus of the field, $|\mathcal{E}|$,

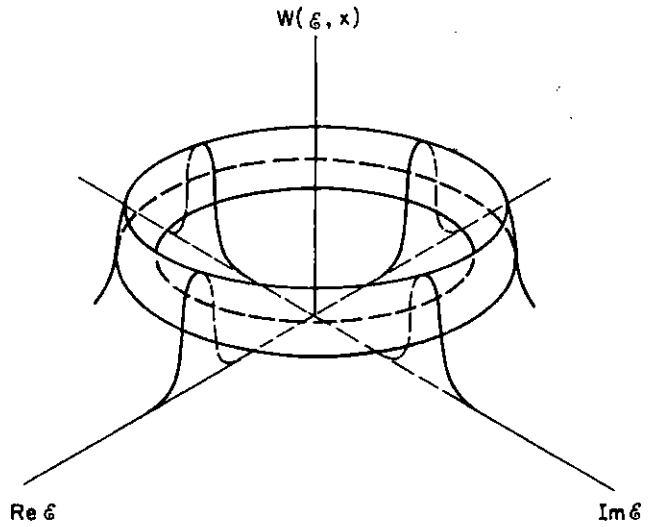

Figure 14

has only a very small probability for taking on values either appreciably smaller or larger than its root-mean-square value, $\{<|\mathcal{E}|^2>\}^{\frac{1}{2}}$.

The shape of the function $W(\mathcal{E}, x)$ furnishes an elementary insight into the origin of the photon correlation effect which was discovered by Hanbury Brown and Twiss by means of the experiment described in Lecture VIII. Let us consider the two-fold coincidence counting rate for photons when the two detectors D_1 and D_2 of Fig. 9 occupy precisely symmetrical positions relative to the half-silvered mirror m, and when the detectors are adjusted so that they register coincidences with no time delay. Since the arrangement is one in which the counters, in effect, occupy the same position and are sensitive at the same time, the coincidence rate is given by a correlation function of the form

$$G^{(2)}(x\,x,\ x\,x) = <|\mathcal{E}(x)|^4>. \tag{14.52}$$

Now, according to Eqs. (14.50) or (14.51), for all chaotic light sources we should find

$$<|\mathcal{E}(x)|^4> = 2<|\mathcal{E}(x)|^2>^2 = 2\{G^{(1)}(x,\ x)\}^2. \tag{14.53}$$

The amount by which $G^{(2)}(x\,x,\ x\,x)$ exceeds $\{G^{(1)}(x,\ x)\}^2$ is a measure of the non-random tendency of the photons to be recorded as simultaneously arriving pairs; i.e., it is a measure of the height of the "bump" on the coincidence rate curve shown in Fig. 10. Since the coincidence rate for zero time delay is twice

the background or accidental coincidence rate, the correlation effect is not a small one. (The original observations of the effect were made difficult by the relatively long response times of the counting systems compared with the time interval over which the correlation persists.)

To see the nature of the photon correlation effect for other types of distributions $W(\mathcal{E}, x)$, let us note that it is proportional to

$$G^{(2)}(x\,x,\ xx) - \{G^{(1)}(x,\ x)\}^2 = <|\mathcal{E}(x)|^4> - <|\mathcal{E}(x)|^2>^2 = \int W(\mathcal{E}, x)\{|\mathcal{E}|^2 - <|\mathcal{E}|^2>\}^2\, d^2\mathcal{E}. \tag{14.54}$$

One of the curious quantum mechanical properties of this expression is that, although it resembles a statistical variance for the quantity $|\mathcal{E}|^2$, it may actually take on negative as well as positive values. That is true since $W(\mathcal{E}, x)$ as we have noted, is not strictly speaking a probability distribution. It is not difficult to find states of the field for which W takes on negative values at least locally and for which the average (14.54) is consequently negative. When the field is in such states photon coincidences will be recorded with less than the random background rate by the Hanbury Brown-Twiss detection apparatus, an effect which is the reverse of the one observed for natural radiation sources.

Whenever the field is generated by an essentially classical source, i.e., one with predetermined behavior, it will be possible, as we have seen Lecture XII, to construct a P-representation for the density operator with a non-negative weight function $P(\{\alpha_k\})$. Then the function $W(\mathcal{E}, x)$ defined by Eq. (14.44) will likewise take on no negative values. We may thus state that for all classically generatable fields, the Hanbury Brown-Twiss correlation is positive,

$$G^{(2)}(x\,x,\ x\,x) - \{G^{(1)}(x,\ x)\}^2 \geq 0. \tag{14.55}$$

If the correlation effect is to vanish for fields of this type we must evidently have

$$W(\mathcal{E}, x)\{|\mathcal{E}|^2 - <|\mathcal{E}|^2>\}^2 = 0 \tag{14.56}$$

for all $\mathcal{E}$. The function $W(\mathcal{E}, x)$ can therefore only take on non-vanishing values at points lying on the circle $|\mathcal{E}|^2 = <|\mathcal{E}|^2>$. If the function $W(\mathcal{E}, x)$, in other words, is of a form which allows no amplitude modulation of the field, the correlation effect will vanish and conversely. In fact in that limit we have more generally

$$G^{(n)}(x\cdots x) = <|\mathcal{E}|^{2n}> = <|\mathcal{E}|^2>^n = \{G^{(1)}(x,\ x)\}^n \tag{14.57}$$

and all n-fold coincidence experiments show an absence of any tendency toward statistical correlations.

A number of the published discussions of the Hanbury Brown-Twiss effect explain it as being caused by the fact that photons are Bose particles and consequently have a certain tendency to cluster. That such explanations are far from complete is made evident by the fact that the quantum mechanical form of the effect may have either sign; it may constitute an anticorrelation or "replusion," rather than a positive correlation or "clumping." Furthermore the fact that classical fields have only a positive correlation effect is a clear demonstration that the average quantities one evaluates by means of the correlation functions (even where the P-representation exists) are not always equivalent in quantum theory and classical theory. The variety of fields encountered in the quantum theory is simply much larger than that allowed by classical theory.

It should be evident that the measurement of the photon correlation effect, at least at zero delay time, simply furnishes a measure of the amount of random

amplitude modulation present in fields with positive $W(\mathcal{E}, x)$. The effect should be nearly absent from the field generated by a well stabilized oscillator. In particular since a gas laser operating well above its threshold is presumably quite a stable oscillator, any Hanbury Brown-Twiss correlation found in its beam should be quite small in magnitude.

The fact that a photon correlation experiment, or its analogue in the radio-frequency region, an intensity correlation experiment, can furnish a simple way of telling whether a radiation field comes from a natrual source or a man-made one could have some interesting if rather far-fetched astronomical consequences. If intelligent beings elsewhere in the galaxy want to communicate with us, it seems reasonable to suppose that they would use amplitude-stabilized oscillators of some sort as radiators. In that case their signals, as we have seen, would have an unmistakable character even when no message was being transmitted. In fact the unmodulated signal could be easier to distinguish from background noise than the modulated one.

QUASIPROBABILITY DISTRIBUTION FOR THE FIELD AMPLITUDES AT TWO SPACE-TIME POINTS

A number of the correlation functions and other expectation values which interest us depend on the fields at two different space-time points x_1 and x_2. These averages may be expressed, when the P-representation exists, in the general form

$$\int P(\{\alpha_k\})\ \ F(\mathcal{E}(x_1\{\alpha_k\}),\ \mathcal{E}(x_2\{\alpha_k\}) \prod_k d^2\alpha_k, \tag{14.58}$$

where the function F is suitably defined for each case. Two familiar examples of such averages are the first order correlation function $G^{(1)}(x_1, x_2)$, for which we would choose

$$F = \mathcal{E}^*(x_1\{\alpha_k\})\ \mathcal{E}(x_2\{\alpha_k\}), \tag{14.59}$$

and the delayed coincidence counting rate, $G^{(2)}(x_1x_2, x_2x_1)$, for which we would choose

$$F = |\mathcal{E}(x_1\ \{\alpha_k\})|^2 \quad |\mathcal{E}(x_2\ \{\alpha_k\})|^2. \tag{14.60}$$

Now, if we define a species of distribution function $W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)$, for the complex field amplitudes at the two points by means of the relation

$$W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2) = \int P(\{\alpha_k\})\,\delta^{(2)}(\mathcal{E}_1 - \mathcal{E}(x_1\{\alpha_k\}))\,\delta^{(2)}(\mathcal{E}_2 - \mathcal{E}(x_2\{\alpha_k\}))\prod_k d^2\alpha_k, \tag{14.61}$$

then an average quantity of the form (14.58) is given by the integral

$$\int W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)\ F(\mathcal{E}_1, \mathcal{E}_2)\,d^2\mathcal{E}_1 d^2\mathcal{E}_2. \tag{14.62}$$

The function $W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)$, more specifically, is a quasiprobability distribution which plays the same role in averaging functions of two space-time variables as the function $W(\mathcal{E}, x)$, which we discussed earlier, plays in the calculation of averages for a single space-time point. We may, in fact obtain $W(\mathcal{E}, x)$ from the two-point function by integrating over either of the field variables,

$$\begin{aligned} W(\mathcal{E}, x) &= \int W(\mathcal{E}x,\ \mathcal{E}'x')\ d^2\mathcal{E}' \\ &= \int W(\mathcal{E}'x',\ \mathcal{E}x)\ d^2\mathcal{E}'. \end{aligned} \tag{14.63}$$

When the function $P(\{\alpha_k\})$ factorizes into a product of independent weight functions, one for each mode, and when the number of excited modes is large, it is easy to show, again by techniques similar to those used in section VIII of the reprinted paper, that $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$ assumes a Gaussian form in the two complex amplitude variables $\mathcal{E}_1$ and $\mathcal{E}_2$. To carry out the derivation we simply show that the double Fourier transform of $W(\mathcal{E}_1x_1, \mathcal{E}_2x_2)$ with respect to the amplitude variables $\mathcal{E}_1$ and $\mathcal{E}_2$ is asymptotically Gaussian in form when the number of excited modes becomes infinite. Inversion of the transform then yields a result which, for the case of stationary fields, can be written as

$$W(\mathcal{E}_1x_1, \mathcal{E}_2x_2) = \frac{1}{\pi^2 G^{(1)}(x_1x_1)G^{(1)}(x_2x_2)\{1 - |g^{(1)}(x_1x_2)|^2\}} \times$$

$$\exp\left\{-\frac{\dfrac{|\mathcal{E}_1|^2}{G^{(1)}(x_1x_1)} + \dfrac{|\mathcal{E}_2|^2}{G^{(1)}(x_2x_2)} - 2\,\mathrm{Re}\,\dfrac{\mathcal{E}_1\mathcal{E}_2^* g^{(1)}(x_1x_2)}{\{G^{(1)}(x_1x_1)G^{(1)}(x_2x_2)\}^{\frac{1}{2}}}}{1 - |g^{(1)}(x_1x_2)|^2}\right\} \tag{14.64}$$

where $g^{(1)}$ is the normalized form of the first order correlation function defined by Eq. (7.5). As a simple check of this result it is easy to verify that the average of the function (14.59) is

$$\{G^{(1)}(x_1x_1)G^{(2)}(x_2x_2)\}^{\frac{1}{2}}\, g^{(1)}(x_1x_2) = G^{(1)}(x_1x_2) \tag{14.65}$$

as required, and that the average of the function (14.60) is indeed

$$G^{(1)}(x_1,x_1)G^{(1)}(x_2,x_2) + |G^{(1)}(x_1,\ x_2)|^2 = G^{(2)}(x_1x_2,\ x_2x_1). \tag{14.66}$$

The function $W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)$ plays a role in the theory which is analogous to that of a probability density for a compound event, i.e., finding the field $\mathcal{E}_1$ at $x_1 = (r_1, t_1)$ and $\mathcal{E}_2$ at $x_2 = (r_2, t_2)$. In probability theory it is often of interest, in dealing with such compound events, to imagine that the first part of the event has already taken place and to calculate the probability that the compound event is then completed. We may define an analogue of such a conditioned probability function by means of the relation

$$W(\mathcal{E}_1x_1 \mid \mathcal{E}_2x_2) = \frac{W(\mathcal{E}_1x_1,\ \mathcal{E}_2x_2)}{W(\mathcal{E}_1,\ x_1)}, \tag{14.67}$$

where $W(\mathcal{E}_1,\ x_1)$ is the function defined by Eq. (14.44). The function $W(\mathcal{E}_1x_1 \mid \mathcal{E}_2x_2)$ is analogous to a probability density for the field amplitude to have values in the neighborhood of $\mathcal{E}_2$ at $x_2 = (r_2, t_2)$, given that it had the value $\mathcal{E}_1$ at $x_1 = (r_1, t_1)$. We shall call the function the conditioned quasiprobability density; it is, strictly speaking, only measurable as a probability density in the classical or strong field limit.

When we calculate the ratio of the functions given by Eqs. (14.64) and (14.48) we find the result

$$W(\mathcal{E}_1x_1 \mid \mathcal{E}_2x_2) = \frac{1}{\pi}\,\frac{1}{G^{(1)}(x_2x_2)\{1 - |g^{(1)}(x_1x_2)|^2\}} \times \tag{14.68}$$

$$\exp\left\{-\frac{\left|\dfrac{\mathcal{E}_2}{\{G^{(1)}(x_2x_2)\}^{\frac{1}{2}}} - \dfrac{\mathcal{E}_1}{\{G^{(1)}(x_1x_1)\}^{\frac{1}{2}}}\, g^{(1)}(x_1x_2)\right|^2}{1 - |g^{(1)}(x_1x_2)|^2}\right\}$$

for the conditioned quasiprobability distribution. The field $\mathcal{E}_2$ in other words, has a Gaussian distribution about the mean value

$$\langle \mathcal{E}_2 \rangle = \mathcal{E}_1 \frac{G^{(1)}(x_1 x_2)}{G^{(1)}(x_1 x_1)} \tag{14.69}$$

with a dispersion proportional to $G^{(2)}(x_2 x_2)\{1 - |g(x_1 x_2)|^2\}$, which vanishes for x_2 near x_1 and tends to approach $G^{(2)}(x_2 x_2)$ as x_2 recedes from x_1. We shall examine these expressions more closely once we have illustrated the evaluation of the correlation functions on which they depend.

Lecture XV ELEMENTARY MODELS OF LIGHT BEAMS

Since our results to this point have all been stated in fairly general terms, it may be of help to discuss an illustrative example or two. Let us consider, as a particularly simple example, a stationary light beam which may be thought of as a plane wave progressing along the positive y-axis. We shall allow the beam to have an arbitrary frequency bandwidth, but shall take it to have a specific polarization $\hat{e}$. The first order correlation function for the beam may then be evaluated as a sum over plane wave mode functions by means of Eq. (14.13). The index which labels the mode functions in this case may be taken to k_y, the y-component of the propagation vector. (The other components vanish.) Since the values of k_y are densely distributed, when the size L of the quantization volume is large, the sum over k_y is equivalent to a one-dimensional integration

$$\sum_{k_y} \rightarrow \frac{L}{2\pi} \int dk_y \ldots .$$

When the mode functions given by Eq. (R 2.9) are substituted in Eq. (14.13) and the sum is replaced by an integral, we find

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{\hbar c}{4\pi} \int_0^\infty \frac{\langle n_k \rangle}{L^2} k \exp\{-i[k_y (y_1 - y_2) - \omega_k (t_1 - t_2)]\} dk_y , \tag{15.1}$$

where $G^{(1)}$ is understood to be a correlation function for the field components in the direction $\hat{e}$, as in Eq. (4.21). Since the beam contains no backward travelling waves, (which would be represented by negative values of k_y,) we may write the integral equally well as one over the frequency variable $\omega_k = ck_y$. Then if we introduce the parameter

$$s = t_1 - t_2 + \frac{1}{c}(y_1 - y_2) \tag{15.2}$$

to express the space-time interval which occurs as an argument, we may write

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{1}{4\pi c} \int_0^\infty \frac{\langle n_k \rangle \hbar \omega_k}{L^2} e^{i\omega_k s} d\omega_k . \tag{15.3}$$

The expression $\langle n_k \rangle \hbar \omega_k$, which occurs in the integrand of Eq. (15.3), is the average energy of excitation of the k-th mode. Let us assume, as an example, that our beam has a spectral profile of the Lorentz form by writing

$$\frac{\langle n_k \rangle \hbar \omega_k}{cL^2} = \frac{2\gamma}{(\omega - \omega_0)^2 + \gamma^2} U. \tag{15.4}$$

Here ω_0 is the central frequency, γ is the half-width at half height, and the constant U is a measure of the intensity of the beam. Since the frequency ω_0 is typically much larger than γ, only a very small numerical error is made in the integration over the spectral porfile if the lower limit $\omega = 0$ in Eq. (15.3) is replaced by $\omega = -\infty$. By making this approximation and letting $\omega' = \omega - \omega_0$ we find

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{\gamma}{2\pi} U e^{i\omega_0 s} \int_{-\infty}^{\infty} \frac{e^{i\omega' s}}{\omega'^2 + \gamma^2} d\omega' . \tag{15.5}$$

The singularities of the function

$$\frac{1}{\omega'^2 + \gamma^2} = \frac{1}{2i\gamma} \left\{ \frac{1}{\omega' - i\gamma} - \frac{1}{\omega' + i\gamma} \right\} \tag{15.6}$$

are a pair of simple poles lying at $\pm i\gamma$ in the complex ω'-plane. The integral in Eq. (15.5) can be written as a contour integral around a closed path in the ω'-plane in either of two simple ways, depending on the sign of the variable s. For $s > 0$ the contour may be closed by means of an infinite semicircle in the upper half plane ($\mathrm{Im}\ \omega' > 0$); for $s < 0$ it may be closed by a semicircle in the lower half plane. Since the integrals along both semicircles vanish, we find by applying the residue theorem

$$\int_{-\infty}^{\infty} \frac{1}{2i\gamma} \left\{ \frac{1}{\omega - i\gamma} - \frac{1}{\omega + i\gamma} \right\} e^{i\omega s} d\omega = 2\pi i \begin{cases} \frac{1}{2i\gamma} e^{-\gamma s}, & s > 0 \\ \frac{1}{2i\gamma} e^{\gamma s}, & s < 0. \end{cases} \tag{15.7}$$

The first order correlation function, according to Eq. (15.5), is therefore given by

$$G^{(1)}(y_1 t_1, y_2 t_2) = \frac{1}{2} U e^{i\omega_0 s - \gamma |s|} . \tag{15.8}$$

The intensity of the field is found by letting $y_1 = y_2$ and $t_1 = t_2$. For these values of the coordinates, which correspond to $s = 0$, we have

$$G^{(1)}(y_1 t_1, y_1 t_1) = \frac{1}{2} U. \tag{15.9}$$

This is the average of the squared magnitude of the complex field $E^{(+)}$. It is easy to see, if we recall the formulae of elementary electrodynamics, that the parameter U is equal to the average total of the electric and magnetic energy densities for the field.

The correlation function given by Eq. (15.8) shows that our light beam exhibits approximate first order coherence when its frequency band width γ is sufficiently small. Thus, when we have

$$\frac{1}{\gamma} \gg |s| = \left| t_1 - t_2 - \frac{1}{c}(y_1 - y_2) \right| , \tag{15.10}$$

the factor $e^{-\gamma|s|}$ in Eq. (15.8) may be approximated by unity, and the remainder of the expression for the correlation function may be written in the appropriate factorized form. As an alternative way of discussing first order coherence we note that the normalized form of the correlation function is

$$g^{(1)}(y_1 t_1, y_2 t_2) = \frac{G^{(1)}(y_1 t_1, y_2 t_2)}{\{G^{(1)}(y_1 t_1, y_1 t_1) G^{(1)}(y_2 t_2, y_2 t_2)\}^{\frac{1}{2}}}$$

$$= \exp[\, i\omega_0 s - \gamma |s| \,]. \tag{15.11}$$

This function indeed has absolute magnitude close to unity as long as $\gamma|s|$ is sufficiently small.

A good deal of attention has been directed experimentally to the problem of developing light sources with narrow line width. In the best of these sources of the ordinary gas discharge or chaotic variety γ is of the order of 10^9 cycles per second. In ordinary laboratory sources it is often of order 10^{11} cycles per second or larger. The corresponding coherence ranges are 30 cm. and .3 cm. respectively.

Although we have been discussing the way in which monochromaticity may imply coherence, it may be worth recalling that it is not a necessary condition even for first order coherence. The coherence condition only becomes linked to a requirement of monochromaticity when we restrict our consideration to stationary fields, as we noted in connection with Eq. (7.24). For the case of stationary laser beams, the range of first order coherence is determined by the spectral bandwidth just as for ordinary sources. For the case of gas lasers it is possible to reduce the band width γ to values of the order of 10^3 cycles per second without too much difficulty, and it seems possible to achieve frequency stabilization to within about 10 cycles per second over brief intervals. The coherence ranges corresponding to these band widths are 300 km, and 30,000 km. respectively.

Before we can calculate the second and higher order correlation functions for our light beam, we must specify its statistical nature somewhat further. It is at this point that the descriptions of beams generated by natural sources and those generated by coherent sources become qualitatively different. Let us assume that our source is of the usual chaotic variety. Then the higher order correlation functions may all be expressed as sums of products of first order correlation functions, as we have seen in Eq. (14.36). The spectral density function of our plane wave beam, in other words, completely determines the statistical properties of the field. In particular the delayed coincidence rate for counting pairs of photons is given by

$$G^{(2)}(y_1t_1,\ y_2t_2,\ y_2t_2,\ y_1t_1) = G^{(1)}(y_1t_1,\ y_1t_1)\ G^{(1)}(y_2t_2, y_2t_2) + |G^{(1)}(y_1t_1, y_2t_2)|^2$$

$$= G^{(1)}(y_1t_1,\ y_1t_1)\ G^{(1)}(y_2t_2,\ y_2t_2)\{1 + |g^{(1)}(y_1t_1, y_2t_2)|^2\}$$

$$= \left(\tfrac{1}{2}U\right)^2 \{\,1 + e^{-2\gamma|s|}\,\}\ . \qquad (15.12)$$

The presence of the term $e^{-2\gamma|s|}$ in this expression shows that the beam can never possess second order coherence. Furthermore when we plot the coincidence rate against s as in Fig. 15 we see that that term constitutes the "bump" on the Hanbury Brown-Twiss correlation curve, i.e. the deviation of the curve from the accidental or background coincidence rate. The experimental curve shown earlier in Fig. 10 corresponds to a curve of the form shown here after the resolution properties of the counter system have been folded in.

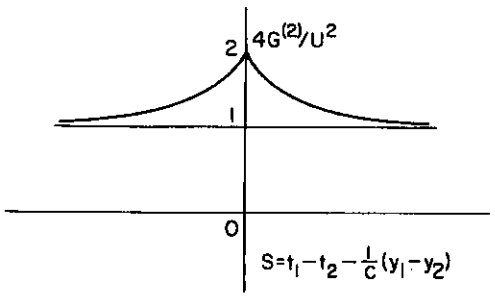

Figure 15

We have noted in the last lecture that the origin of the correlation effect lies in the random amplitude modulation of our light beam. Thus the factors of n! by which the n-fold coincidence rate (at zero time delay) exceeds the random coincidence rate are easily explained in terms of the moments of the Gaussian amplitude distribution $W(\mathcal{E}, x)$ given by Eq. (14.48). To understand the behavior of the correlation effect for non-vanishing time delays, and to see, for example, why the effect disappears for $|s| \gg 1/2\gamma$, we may make use of the quasiprobability distributions defined for pairs of values of the field amplitude in the last lecture. When we substitute the values given by Eqs. (15.8) and (15.11) for the correlation functions into the expression (14.65) for the conditioned quasiprobability function $W(\mathcal{E}_1x_1|\mathcal{E}_2x_2)$, we find

$$W(\mathcal{E}_1y_1t_1|\mathcal{E}_2y_2t_2) = \frac{1}{\frac{1}{2}\pi U(1 - e^{-2\gamma|s|})} \exp\left\{-\frac{|\mathcal{E}_2 - \mathcal{E}_1 e^{i\omega_0 s - \gamma|s|}|^2}{\frac{1}{2}U\ (1 - e^{-2\gamma|s|})}\right\}. \qquad (15.13)$$

This function is to be interpreted as the distribution of values of the field amplitude $\mathcal{E}_2$ at y_2t_2, when the amplitude is known to take on the value $\mathcal{E}_1$ at y_1t_1. When the parameter s vanishes, the mean radius of the Gaussian peak of this expression vanishes and the distribution reduces to the delta function $\delta^{(2)}(\mathcal{E}_2 - \mathcal{E}_1)$. As $|s|$ increases from zero, the mean value of $\mathcal{E}_2$, which is given by $\mathcal{E}_1 e^{i\omega_0 s - \gamma|s|}$, describes an exponential spiral in the complex $\mathcal{E}_2$-plane while relaxing to the value zero. The spiral which corresponds to $s < 0$ is shown in exaggerated form in Fig. 16. At the same time the mean squared radius of the Gaussian peak of the

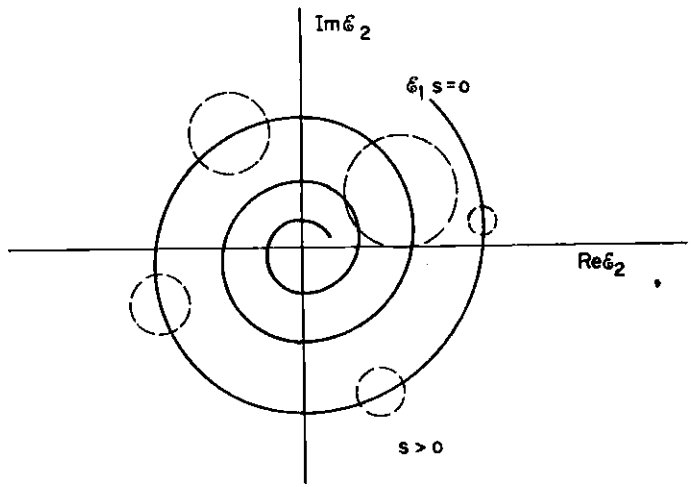

Figure 16

distribution increases to the asymptotic value (1/2)U. For values of $|s|$ much greater than $1/\gamma$ the conditioned distribution (15.13) relaxes to a form centered on the origin, which is simply the unconditioned distribution $W(\mathcal{E}_2, y_2t_2)$ given by Eq. (14.48). The time $1/\gamma$ is a relaxation time for the field amplitude distributions. Our knowledge of $\mathcal{E}_1$ ceases to have much influence on the distribution of $\mathcal{E}_2$ for $|s| > 1/\gamma$. It is not surprising then that for intervals for which $|s| \gg 1/\gamma$ the two-photon coincidence rate, which is given by

$$G^{(2)}(y_1t_1y_2t_2, y_2t_2y_1t_1) = \int W(\mathcal{E}_1y_1t_1,\ \mathcal{E}_2y_2t_2)\,|\mathcal{E}_1|^2\,|\mathcal{E}_2|^2 d^2\mathcal{E}_1 d^2\mathcal{E}_2$$

$$= \int W(\mathcal{E}_1y_1t_1)\,W(\mathcal{E}_1y_1t_1\,|\,\mathcal{E}_2y_2t_2)\,|\mathcal{E}_1|^2\,|\mathcal{E}_2|^2 d^2\mathcal{E}_1 d^2\mathcal{E}_2\ , \tag{15.14}$$

reduces to the factorized form

$$G^{(2)}(y_1t_1y_2t_2,\ y_2t_2y_1t_1) = G^{(1)}(y_1t_1,\ y_1t_1)\,G^{(1)}(y_2t_2,\ y_2t_2).$$

The tendency toward photon coincidences is wiped out, in other words, when the interval $s = t_1 - t_2 - c^{-1}(y_1 - y_2)$ becomes large because the field amplitudes $\mathcal{E}(y_1t_1)$ and $\mathcal{E}(y_2t_2)$ cease to be statistically correlated.

To see how the full time dependence of the coincidence rate emerges from the integral (15.14), we note that when the conditioned distribution function is given by Eq. (15.13), the average value of $|\mathcal{E}_2|^2$ when $\mathcal{E}_1$ is fixed is

$$\int W(\mathcal{E}_1y_1t_1\,|\,\mathcal{E}_2y_2t_2)\,|\mathcal{E}_2|^2 d^2\mathcal{E}_2 = |\mathcal{E}_1|^2 e^{-2\gamma|s|} + \tfrac{1}{2}U(1 - e^{-2\gamma|s|}). \tag{15.16}$$

When this expression is multiplied by $|\mathcal{E}_1|^2$ and averaged, as in Eq. (15.14), over the Gaussian form for $W(\mathcal{E}_1y_1t_1)$, we find

$$G^{(2)}(y_1t_1,\ y_2t_2,\ y_2t_2,\ y_1t_1) = (\tfrac{1}{2}U)^2\left\{2e^{-2\gamma|s|} + 1 - e^{-2\gamma|s|}\right\}$$

$$= (\tfrac{1}{2}U)^2\left\{1 + e^{-2\gamma|s|}\right\}, \tag{15.17}$$

which verifies the value of the coincidence rate found earlier in Eq. (15.12).

The values we have derived for the correlation functions have all been based on the assumption that the energy spectrum of our light beam has the Lorentz shape. The corresponding results are easily derived for other spectra for which the Fourier transform of the energy distribution is known. Other simple, smooth representations of the profile of a spectrum line, for example, lead to results which are qualitatively similar to those for the Lorentz line.

Since the photon correlation effect extends over delay times of the order of the inverse band width, γ, it might appear that this time can be stretched out by a factor of a million or more by using the extremely monochromatic light of the laser rather than light from natural sources. The error in such reasoning lies in the fact that the statistical properties of the laser beam are quite different from those of the chaotically generated beams we have been discussing. Lasers, when they are operating most monochromatically, generate beams with very little amplitude modulation, and for these, as we have seen in the last lecture, there would be virtually no photon correlation effect at all.

MODEL FOR IDEAL LASER FIELDS

For fields generated by chaotic sources, knowledge simply of the average occupation numbers $< n_k >$ is sufficient to determine the density operator ρ, and from it all of the statistical properties of the field. However if our source is not chaotic in nature, we cannot expect that there will exist any self-evident way of finding the density operator for the field it generates without analyzing the mechanism by which it radiates in some detail. The only reliable method we have of constructing density operators, in general, is to devise theoretical models of the system under study and to integrate corresponding Schrödinger equation, or equivalently to solve the equation of motion for the density operator. These assignments are formidable ones for the case of the laser oscillator and have not been carried out to date in quantum mechanical terms. The greatest part of the difficulty lies

in the mathematical complications associated with the nonlinearity of the device. The nonlinearity plays an essential role in stabilizing the field generated by the laser. It seems unlikely, therefore, that we shall have a quantum mechanically consistent picture of the frequency bandwidth of the laser or of the fluctuations of its output until further progress is made with these problems.

If we are willing to overlook the noise and band width problems for the moment, and to confine our discussion to the case of an ideally monochromatic laser, then it is not difficult to find a representation for the density operator of the beam it generates. The radiation field is coupled within the laser to the electric dipole vectors of all of the atoms of the active medium. These atoms have a polarization which oscillates with the field and at the same time radiates energy into it. If we view the active medium as a whole, we see that it has an oscillating polarization density of macroscopic proportions, i.e., all neighboring atoms contribute similarly to the total polarization density. If we remember that the time derivative of a polarization density is, in effect, a current distrbution, then we may think of the field as being radiated by the oscillating current distribution. When the laser is operating well above its threshold there is nothing weak about this current distribution; it is essentially of classical magnitude. Furthermore, if the laser has the ideal stability we have assumed, the current simply oscillates steadily in a perfectly predictable way. We may, in other words, to an excellent approximation, describe the bound current in the active medium as a c-number current density.

The general problem of finding the fields radiated by prescribed current distributions has been solved in Lecture XII. The most important property of the solution is that radiation by a known current distribution always brings the field to a coherent state (assuming that no other radiation was present initially). If the current oscillates with a single frequency, only the field modes with precisely that frequency will be excited. If we assume, for simplicity, that the geometry of our system favors the excitation of only one mode of the field, then the density operator for the field may be written in the form

$$\rho = |\alpha><\alpha|, \tag{15.18}$$

where $|\alpha>$ is a coherent state for the excited mode, and the amplitude α is given by an integral of the form (12.20) taken over the bound current distribution.

Let us write the complex field eigenvalue which corresponds to the amplitude α as

$$\mathcal{E}(rt) = i\left(\frac{\hbar\omega}{2}\right)^{\frac{1}{2}} u(r)\, e^{-i\omega t}\,\alpha\ . \tag{15.19}$$

Then, since the density operator (15.18) corresponds to a pure coherent state, the correlation functions of all orders will factorize to the form of Eq. (8.5), i.e., the beam will possess full coherence. It follows then that the n-fold delayed coincidence rates will factorize to the form

$$G^{(2)}(x_1 \ldots x_n,\ x_n \ldots x_1) = \prod_{j=1}^{n} G^{(1)}(x_j,\ x_j), \tag{15.20}$$

and no photon coincidence correlations of any order will be detectable in the ideal laser beam.

The argument which led to the density operator (15.18) for the laser beam assumed that the oscillating current distribution is known precisely i.e., that we know its phase of oscillation as well as its amplitude. In practice our knowledge about quantities which oscillate at extremely high frequencies rarely includes any information about their absolute phase. (This is due more to the absence of a suitable clock to use as a reference standard than it is to any difficulty of principle in defining or measuring the phase of essentially classical quantities such as the

bound current in the laser.) When we lack any knowledge of the phase of oscillation of the current, the density operator should be written in an appropriately specialized form of Eq. (12.30). It is clear that this form is simply the expression (15.8) for the density operator averaged over the phase of the complex amplitude α, i.e.,

$$\rho = \int_0^{2\pi} | \, |\alpha| e^{i\theta} > < |\alpha| e^{i\theta} | \frac{d\theta}{2\pi}$$

$$= \int \frac{1}{2\pi|\alpha|} \delta(|\beta| - |\alpha|) \, |\beta><\beta| d^2\beta . \qquad (15.21)$$

These forms of the density operator depend on α only through its absolute value, and hence represent stationary fields. They represent mixed rather than pure states of the field, but as we have noted in the last lecture, mixtures corresponding to averaging an overall phase variable do not alter the coherence properties of the field. It is easy to verify that the correlation functions which are derived from the density operator (15.21) are identical to those which follow from (15.18).

The explicit construction of the density operator for an ideal laser beam shows that no photon correlations are to be detected in such a beam. The reason for the absence of such correlations is evident from the analysis of the last lecture. The quasiprobability function $W(\mathcal{E}, x)$ which corresponds to the stationary density operator (15.21) is immediately seen from Eqs. (14.44) and (15.19) to be

$$W(\mathcal{E}, x) = \frac{1}{2\pi(\frac{\hbar\omega}{2})^{\frac{1}{2}}|u(r)\alpha|} \delta(|\mathcal{E}| - (\frac{\hbar\omega}{2})^{\frac{1}{2}}|u(r)\alpha|) . \qquad (15.22)$$

This function vanishes everywhere in the complex $\mathcal{E}$-plane except on a circle where the delta function is singular. It describes a field which undergoes no amplitude modulation at all, and that is the basic reason for the absence of photon correlations in an ideal laser beam.

It is also possible, by making use of the correspondence principle, to see the origin of this property of coherently radiated beams more directly. We shall simplify our picture of the laser by regarding it simply as an oscillating charge distribution which radiates much as an antenna does. The charge, we assume, has only a single mode of vibration whose amplitude is, in effect, that of a harmonic oscillator. Since the electric polarization of this oscillator assumes macroscopic proportions we must regard the oscillator coordinate as an essentially classical quantity; i.e., the oscillator is typically in highly excited quantum states which have enormous quantum numbers.

When the oscillator is decoupled from whatever mechanism has excited it and allowed to radiate spontaneously, its amplitude of vibration will decrease quite slowly in relation to the oscillation period. Since the behavior of the oscillator is essentially classical, the current due to its moving charge distribution is quite predictable. As we have noted earlier, the radiation by such a current brings the field to a coherent state. If, on the other hand, we look at the oscillator from a quantum mechanical standpoint, we may think of it as making transitions downward in energy, step by step, passing through states with quantum numbers n, n - 1, n - 2 $\cdots$ where $n \gg 1$. The length of time the oscillator spends in each of these states is distributed exponentially and, since n is so large, the average lifetimes of the states do not vary significantly from one state to the next. Each transition is accompanied by the emission of a photon. We are therefore not surprised to find that when the photons are detected by a counter, the intervals between their successive arrival times are exponentially distributed. This exponential distribution of time intervals indicates the absence of any tendency toward pair or higher order correlations. It is the characteristic distribution for the intervals between totally

uncorrelated events which happen at a fixed average rate. It is clear that where two or more counters are used there will be no time-dependent correlations of their outputs.

MODEL OF A LASER FIELD WITH FINITE BANDWIDTH

An acutal laser beam, in contrast to the ideal variety we have just discussed, will never be precisely monochromatic. Its frequency is bound to vary more or less randomly over a narrow range due to disturbances which have their origin both inside and outside the laser itself. We shall construct a simple model of a laser field with finite frequency bandwidth by assuming that the mechanicsm which disturbs the laser is essentially stochastic is nature.

Let us assume, for simplicity, that the laser excites only a single mode of the electromagnetic field which has frequency ω_0. Then the field Hamiltonian for that mode is

$$H_0 = \hbar\omega_0 a^\dagger a$$

and, in the absence of any perturbing influences, the time-dependent operators $a(t)$ and $a^\dagger(t)$ are given in terms of the time-independent ones, a and $a^\dagger$, by

$$a(t) = a\, e^{-i\omega_0 t} \qquad (15.23)$$

$$a^\dagger(t) = a^\dagger e^{i\omega_0 t} ,$$

The completely harmonic behavior of the oscillating field will be perturbed by various interactions of the field with other systems. We shall assume that the effect of these interactions can be represented by the addition of a term to the field Hamiltonian which depends on one or more random functions of time, $f(t)$. If we write this stochastic addition to the Hamiltonian as $H_f(t)$, the total field Hamiltonian becomes

$$H = H_0 + H_f(t) . \qquad (15.24)$$

To see the influence of the stochastic term most clearly we shall solve the Schrödinger equation in the interaction representation. The interaction Hamiltonian is then

$$H_f'(t) = e^{\frac{i}{\hbar}H_0 t} H_f(t)\, e^{-\frac{i}{\hbar}H_0 t} . \qquad (15.25)$$

We define the unitary operator $U_f(t, t')$ as the solution of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} U_f(t, t') = H_f'(t)\, U_f(t, t') \qquad (15.26)$$

which obeys the initial condition

$$U_f(t', t') = 1. \qquad (15.27)$$

Then, if we write the state vector of the field at time t as $|t>$, we see that it evolves according to the transformation

$$|t> = U_f(t, t')\, |t'> .$$

The equation of motion for the density operator in the interaction representation, which we shall write as $\rho_I(t)$, is

$$i\hbar \frac{\partial}{\partial t} \rho_I(t) = [H_f'(t), \rho_I(t)] . \qquad (15.28)$$

The solution for the time development of the density operator may be written in terms of the unitary operator U_I as

$$\rho_I(t) = U_I(t, t')\,\rho_I(t')\,U_I^{-1}(t, t'). \tag{15.29}$$

The expressions for the field correlation functions which we have discussed earlier in these lectures have all been constructed according to Heisenberg picture of quantum mechanics in which the state vectors and the density operator are independent of time. When these vary with time, as in the interaction representation, the expectation values we require must be constructed somewhat differently. The required expressions can be found by starting with the form the expectation values take in the Heisenberg representation and carrying out the unitary transformation to the interaction representation.

Let us consider two arbitrary operators which take the time-dependent forms $L(t)$ and $M(t)$ in the Heisenberg representation. An example of the kind of statistical average which is used in the construction of the correlation functions is the averaged product which may be written as $\langle L(t)M(t')\rangle_I$. The subscript on the average means that it is computed for a particular behavior of the random function $f(t)$ on which the stochastic Hamiltonian dependens. The average, when evaluated in the Heisenberg representation, is clearly

$$\langle L(t)M(t')\rangle_I = \mathrm{Tr}\{L(t)M(t')\rho\} \tag{15.30}$$

where ρ is the time-independent Heisenberg density operator.

One of the ways of defining the Heisenberg representation (which is unitarily equivalent to all other ways) is to let the fixed Heisenberg state vector for the system be identical to the state vector in the interaction representation at a particular time t_0. Then the relation

$$|t\rangle = U_I(t, t_0)\ |t_0\rangle \tag{15.31}$$

expresses the unitary transformation from Heisenberg states $|t_0\rangle$ to states $|t\rangle$ in the interaction representation. The corresponding transformations of the operators L, M and ρ are

$$\begin{aligned} L_I(t) &= U_I(t, t_0)\ L(t)\,U_I^{-1}(t, t_0) \\ M_I(t) &= U_I(t, t_0)\ M(t)\ U_I^{-1}(t, t_0) \\ \rho_I(t) &= U_I(t, t_0)\ \rho\ U_I^{-1}(t, t_0), \end{aligned} \tag{15.32}$$

where the subscripts i denote the forms of the operators in the interaction representation. When the inverted forms of these relations are used to express the operators in Eq. (15.30) we find

$$\langle L(t)M(t')\rangle_I = \mathrm{Tr}\{U_I^{-1}(t, t_0)\,L_I(t)\,U_I(t, t_0)\,U_I^{-1}(t', t_0)\,M_I(t')\times \rho_I(t')\,U_I(t', t_0)\}. \tag{15.33}$$

Since the time displacement operator U_I obeys the multiplication law

$$U_I(t, t')\,U_I(t', t_0) = U_I(t, t_0), \tag{15.34}$$

the expression for the average may be reduced to the form

$$\langle L(t)M(t')\rangle_I = \mathrm{Tr}\{L_I(t)\,U_I(t, t')\,M_I(t')\rho_I(t')\,U_I^{-1}(t, t')\}. \tag{15.35}$$

The occurrences of the operator U_I in this expression evidently take into account the effect of the disturbance of the field during the interval from t' to t. The disturbance, we are assuming, is a random one and the average (15.35) has been evaluated for some particular way in which it may behave, i.e., it is evaluated for a particular random function $f(t)$. Before the average can be compared with experiments it must again be averaged over a suitable ensemble of random functions $f(t)$. The latter averaging process is simplified by our use of the interaction representation.

Since the products LM which interest us are in normally ordered form it will be extremely convenient to make use of the P-representation for the density operator. We shall therefore only consider the class of stochastic Hamiltonians which preserve the possibility of expressing the density operator by means of the P-representation. We assume, in other words, that $\rho_I(t)$ may be written in the form

$$\rho_I(t) = \int P(\alpha, t)\,|\alpha\rangle\langle\alpha|\,d^2\alpha \tag{15.36}$$

at all times t.

If the density operator at time t' corresponds to the pure coherent state $|\alpha\rangle$, i.e.

$$\rho_I(t') = |\alpha\rangle\langle\alpha|, \tag{15.37}$$

then, according to Eq. (15.29), at time t it will be

$$\begin{aligned} \rho_I(t) &= U_I(t, t')\,\rho_I(t')\,U_I^{-1}(t, t') \\ &= U_I(t, t')\,|\alpha\rangle\langle\alpha|\,U_I^{-1}(t, t'). \end{aligned} \tag{15.38}$$

Now, according to Eq. (15.36), this operator too will have a P-representation for which we may introduce the special notation

$$\rho_I(t) = \int P(\alpha t'|\beta t)\ |\beta\rangle\langle\beta|\,d^2\beta\ . \tag{15.39}$$

The function $P(\alpha t'|\beta t)$ is evidently a conditioned quasiprobability function. It corresponds in the classical limit to a probability distribution for the complex amplitude β at time t, when we are given the knowledge that it had (or will have) the value α at time t'.

To illustrate the use of these relations in evaluating statistical averages, let us consider the average of the product $a^\dagger(t)\,a(t')$ which occurs in the first order correlation function. If we substitute $L(t) = a^\dagger(t)$ and $M(t) = a(t)$ into Eq. (15.35) we find, by using Eq. (15.23)

$$\langle a^\dagger(t)\,a(t')\rangle_I = \mathrm{Tr}\{a^\dagger e^{i\omega_0 t}\,U_I(t, t')\,a\,e^{-i\omega_0 t'}\rho_I(t')\,U_I^{-1}(t, t')\}. \tag{15.40}$$

Next we make use of Eq. (15.36) for the density operator, and the fact that $|\alpha\rangle$ is an eigenstate of a to write

$$\langle a^\dagger(t)\,a(t')\rangle_I = \mathrm{Tr}\{U_I(t, t')\int P(\alpha t')\,\alpha|\alpha\rangle\langle\alpha|d^2\alpha U_I^{-1}(t, t')\,a^\dagger\}\,e^{i\omega_0(t-t')}. \tag{15.41}$$

The unitary transformation inside the brackets may now be carried out by using Eq. (15.39) to represent the density operator indicated in Eq. (15.38). We then have

$$\begin{aligned} \langle a^\dagger(t)\,a(t')\rangle_I &= \mathrm{Tr}\{\int P(\alpha t')\,\alpha P(\alpha t'|\beta t)\,|\beta\rangle\langle\beta|\,\beta^* d^2\alpha d^2\beta\}\,e^{i\omega_0(t-t')} \\ &= \int P(\alpha t')\,P(\alpha t'|\beta t)\,\alpha\beta^*\,d^2\alpha\ d^2\beta\ e^{i\omega_0(t-t')}\ . \end{aligned} \tag{15.42}$$

The latter expression for the average bears a close resemblance to forms which occur in the classical theory of continuous Markoff processes. We must now remember that the average we have constructed corresponds to some particular behavior of the random Hamiltonian. The quantity to be compared with experiment is not any one such value, but the average of all such values taken over a suitable ensemble of random functions f(t). We may write this average as

$$< a^\dagger(t)\, a(t') >= \int < P(\alpha t')\, P(\alpha t' \,|\beta t) >_{\substack{\text{av.} \\ \text{over f}}} \alpha\beta^*\, d^2\alpha d^2\beta e^{\,i\omega_0(t-t')} \,. \qquad (15.43)$$

The foregoing equations furnish us with a fairly general framework for discussing the influence of random disturbances on the oscillations of the field. We shall now use this formalism in constructing a simple model of a laser beam of finite bandwidth.

Surely the simplest way to give the oscillating mode of the field a finite frequency bandwidth is to assume that its frequency is a random function of time. We may do this by writing the total field Hamiltonian of Eq. (15.24) as

$$H = \hbar[\,\omega_0 + f(t)\,]\, a^\dagger a \,, \qquad (15.44)$$

where f(t) is a random function of some sort whose ensemble average, $<f(t)>$, vanishes.

Since the random Hamiltonian is evidently

$$H_f(t) = \hbar f(t)\, a^\dagger a, \qquad (15.45)$$

and it commutes with $H_0 = \hbar\omega a^\dagger a$, the interaction Hamiltonian according to Eq. (15.25) is simply H_f itself.

The Schrödinger equation (15.26) then takes the form

$$i\frac{\partial}{\partial t} U_f(t,\, t') = f(t)\, a^\dagger a U_f(t,\, t') \,. \qquad (15.46)$$

Its solution is simply an exponential function which may be written in the form

$$U_f(t,\, t') = e^{-ia^\dagger a \phi(tt')} \qquad (15.47)$$

where ϕ is defined by

$$\phi(tt') = \int_{t'}^{t} f(t'')\, dt'' \,. \qquad (15.48)$$

To see the effect of the transformation U_f on the states of the field, let us suppose that the field is in the coherent state $|\alpha>$ at time t'. Then at time t the state will be

$$\begin{aligned} |t> &= U_f(t,\, t')\, |\alpha> \\ &= e^{-ia^\dagger a \phi(tt')}\, |\alpha> \\ &= e^{-\frac{1}{2}|\alpha|^2} e^{-ia^\dagger a \phi(tt')} \sum_{n=0}^{\infty} \{\alpha^n/(n!)^{\frac{1}{2}}\} |n> \\ &= e^{-\frac{1}{2}|\alpha|^2} \sum \frac{\alpha^n}{(n!)^{\frac{1}{2}}} e^{-in\phi(tt')}\, |n> \\ &= |\alpha e^{-i\phi(tt')} > . \end{aligned} \qquad (15.49)$$

The particular random Hamiltonian we have assumed just transforms one coherent state into another for which the amplitude parameter differs from the original one

by a phase factor. There is evidently no amplitude modulation in this model at all.

When we use Eq. (15.46) to construct the density operator represented by Eqs. (15.38) and (15.39) we find

$$|\alpha e^{-i\phi(tt')}><\alpha e^{-i\phi(tt')}| \;= \int P(\alpha t' \,|\beta t)\, |\beta><\beta|\, d^2\beta, \qquad (15.50)$$

from which we see that we may take the conditioned quasiprobability density to be simply the delta function

$$P(\alpha t' \,|\beta t) \;= \delta^{(2)}(\beta - \alpha e^{-i\phi(tt')}) \,. \qquad (15.51)$$

If we introduce the phases of the amplitudes α and β via the definitions

$$\begin{aligned} \alpha &= |\alpha|\, e^{i\theta_0} \\ \beta &= |\beta|\, e^{i\theta} \,, \end{aligned} \qquad (15.52)$$

then the two-dimensional delta function (15.51) can be written in terms of a product of two one-dimensional ones as

$$P(\alpha t' \,|\beta t) = \frac{1}{|\alpha|}\, \delta(|\beta| - |\alpha|)\; \delta(\theta - \theta_0 + \phi(tt')) \,. \qquad (15.53)$$

This function describes the evolution of the state of the field from the coherent state $|\alpha>$ at time t', when we are given any particular random function f(t). To find the state at time t which is typical of the set of possible random functions, we must average Eq. (15.53) over the ensemble of functions f(t). We may write this average as

$$P_{av}\,(\alpha t' \,|\beta t) = \frac{1}{|\alpha|}\; \delta(|\beta| - |\alpha|) < \delta(\theta - \theta_0 + \phi(tt'))>_{\text{av over f}} \,. \qquad (15.54)$$

Now, if we recall that the function $\delta(\theta)$ has the Fourier series expansion

$$\delta(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\theta} \,, \qquad (15.55)$$

we see that the averaged delta function in Eq. (15.54) may be written as

$$\delta(\theta) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{\,im(\theta-\theta_0)} < e^{\,im\int_{t'}^{t} f(t'')\, dt''} >_{\text{av over f}} \,. \qquad (15.56)$$

We must clearly specify some of the properties of the random functions f(t) before the exponential functions in Eq. (15.56) can be averaged over them.

The different physical processes which may perturb the frequency of our field oscillator require in general that we discuss various kinds of random functions f(t). For the present, however, we shall only consider one of the simpler types of random functions. We shall assume that f(t) is a stationary Gaussian stochastic process, i.e., that at any time t the ensemble of values of f(t) has a fixed Gaussian distribution. Then it is not difficult to show that the averaged exponentials in Eq. (15.56) are given by

$$< \exp\{\, im\int_{t'}^{t} f(t'')\, dt''\} >_{\substack{\text{av} \\ \text{over f}}} = \exp\{-\frac{1}{2} m^2 \int_{t'}^{t}\int_{t'}^{t} < f(t'')\, f(t''')> dt''\, dt'''\}, \qquad (15.57)$$

where the ensemble average $< f(t'')\, f(t''') >$ is simply the auto-correlation function of the random process f(t).

Let us assume, simply as an illustration, that the function f(t) fluctuates so

rapidly that its autocorrelation function can be taken to have the form

$$\langle f(t'')f(t''')\rangle = 2\zeta\,\delta(t'' - t''') \tag{15.58}$$

where ζ is a positive constant. Then the averaged exponential in Eq.(15.57) reduces to

$$\langle \exp\{im\int_{t'}^{t} f(t'')dt''\}\rangle_{\substack{av\\over\ t}} = \exp\{-m^2\zeta|t - t'|\}, \tag{15.59}$$

and the averaged delta function in Eq. (15.56) becomes

$$\langle \delta(\theta - \theta_0 + \phi(tt'))\rangle_{\substack{av\\over\ t}} = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\theta-\theta_0)-m^2\zeta|t-t'|}\ . \tag{15.60}$$

It is interesting to note that this function is simply the Green's function of the partial differential equation for the diffusion of heat on a circular ring, i.e., it satisfies the equation

$$\left(\frac{\partial}{\partial t} - \zeta\frac{\partial^2}{\partial\theta^2}\right)\langle \delta(\theta - \theta_0 + \phi)\rangle_{av} = 0$$

for $t > t'$ and reduces to $\delta(\theta - \theta_0)$ for $t = t'$. It is clear then that the conditioned quasiprobability function (15.54), which we may write as

$$P_{av}(\alpha\, t'|\beta t) = \frac{1}{2\pi|\alpha|}\,\delta(|\beta| - |\alpha|)\sum_{m=-\infty}^{\infty} e^{im(\theta-\theta_0)-m^2\zeta|t-t'|} \tag{15.61}$$

describes a kind of random phase modulation in which the phase variable $\theta = \arg\beta$ "diffuses" away from its initial value, θ_0.

The reciprocal of the diffusion constant ζ defines a relaxation time for the phase variable. For time intervals $t - t'$ which greatly exceed $1/\zeta$ the distribution (15.61) reduces to a constant, circularly symmetric form; the phase θ becomes completely random.

Let us now return to the question of evaluating the first order correlation function for the field. According to Eq. (15.43) we may construct the function as soon as we have evaluated the average

$$\langle P(\alpha,\ t')\,P(\alpha t'|\,\beta t)\rangle_{\substack{av\\over\ t}}\ . \tag{15.62}$$

We shall assume that we have no knowledge of the initial phase of oscillation of the field. Since the random perturbation of the field only shifts its phase, the phase remains uniformly distributed at all times; i.e., we never know more about the phase than we did initially. The density operator which represents the field is therefore stationary. The function $P(\alpha,\ t)$ in Eq. (15.36) depends on α only through its absolute value and is independent of t, and of the behavior of the function f(t) as well. In this most frequently occurring case, the function $P(\alpha,\ t')$ may be written as $P(|\alpha|)$ and removed from the averaging brackets in the expression (15.62). That expression then reduces to the form

$$P(|\alpha|)\ P_{av}(\alpha t'|\beta t)\ , \tag{15.63}$$

where the second factor is given by Eq. (15.61).

Now it is evident from Eq. (15.61) that

$$\int P_{av}(\alpha t'\ |\beta t)\ \beta^* d^2\beta = \frac{1}{2\pi|\alpha|}\int_0^{\infty}\delta(|\beta| - |\alpha|)\,|\beta|^2 d|\beta|\int_0^{2\pi} e^{-i\theta}\ \times$$

$$\sum e^{im(\theta-\theta_0)-m^2\zeta|t-t'|}\ d\theta$$

$$= |\alpha|\,e^{-i\theta_0-\zeta|t-t'|} = \alpha^* e^{-\zeta|t-t'|}\ . \tag{15.64}$$

On substituting the expression (15.63) into the correlation function (15.43) and making use of the integral just evaluated we find

$$\langle a^\dagger(t)a(t')\rangle = \int P(|\alpha|)\,|\alpha|^2 d^2\alpha\, e^{i\omega_0(t-t')-\zeta|t-t'|}$$

$$= \langle|\alpha|^2\rangle\, e^{i\omega_0(t-t')-\zeta|t-t'|}\ , \tag{15.65}$$

where the symbol $\langle|\alpha|^2\rangle$ has been used for the mean squared amplitude of excitation, or equivalently the average number of photons in the mode.

If we assume that the mode function $u(r)$ for the field does not change as a result of the perturbation, then the full space-time dependence of the first order correlation function may be found by multiplying the expression (15.65) by a product of the form $u^*(r)\,u(r')$. According to Eq. (R 10. 17), which is a quantum mechanical form of the Wiener-Khintchine theorem, the energy spectrum of the field will be porportional to the Fourier transform of the correlation function (15.65). When we calculate the transform we find

$$\int_{-\infty}^{\infty}\langle a^\dagger(0)a(t')\rangle e^{i\omega t'}dt' = \langle|\alpha|^2\rangle\int_{-\infty}^{\infty} e^{i(\omega-\omega_0)t'-\zeta|t'|}\,dt'$$

$$= \langle|\alpha|^2\rangle\ \frac{2\zeta}{(\omega - \omega_0)^2 + \zeta^2}\ . \tag{15.66}$$

Our phase diffusion model thus has an energy spectrum of Lorentzian shape, and the diffusion constant ζ is its half-width.

From a spectroscopic standpoint, the field we are describing could not be distinguished from the chaotically generated field of Lorentzian line shape which we discussed earlier, if we happened to have $\zeta = \gamma$. The fundamentally different nature of these two fields is best expressed by means of their higher order correlation functions. These functions may be evaluated for the phase diffusion model through simple extensions of the methods we have developed, but we shall not do so here. One fairly obvious result, however, is worth mentioning. Since the random phase modulation we have described carries no amplitude modulation with it, it will not introduce any photon coincidence correlations.

There are a number of ways in which the simple phase diffusion model which we have presented as an illustration can be generalized and made more realistic. We may easily remove, for example, the assumption that the stochastic process f(t) has a vanishingly small relaxation time. Furthermore we may consider other types of stochastic processes than Gaussian ones. Finally, we may consider other forms of the random Hamiltonian than (15.45) and attempt in that way to account for some of the effects of random amplitude modulation as well as phase modulation.

Lecture XVI — INTERFERENCE OF INDEPENDENT LIGHT BEAMS

One of the questions having to do with coherence which has given rise to much discussion and a certain amount of confusion recently is that of interference between independent light beams. That such interference phenomena can exist should come as no great surprise; they have been observed long ago with radio waves of fixed frequency. If we have had to wait until recently [1] to see such phenomena at optical frequencies, the delay has been wholly due to instrumental difficulties.

The problems which have arisen in the discussion of these interference phenomena concern the precise way in which they should be understood and described. It would be quite difficult to say how much of the misunderstanding we have mentioned is simply semantic in nature and how much is more deeply conceptual. There

is, for example, nothing intrinsically quantum mechanical about the interference of independent beams. Yet the fact that altogether different sets of quanta must somehow interfere with one another seems to have contributed greatly to the confusion. We shall not recount the history of this subject here but shall only discuss a few of the simplest possible examples of the interference phenomenon.

The simplest sort of experimental arrangement we can have is essentially that illustrated in Fig. 17. Two independent laser sources (or possibly other types of sources), L_1 and L_2 project their beams in directions which are nearly parallel, but slightly convergent. The beams fall upon overlapping areas of a screen Σ. If the light intensities are high enough, or we have sufficient time available to record over a long period, we may let our detector be a photographic film in the plane Σ. If the conditions do not favor photography, on the other hand, we might use a mosaic of photon counters in the plane Σ. In either case we will look for interference fringes in the area of overlap of the beams.

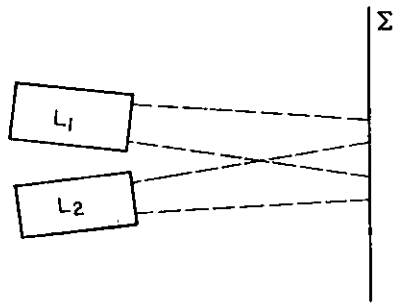

Figure 17

Let us assume that the way in which each light source excites the field can be described in the P-representation by means of functions $P_1(\{\alpha_{1k}\})$ and $P_2(\{\alpha_{2k}\})$. The single P-function which describes the superposed fields is then given, according to Eq. (R7.18) or (R9.15), by

$$P(\{\alpha_k\}) = \int P_1(\{\alpha_{1k}\}) P_2(\{\alpha_{2k}\}) \prod_k \delta^{(2)}(\alpha_k - \alpha_{1k} - \alpha_{2k})\, d^2\alpha_{1k}\, d^2\alpha_{2k}. \tag{16.1}$$

The average intensity of the superposed fields at any space-time point x is given by the first order correlation function

$$\begin{aligned} G^{(1)}(x, x) &= \int P(\{\alpha_k\}) \left| \mathcal{E}(x, \{\alpha_k\}) \right|^2 \prod_k d^2\alpha_k \\ &= \int P_1(\{\alpha_{1k}\})\, P_2(\{\alpha_{2k}\}) \left| \mathcal{E}(x, \{\alpha_{1k} + \alpha_{2k}\}) \right|^2 \prod_k d^2\alpha_{1k} d^2\alpha_{2k}. \end{aligned} \tag{16.2}$$

In reaching the second of these expressions we have made use of Eq. (16.1) and have carried out the integrations over the variables $\{\alpha_k\}$. Now let us note that the eigenvalue field $\mathcal{E}(x, \{\alpha_k\})$ depends linearly upon the amplitudes α_k so that we have

$$\mathcal{E}(x, \{\alpha_{1k} + \alpha_{2k}\}) = \mathcal{E}(x, \{\alpha_{1k}\}) + \mathcal{E}(x, \{\alpha_{2k}\}), \tag{16.3}$$

a statement which corresponds to the classical superposition principle. If we substitute this relation in Eq. (16.2), and let the symbols $\{G^{(1)}(x, x)\}_j$ with $j = 1, 2$, be the intensities which would be produced by either source in the absence of the other, then we may write the total intensity as

$$\begin{aligned} G^{(1)}(x, x) &= \{G^{(1)}(x, x)\}_1 + \{G^{(1)}(x, x)\}_2 \\ &+ 2\mathrm{Re}\left\{ \int P_1(\{\alpha_{1k}\})\, \mathcal{E}^*(x, \{\alpha_{1k}\}) \prod_k d^2\alpha_{1k} \int P_2(\{\alpha_{2k}\})\, \mathcal{E}(x, \{\alpha_{2k}\}) \prod_k d^2\alpha_{2k} \right\}. \end{aligned} \tag{16.4}$$

The third term of this sum is evidently an interference term. We must next ask when it contributes to the observed intensities and when it does not.

We have noted in Section VII of the reprinted paper that any light beam described in the P-representation can be regarded as the superposition of two fields, one of which corresponds to a pure coherent state and the other of which is of the unphased form, i.e., it has vanishing expectation value for the complex field strength. When each of the fields generated by the two sources is analyzed in this way, it becomes clear that the unphased components of the fields will not contribute to the interference term in Eq. (16.4). The interference term will, in fact, vanish completely unless the field generated by each of the two sources has a non-zero coherent component.

The most elementary kind of example in which the interference term is different from zero is one in which the two sources acting separately bring the field to coherent states represented by

$$\begin{aligned} P_1(\{\alpha_{1k}\}) &= \prod_k \delta^{(2)}(\alpha_{1k} - \beta_{1k}) \\ P_2(\{\alpha_{2k}\}) &= \prod_k \delta^{(2)}(\alpha_{2k} - \beta_{2k}). \end{aligned} \tag{16.5}$$

Then the interference term of Eq. (16.4) reduces to

$$2\mathrm{Re}\{\mathcal{E}^*(x, \{\beta_{1k}\})\, \mathcal{E}(x, \{\beta_{2k}\})\}. \tag{16.6}$$

The analysis of this term may be simplified by assuming that the two sources are ideal lasers which are similar in construction and that each excites only a single plane wave mode. The two plane wave modes are then not identical since their propagation vectors are not quite parallel, but they have the same frequency. Under these conditions it is easy to see that the interference term (16.6) describes stationary intensity fringes which are seen on the screen in the area in which the two beams overlap. The fringes are perpendicular to the plane which contains the two propagation vectors and may be made narrow or broad by making the angle between the beams large or small.

Let us suppose that the single mode excited by source 1 has amplitude β_1 and that excited by source 2 has amplitude β_2. Then, since the plane wave mode functions are intrinsically complex, it is clear that the position of the fringe system on the screen Σ (i.e., its displacement in the direction perpendicular to the fringes) will depend on the phase difference of the complex amplitudes β_1 and β_2. If the geometry of the experiment is sufficiently well determined, then by observing the fringe system we may measure the phase difference.

No difficulty of principle stands in the way of our actually carrying out experiments of the type we have just described with two laser beams. But in practice we never have the complete knowledge of the excitation amplitudes which we assumed, for example, in constructing Eqs. (16.5) and (16.6). As we have remarked many times earlier, we are almost always lacking knowledge of overall phase parameters. As long as this is so we do not know the phases of oscillation of our lasers, and the only way we can honestly represent the density operators for the modes they excite is by means of the functions

$$P_j(\alpha_j) = \frac{1}{2\pi|\beta_j|}\, \delta(|\alpha_j| - |\beta_j|) \tag{16.7}$$

for $j = 1, 2$. These functions represent the stationary density operators which are

obtained, as in Eq. (15.21), by averaging the coherent states over phase. But the P-functions (16.7) are of the unphased variety; they correspond to vanishing averaged complex fields. When the descriptions of our two sources are stationary, in other words, the interference term in Eq. (16.4) vanishes identically.

If this result is taken to mean that there are no fringes to be seen on the screen, then our ignorance of the phase parameters has somehow wiped out a large scale physical phenomenon. To bring the paradox of such a conclusion into sharper focus it is possible to argue that each of our laser sources is essentially classical in nature and really has a well defined phase of oscillation. Consequently the fringes should be visible on the screen both to people who do and who don't know the phases alike.

To see that we have not really encountered any fundamental dilemma we must recall that density operators are constructed for the purpose of describing ensembles of quantum mechanical experiments. The need to repeat experiments upon many similarly prepared systems arises for reasons which are quite basic to quantum mechanics. The quantities measured in general fluctuate unpredictably from one system to another, even when all the systems are prepared in precisely the same quantum state. When the quantum state itself is random there is still a further reason for carrying out experiments on a large number of systems and averaging their results.

The two P-functions given by Eq. (16.5) represent, for example, pure states of the field. In any single experiment carried out with two sources for which all the excitation amplitudes and phases are known, we would probably detect a more-or-less noisy form of the interference pattern we have been discussing. The interference pattern would assume the smooth form given by Eq. (16.6) only after we had averaged over many experiments performed with identically prepared sources.

Now when we have no knowledge of the phases of oscillation of our two laser sources, our formalism describes an ensemble of experiments in which the phases are allowed to be completely random. It is true that the contribution of the interference effect to the average intensity for this ensemble vanishes. But one can not conclude from the vanishing of the ensemble average that the fringes do not show up in the individual experiments. This experiment is one in which the members of the ensemble are individually quite unlike their ensemble average. Each of the experiments will exhibit a stationary fringe pattern on the screen, just as when the oscillation phases are known. But since the phases are random, the displacement of the pattern will vary randomly from one experiment to the next. It is the averaging over the random displacement which wipes away the fringes in the ensemble average.

A question we might now ask is how we can use the density operator formalism at all to make statistical statements about the fringe pattern. When the sources are stationary it has appeared to tell us nothing but that the ensemble average of the interference intensity vanishes at every point on the screen. Let us imagine that we are performing the experiment with a pair of lasers chosen from our random phase ensemble. To determine that there is indeed an interference pattern on the screen we must measure the intensity at a considerable number of points on the screen. We do not prepare the system anew for each of these measurements; they are carried out for a single preparation of the lasers. Now just the first of the intensity measurements at a known point on the screen goes a long way toward determining the phase difference of the two lasers. It determines a linear combination of the sine and cosine of the phase difference of the amplitudes β_1 and β_2 which restricts the phase difference to either of two discrete values. Measurment of the intensity at another point then determines the phase difference.

Once we have used intensity measurements at a couple of points to determine the phase difference we can predict the appearance of the rest of the interference pattern in an ensemble average sense. Of course the ensemble in this case is no longer the one we began with, though it still remains a stationary one. Our initial

intensity measurements furnish us with information which requires that we reduce the size of our initial ensemble by retaining only those experiments in which the phase difference is found to be nearly the same. This reduced ensemble will be described by a stationary density operator since a phase factor common to the amplitudes β_1 and β_2 of a pair of degenerate modes remains completely random. Let us suppose that we find the phase difference of the two beams to be

$$\arg \beta_1 - \arg \beta_2 = \theta. \tag{16.8}$$

Then the selection process by which we reduce the ensemble to one appropriate to experiments for fixed θ can be represented by inserting a factor

$$\frac{1}{2\pi}\,\delta(\arg \alpha_1 - \arg \alpha_2 - \theta) \tag{16.9}$$

into the integrand of the P-function (16.1). Once we have located the fringe pattern by experimentally determing its unpredictable position, we have no difficulty in constructing a stationary density operator which predicts the average intensities in the pattern.

The idea of reducing the size of our ensemble to reflect the acquisition of knowledge about a system should not be too unfamiliar. In any multi-step game of chance, for example, the odds for winning, which one hopes are even initially, change as one completes each move. The initial odds are calculated by using the complete ensemble of possible games, but the odds calculated at the later states use only the reduced ensembles appropriate to the information which was revealed by the earlier moves.

Another sense, though a rather different one, in which the use of the stationary density operator furnishes information about the randomly placed interference pattern may be seen by discussing the second order correlation function. It is easy to show that the two-fold coincidence counting rate

$$G^{(2)}(\mathrm{r}t\,\mathrm{r}'t',\ \mathrm{r}'t'\,\mathrm{r}t) = \int P(\{\alpha_k\})\,|\,\mathcal{E}(\mathrm{r}t\{\alpha_k\})\,|^2\,|\,\mathcal{E}(\mathrm{r}'t\{\alpha_k\})\,|^2 \prod_k d^2\alpha_k \tag{16.10}$$

contains a term which oscillates as a function of the positions $\mathbf{r}$ and $\mathbf{r}'$ on the screen. This type of interference term may be derived by means of essentially the same argument as we used in discussing the intensity interference experiments in Lecture II. The oscillation of the intensity correlation function must evidently reflect oscillation of the intensity itself. Furthermore since the unknown phase angles of β_1 and β_2 cancel out of the second order correlation function nothing need be known about them to calculate it.

However a simple measurement of the intensity of a random fringe pattern (e.g., by examining a photograph) is not the same as a measurement of $G^{(2)}$, and there is no simple way of concluding in general from a knowledge of $G^{(2)}$ what the intensity pattern of the random fringe system should be. Thus, while $G^{(2)}$ and the other even order correlation functions are useful in their own right, they offer no alternative way of discussing the fringe intensities. If we want the intensities we must derive them from the density operators for appropriately reduced ensembles.

We have assumed to this point that our light sources are ideal noise-free lasers. We now ask what happens when the random modulation of the devices is taken into account. Since the most important of the parameters in determing the two-beam interference pattern is the phase of oscillation of the laser, we can secure a good idea of what goes on by using the phase diffusion model to represent the laser beams. According to that model, the phase of a laser beam wanders appreciably over time-intervals long compared to a relaxation time $1/\zeta$, and remains relatively fixed over time intervals which are much shorter in length.

When the two laser beams are represented by such models, the light intensities

we record on the screen will depend on the length of time we require to make our measurements. If the intensities are sufficiently great that we can record them in a time short compared to $1/\zeta$, then the two beams will retain nearly the initial values of their phases while the measurements are being made. A randomly situated fringe pattern of the sort we have already discussed should then show up. But a similar measurement made, say, half a relaxation time later would reveal a differently placed set of fringes, corresponding to the fluctuation that had taken place in the phase difference of the two beams.

If we could follow the fringe intensity as a function of time, we should see the parallel fringe system execute a sort of random wandering back and forth on the screen. If we were to try recording the intensities on the screen by integrating these over a period much longer than the relaxation time we would find that the fringe structure is washed out and only a uniform intensity remains.

Laser sources are convenient ones for such two-beam experiments, because they are intense, and monochromatic enough to have relatively long relaxation times. It is also quite possible, in principle, to carry out such experiments with beams from ordinary chaotic sources. The random amplitude modulation of these beams will mean that the fringes fluctuate greatly in contrast as well as in position. The relaxation time for these variations will be the inverse frequency bandwidth of the sources. If such fringes have not been photographed to date, it is because exposure times shorter than 10^{-10} sec. would be necessary.

REFERENCES

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Lecture XVII PHOTON COUNTING EXPERIMENTS

The number of photons which a counter records in any interval of time fluctuates randomly. In a simple type of counting experiment we might imagine that the counter is exposed to the field for a fixed interval of time t. Then, by repeating the experiment many times, we should find a distribution function for the number of counts received in that interval. Although the average number of counts is frequently all that we require, the way in which the number fluctuates about its average value can be fully understood only when we know the distribution function and its moments. In this lecture we shall discuss ways of predicting the distribution function and the relation between the form of the distribution and the coherence of the field.

Let us first recall some of the results we established in Lecture V. We calculated there the probability that in an interval of time from t_0 to t all n atoms of a hypothetical n-atom photodetector undergo photoabsorption transitions which are registered as photon counts. When we eliminate the tensor indices by assuming the field to be fully polarized, this probability is given by Eq. (5.8), i.e., we have

$$p^{(n)}(t) = \int_0^t \cdots \int_0^t \prod_{j=1}^{n} S(t_j'' - t_j') \, G^{(n)}(r_1 t_1' \cdots r_n t_n', \, r_n t_n'' \cdots r_1 t_1'') \times \prod_{j=1}^{n} dt_j' \, dt_j'', \tag{17.1}$$

where the sensitivity function S is defined by Eqs. (4.12) and (4.10), and we have set $t_\sigma = 0$. If our detector happens to be of the broadband variety, we may use Eq. (4.14) to reduce the number of time integrations in this integral from 2n to n, but this reduction is not a necessary one for the arguments to follow.

We must now consider a more realistic model of a counter which contains an enormous number of atoms, say $N \sim 10^{20}$, which are capable of detecting photons by undergoing photoabsorption processes. Needless to say, it will virtually never happen that all N of these atoms do undergo absorption processes in any finite interval of time. The total number of photoabsorptions is much smaller as a rule, and we shall try to use Eq. (17.1) to find its distribution law.

The total number of photocounts recorded in any interval of time may be regarded as a sum of random variables, one for each atom of the detector. To do this, let us introduce the random variable z_j for the j-th atom, which takes on the values

$$z_j = \begin{cases} 0 \text{ if no photoabsorption process is recorded for the j-th atom} \\ 1 \text{ if a photoabsorption process is recorded for the j-th atom.} \end{cases} \tag{17.2}$$

Then the random variable which represents the total number of counts will be

$$C = \sum_{j=1}^{n} z_j . \tag{17.3}$$

Associated with each final state of the system i.e., any set of values $z_1 \cdots z_N$, there is a probability function $\mathcal{P}(z_1 \cdots z_N, t)$. The statistical average of any function of the z_j's is then found by averaging the function over the probability distribution. For example, the average number of counts is given by

$$\langle C \rangle = \sum_{\{z_j = 0, 1\}} \sum_{j=1}^{N} z_j \, \mathcal{P}(z_1 \cdots z_N, t), \tag{17.4}$$

where the final summation is over the values 0 and 1 for the entire set of variables z_j. We shall write such sums in the future as sums over $\{z_j\}$. We next introduce the reduced probability function for the j-th atom which we define as

$$p_j(z_j, t) = \sum_{\{z_k, \, k \neq j\}} \mathcal{P}(z_1 \cdots z_N, t). \tag{17.5}$$

The average number of counts may be written in terms of the reduced probabilities p_j as

$$\begin{aligned} \langle C \rangle &= \sum_{\{z_k\}} \sum_{j=1}^{N} z_j p_j(z_j, t) \\ &= \sum_{j=1}^{N} p_j(1, t) . \end{aligned} \tag{17.6}$$

The probability $p_j(1, t)$ which occurs in the latter expression is clearly equal to the one-atom transition probability $p^{(1)}(t)$ evaluated for the j-th atom. That probability is given by Eq. (17.1) for n = 1, with $r_1 = r_j$, and we shall write it as $p^{(1)}{}_j(t)$. The average number of counts is thus

$$\langle C \rangle = \sum_{j=1}^{N} p^{(1)}{}_j(t) . \tag{17.7}$$

We shall now introduce a generating function which will enable us to solve simultaneously for the unknown distribution of photocounts and for its moments. We could, of course, find the moments directly by generalizing the way in which $\langle C \rangle$ was obtained, but the present method has the advantage of enabling us to obtain all the quantities of interest from a single function. The generating function we choose is

$$Q(\lambda, t) = \langle (1 - \lambda)^{C} \rangle, \tag{17.8}$$

where C is the random integer given by Eq. (17.3), the brackets indicate an ensemble average, and the variable λ is intended simply to be a useful parameter.

If we write Q as a sum over the integer values which C may take on we have an expansion of the form

$$Q(\lambda, t) = \sum_{m=0}^{N} (1 - \lambda)^m \, p(m, t), \tag{17.9}$$

where $p(m, t)$ is the probability that the counter has recorded m photocounts at the time t. It is clear that if $Q(\lambda, t)$ is known $p(m, t)$ can be obtained by differentiation,

$$p(m, t) = \frac{(-1)^m}{m!} \left[\frac{d^m}{d\lambda^m} Q(\lambda, t) \right]_{\lambda = 1}, \tag{17.10}$$

since Eq. (17.9) may be regarded as a Taylor expansion for Q about $\lambda = 1$.

If, on the other hand, we expand $Q(\lambda, t)$ in a power series about $\lambda = 0$ we have

$$Q(\lambda, t) = \sum_{n=0}^{N} \frac{\lambda^n}{n!} \left[\frac{d^n}{d\lambda^n} Q(\lambda, t) \right]_{\lambda = 0}. \tag{17.11}$$

The derivatives which occur in this expansion are given by

$$\begin{aligned} (-1)^n \left[\frac{d^n}{d\lambda^n} Q(\lambda, t) \right]_{\lambda = 0} &= \left\langle \frac{C!}{(C - n)!} \right\rangle \\ &= \langle C(C - 1) \cdots (C - n + 1) \rangle. \end{aligned} \tag{17.12}$$

The averages on the right of this equation are known as factorial moments. They are simple linear combinations of the ordinary moments $\langle C^n \rangle$ of the distribution of photocounts. It is clear from these relations that a knowledge of the generating function enables us to find both the probability distribution and its moments. We must next show how it is possible to evaluate the generating function in terms of the photoabsorption probabilities $p^{(n)}(t)$.

First let us note that $Q(\lambda, t)$ can be written as

$$\begin{aligned} Q(\lambda, t) &= \sum_{\{z_k\}} \wp(z_1 \cdots z_N t) \, (1 - \lambda)^{\sum_{j=1}^{N} z_j} \\ &= \sum_{\{z_k\}} \wp(z_1 \cdots z_N t) \prod_{j=1}^{N} (1 - \lambda)^{z_j}. \end{aligned} \tag{17.13}$$

The latter form, however, may be simplified by using the identity

$$(1 - \lambda)^{z_j} = 1 - z_j \lambda, \tag{17.14}$$

which holds because z_j takes on only the values zero and one. With this simplification, Eq. (17.13) becomes

$$Q(\lambda, t) = \sum_{\{z_j\}} \wp(z_1 \cdots z_N, t) \prod_{j=1}^{N} (1 - \lambda z_j). \tag{17.15}$$

When the N-fold product in this expression is expanded in powers of λ, we have

$$Q(\lambda, t) = \sum_{n=0}^{N} (-\lambda)^n \sum_{\{z_j\}} \sum_{\substack{\text{n-fold} \\ \text{combinations}}} z_{j_1} \cdots z_{j_n} \, \wp(z_1 \cdots z_n, t), \tag{17.16}$$

where the first sum is taken over all the ways of choosing n atoms from the set of N.

If we now define the n-fold joint probability that atoms $j_1 \cdots j_n$ all undergo photoabsorption processes as

$$p^{(n)}{}_{j_1 \cdots j_n}(t) = \sum_{\{z_k\}} z_{j_1} \cdots z_{j_n} \wp(z_1 \cdots z_N t), \tag{17.17}$$

then we may write the generating function in the form

$$Q(\lambda, t) = \sum_{n=0}^{N} (-\lambda)^n \sum_{\text{n-fold combinations}} p^{(n)}{}_{j_1 \cdots j_n}(t). \tag{17.18}$$

Now the number $p^{(n)}{}_{j_1 \cdots j_n}(t)$ has been defined as the probability that each of a particular set of n atoms absorbs a photon, regardless of what all the other atoms do. This probability is simply the expression $p^{(n)}(t)$ given by Eq. (17.1) and evaluated for the particular atoms $j_1 \cdots j_n$. Hence we know all the terms of Eq. (17.18) and the problem is simply to sum them. What we shall do, in fact, is to turn the sums over atoms into volume integrations.

Since the probabilities $p^{(n)}(t)$ are only large for values of n which are extremely small in comparison with N, we may approximate the sums over n-fold combinations by writing

$$\sum_{\text{n-fold combinations}} \approx \frac{1}{n!} \sum_{j_1=1}^{N} \sum_{j_2=1}^{N} \cdots \sum_{j_n=1}^{N}. \tag{17.19}$$

Then the sums over the individual atoms may be carried out as spatial integrations by letting the number of atoms per unit volume be $\sigma(\mathbf{r})$ and writing

$$\sum_{j_1=1}^{N} \cdots = \int d\mathbf{r}_1 \, \sigma(\mathbf{r}_1) \cdots \tag{17.20}$$

We are, in effect, dealing with the limit $N \to \infty$. When the probabilities given by Eq. (17.1) are substituted in the expression (17.18) for the generating function and the sum over combinations of atoms is transformed as we have indicated, we find

$$\begin{aligned} Q(\lambda, t) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int_{t_0}^{t} \cdots \int_{t_0}^{t} \int_{\text{Vol. of Detector}} \cdots \int_{\text{Vol. of Detector}} \\ G^{(n)}(\mathbf{r}_1', t_1' \cdots \mathbf{r}_n', t_n'; \mathbf{r}_n'', t_n'' \cdots \mathbf{r}_1', t_1'') \times \\ \prod_{j=1}^{n} \sigma(\mathbf{r}_j') \, S(t_j'' - t_j') \, d\mathbf{r}_j' \, dt_j' \, dt_j''. \end{aligned} \tag{17.21}$$

To abbreviate this expression a bit, let us define the function

$$V(x', x'') = \sigma(\mathbf{r}') \, \delta(\mathbf{r}' - \mathbf{r}'') \, S(t'' - t'), \tag{17.22}$$

where x indicates both the position r and the time t. Then the expression for the generating function reduces to

$$\begin{aligned} Q(\lambda, t) = \sum_{n=1}^{\infty} \frac{(-\lambda)^n}{n!} \int \cdots \int G^{(n)}(x_1' \cdots x_n', x_n'' \cdots x_1'') \times \\ \prod_{j=1}^{n} V(x_j' \, x_j'') \, d^4x_j' \, d^4x_j''. \end{aligned} \tag{17.23}$$

Since this is a power series expansion about $\lambda = 0$, the factorial moments must be given, according to Eqs. (7.11) and (7.12), by

$$\left\langle \frac{C!}{(C-n)!} \right\rangle = \int \cdots \int G^{(n)}(x_1' \cdots x_n', x_n'' \cdots x_1'') \times \prod_{j=1}^{n} V(x_j' x_j'')\, d^4x_j' d^4x_j'', \tag{17.24}$$

where the integrations are carried out over the sensitive volume of the counter and the time interval from 0 to t.

As an illustration of the usefulness of these results, let us consider the case of a fully coherent field. For such a field we have the factorization

$$G^{(n)}(x_1' \ldots x_n', x_n'' \ldots x_1'') = \prod_{j=1}^{n} G^{(1)}(x_j', x_j''), \tag{17.25}$$

so that the series for $Q(\lambda, t)$ may be summed to the form

$$Q(\lambda, t) = e^{-\lambda \int\int G^{(1)}(x', x'') V(x'x'') d^4x' d^4x''} . \tag{17.26}$$

But from Eq. (17.24) we see that the average number of counts is just

$$\langle C \rangle = \int\int G^{(1)}(x', x'')\ V(x'x'')\ d^4x'\ d^4x''; \tag{17.27}$$

so that the generating function may be written as

$$Q(\lambda, t) = e^{-\lambda \langle C \rangle}, \tag{17.28}$$

Now by using Eq. (17.12) we derive the factorial moments

$$\left\langle \frac{C!}{(C-n)!} \right\rangle = \langle C \rangle^n, \tag{17.29}$$

and by using Eq. (17.10) we find that the probability distribution is

$$p(m, t) = \frac{\langle C \rangle^m}{m!} e^{-\langle C \rangle} , \tag{17.30}$$

i.e., when the field is fully coherent we always have a Poisson distribution for the number of counts.

When the field does not possess full coherence we can nevertheless use the coherent states as a basis for describing it. To illustrate the form the statistical calculations take, we shall use the P-representation for the density operator of the field. The R-representation, which applies more generally, can also be used similarly. In the P-representation $G^{(n)}$ is given by the integral

$$G^{(n)}(x_1 \ldots x_{2n}) = \int P(\{\alpha_k\}) \prod_{j=1}^{n} \mathcal{E}^*(x_j \{\alpha_k\}) \times \prod_{j=n+1}^{2n} \mathcal{E}(x_j \{\alpha_k\}) \prod_k d^2\alpha_k . \tag{17.31}$$

When this expression is substituted into the series (17.23) we find that the series may be summed to the closed form

$$Q(\lambda, t) = \int P(\{\alpha_k\})\ e^{-\lambda \Omega(\{\alpha_k\})} \prod_k d^2\alpha_k, \tag{17.32}$$

where

$$\Omega(\{\alpha_k\}) = \int \mathcal{E}^*(x' \{\alpha_k\})\, \mathcal{E}(x'' \{\alpha_k\}) V(x', x'') d^4x' d^4x''. \tag{17.33}$$

Furthermore we see from Eq. (17.12) that the factorial moments are

$$\left\langle \frac{C!}{(C-n!)} \right\rangle = \int P(\{\alpha_k\})\, \Omega^n(\{\alpha_k\}) \prod_k d^2\alpha_k \tag{17.34}$$

and from Eq. (17.10) that the probability distribution is given by

$$p(m, t) = \int P(\{\alpha_k\}) \frac{\Omega^m(\{\alpha_k\})}{m!} e^{-\Omega(\{\alpha_k\})} \prod_k d^2\alpha_k . \tag{17.35}$$

The probability of counting m photons is evidently a species of average over the corresponding probabilities for an ensemble of Poisson distributions. We hardly need emphasize that the averaging process is not a classical one and that the quasiprobability function P may assume negative values.

As a further illustration of the methods we are discussing let us consider the general case of a chaotically generated field. The density operators of such fields may be represented by means of the Gaussian function

$$P(\{\alpha_k\}) = \prod_k \frac{1}{\pi \langle n_k \rangle} e^{-\frac{|\alpha_k|^2}{\langle n_k \rangle}} . \tag{17.36}$$

Then, since the function Ω is a quadratic form in the variables α_k, it will be possible to evaluate the integral (17.32) for the generating function in full generality.

Before we do this, however, let us introduce some useful notation. We may express the function $\mathcal{E}(x, \{\alpha_k\})$ as a linear form in the variables α_k by using the normal mode expansion

$$\mathcal{E}(x, \{\alpha_k\}) = \sum_k e(x, k) \alpha_k , \tag{17.37}$$

where the functions e are given by Eq. (14.26). If we then define the matrix

$$B_{k'k''} = \int e^*(x'k') V(x'x'') e(x''k'') d^4x' d^4x'', \tag{17.38}$$

we may write the quadratic form Ω as

$$\Omega(\{\alpha_k\}) = \sum_{k'k''} \alpha_{k'}^* B_{k'k''} \alpha_{k''} . \tag{17.39}$$

When this expression for Ω and the Gaussian form for P are substituted in Eq. (17.32) we find that the generating function is given by

$$Q(\lambda, t) = \int \cdots \int \exp\left\{ - \sum_k \frac{|\alpha_k|^2}{\langle n_k \rangle} - \lambda \sum_{k'k''} \alpha_{k'}^* B_{k'k''} \alpha_{k''} \right\} \prod_k \frac{d^2\alpha_k}{\pi \langle n_k \rangle} .$$

If we then introduce the variables

$$\beta_k = \alpha_k / \{\langle n_k \rangle\}^{\frac{1}{2}} \tag{17.40}$$

and define the matrix

$$M_{k'k''} = \{\langle n_{k'} \rangle\}^{\frac{1}{2}} B_{k'k''} \{\langle n_{k''} \rangle\}^{\frac{1}{2}}, \tag{17.41}$$

the integral for the generating function may be simplified to the form

$$Q(\lambda, t) = \int \cdots \int \exp\left\{ - \sum_k |\beta_k|^2 - \lambda \sum_{k'k''} \beta_{k'}^* M_{k'k''} \beta_{k''} \right\} \prod_k \frac{d^2\beta_k}{\pi} . \tag{17.42}$$

Now we can consider the set of numbers β_k as forming the components of a complex vector β. Then if we let M represent the matrix whose components are given by Eq. (17.41), we may write the exponent in the integrand of Eq. (17.42) as the product

$$-\beta^\dagger (1 + \lambda M) \beta.$$

Since the Matrix M is Hermitian it may be diagonalized by carrying out a unitary transformation upon the vector β. Then if we let the eigenvalues of M be $\mathcal{M}_i$, and let the transformed complex coordinates be γ_i, the integral for the generating function reduces to the elementary form

$$Q(\lambda, t) = \int\cdots\int \exp\Big\{-\sum_i (1+\lambda\mathcal{M}_i)|\gamma_i|^2\Big\}\prod_i \frac{d^2\gamma_i}{\pi}$$

$$= \frac{1}{\prod_i (1+\lambda\mathcal{M}_i)} \tag{17.43a}$$

$$= \frac{1}{\det(1+\lambda M)} . \tag{17.43b}$$

It is worth noting that the matrix M must be positive definite, since the quadratic form Ω defined by Eqs. (17.33) or (17.39) is the average number of photons counted in a particular coherent field. Hence the eigenvalues $\mathcal{M}_i$ are positive, and the singularities of the generating function lie on the negative real axis of the variable λ. Since Q is analytic in the half-plane $\mathrm{Re}\lambda \geq 0$, we see that if we are given Q as a power series expansion about either of the points $\lambda = 0$ or $\lambda = 1$, the series expansion about the other of the points may be evaluated, in principle by analytic continuation. This argument shows that the procedure we have been using, of evaluating the generating function by means of its expansion about $\lambda = 0$, actually leads to a unique answer for the probability distribution.

Since the matrix M is in general of infinite rank, neither of the expressions (17.43) is easy to evaluate directly. Let us note, however, that $\det(1+\lambda M)$ may be written as

$$\prod_i (1+\lambda\mathcal{M}_i) = \exp\Big\{\sum_i \log(1+\lambda\mathcal{M}_i)\Big\} .$$

Now for $|\lambda| < (\mathcal{M}_{max})^{-1}$, where $\mathcal{M}_{max}$ is the largest of the eigenvalues $\mathcal{M}_i$, we may expand the logarithm in the exponent in a convergent power series. In this way we see that

$$\det(1+\lambda M) = \exp\Big\{\sum_i (\lambda\mathcal{M}_i - \tfrac{1}{2}\lambda^2\mathcal{M}_i^2 + \cdots)\Big\}$$

$$= \exp\Big\{\mathrm{Tr}\,(\lambda M - \tfrac{1}{2}\lambda^2 M^2 + \cdots)$$

$$= \exp\{\mathrm{Tr}\log(1+\lambda M)\} \tag{17.44}$$

where Tr, as always, stands for the trace. By making use of this identity we can express the generating function as

$$Q(\lambda, t) = e^{-\mathrm{Tr}\log(1+\lambda M)} . \tag{17.45}$$

If we expand the logarithm in powers of λ, we may write this function in the form

$$Q(\lambda, t) = \exp\Big\{\sum_{r=1}^{\infty} \frac{(-\lambda)^r}{r} I_r\Big\} , \tag{17.46}$$

where I_r is defined by

$$I_r = \mathrm{Tr}\{M^r\} . \tag{17.47}$$

If we recall the definition of the matrix M given by Eqs. (17.41) and (17.38), then we see that for r = 1 we have

$$I_1 = \iint \sum_k e^*(x'k)\, e(x''k)\, \langle n_k\rangle\, V(x'x'')\, d^4x'\, d^4x'' .$$

The sum over k in the integrand, according to Eq. (14.32), is simply the first order correlation function. The integral thus reduces to

$$I_1 = \iint G(x', x'')\, V(x'x'')\, d^4x'\, d^4x'' . \tag{17.48}$$

If we compare Eq. (17.46) with Eqs. (14.27) and (14.28) we see that this r = 1 term is of the same form as the exponent of the generating function for the case of a pure coherent field. The lack of coherence for the Gaussian case is reflected by the presence in the exponent of the additional terms with $r \geq 2$. By making further use of the matrix M we can show that the general expression for I_r is the cyclic integral

$$I_r = \int \prod_{j=1}^{r} G^{(1)}(x_j', x_{j+1}'')\, V(x_j' x_j'')\, d^4x_j'\, d^4x_j'' , \tag{17.49}$$

in which the coordinate x_{r+1}'' is to be interpreted as x_1''. For the case of broad-band detectors the definitions (17.22) and (4.14) allow us to simplify this integral to the form

$$I_r = s^r \int_0^t\cdots\int_0^t \prod_j dt_j' \int\cdots\int \prod_j G^{(1)}(r_j' t_j', r_{j+1}' t_j) \sigma(r_j')\, dr_j' . \tag{17.50}$$

To discuss the evaluation of these integrals let us suppose that our counting experiment has particularly simple geometry. We shall assume that our field consists of plane waves travelling in the positive y-direction, so that the first order correlation function is given by Eq. (15.1). This function naturally depends only on the y-coordinates of its spatial arguments. We next assume that the sensitive region of the counter, i.e., its photocathode, is a very thin layer of atoms lying in a plane perpendicular to the y-axis. The function $\sigma(r)$, in other words, is essentially a delta function of the y-coordinate. With these assumptions, which experiments often approximate quite closely in practice, the spatial integrations in Eq. (17.50) become trivial. The functions $G^{(1)}$ are independent of their position variables for all of the points for which $\sigma(r)$ differs from zero.

The time integrals in Eq. (17.50) are considerably less trivial, but we may discuss the forms they take for short times and for long times. If the time t is much smaller than the inverse frequency bandwidth of the radiation present, the functions $G^{(1)}$ will hardly vary at all in the interval from 0 to t. For such times the integral I_r must simply be proportional to t^r. If we write I_1 as wt, where w is a proportionality constant, then the elementary character of the spatial integrations shows that the general result must be

$$I_r = (wt)^r . \tag{17.51}$$

When this result is substituted in Eq. (17.46), we find that the generating function for small values of t is

$$Q(\lambda, t) = \exp\{-\log(1+\lambda wt)\}$$

$$= \frac{1}{1+\lambda wt} . \tag{17.52}$$

The probability distribution for the number of counts is then given, according to Eq. (17.10), by

$$p(m, t) = \frac{(wt)^m}{(1+wt)^{m+1}} . \tag{17.53}$$

The distribution for short times is thus given by a power law not unlike the Planck distribution. The mean number of counts is wt, so that w is simply the average counting rate.

For times t which considerably exceed the inverse bandwith of the radiation field, it is also possible to simplify the integrals I_r. In this case, however, their values depend sensitively on the spectral distribution of the energy present in the field. Let us therefore assume, as an example, that the frequency spectrum has the Lorentz form

$$\langle n_k \rangle \hbar\omega_k = \frac{\text{constant}}{(\omega - \omega_0)^2 + \gamma^2} . \tag{17.54}$$

The time dependence of the first order correlation function is then given by Eq. (15.8). When this function is substituted into the integral (17.50), we see that, because of the cyclical structure of the integrand, all of the I_r will increase linearly with time for $t \gg \gamma^{-1}$. We may again define the average counting rate, w, by writing the integral I_1 as wt. Then it is not difficult to show that the full set of integrals I_r may be written in the form

$$I_r = \frac{t(2\gamma w)^r}{2(r-1)!} \left(-\frac{1}{2\gamma}\frac{d}{d\gamma}\right)^{r-1} \frac{1}{\gamma} \tag{17.55}$$

for $t \gg \gamma^{-1}$.

With these values for the I_r it is possible to sum the series in the exponent of Eq. (17.46) in closed form. When this is done we find that the generating function is

$$Q(\lambda, t) = \exp\{-[(\gamma^2 + 2\gamma w\lambda)^{\frac{1}{2}} - \gamma]t\} . \tag{17.56}$$

When the counting rate w is small compared to the frequency bandwidth i.e., $w \ll \gamma$, then the expression in the exponent may be expanded, and we find that in the lowest approximation the generating function reduces to

$$Q(\lambda, t) = e^{-\lambda wt}. \tag{17.57}$$

This function, as we have seen, leads to a Poisson distribution. It is the distribution we would find if there were no tendency for the photons to arrive in correlated bunches, or for the field amplitude to fluctuate randomly.

To discuss the distribution and moments which follow from the generating function (17.5), it is useful to introduct the set of inverse polynomials

$$\begin{aligned} s_0(\xi) &= s_1(\xi) = 1 \\ s_2(\xi) &= 1 + \frac{1}{\xi} \\ s_3(\xi) &= 1 + \frac{3}{\xi} + \frac{3}{\xi^2} \\ s_4(\xi) &= 1 + \frac{6}{\xi} + \frac{15}{\xi^2} + \frac{15}{\xi^3} \end{aligned} \tag{17.58}$$

The further members of the sequence are given by the recursion formula

$$s_{n+1}(\xi) = -s_n'(\xi) + \left(1 + \frac{n}{\xi}\right) s_n(\xi). \tag{17.59}$$

These polynomials are quite familiar in the theory of Bessel functions. They may also be calculated from the expression

$$s_n(\xi) = e^{\xi}\left(\frac{2\xi}{\pi}\right)^{\frac{1}{2}} K_{n-1/2}(\xi), \tag{17.60}$$

where $K_{n-1/2}$ is a modified Hankel function of half-integral order.

If we now expand the generating function (17.56) in a power series about $\lambda = 1$ and examine its coefficients we find that the probability of receiving m counts in time t is

$$p(m, t) = \frac{1}{m!}\left(\frac{\gamma wt}{\Gamma}\right)^m s_m(\Gamma t)\, e^{-(\Gamma-\gamma)t} , \tag{17.61}$$

where we have written

$$\Gamma = (\gamma^2 + 2w\gamma)^{\frac{1}{2}}. \tag{17.62}$$

The distribution (17.61) has the same mean value, wt, as the Poisson distribution which follows from the generating function (17.57). Its variance, however, is always larger than that of the Poisson distribution because of the photon clumping effect.

The power series expansion of the generating function (17.56) about $\lambda = 0$ is

$$Q(\lambda t) = \sum_{n=0}^{\infty} \frac{(-\lambda wt)^n}{n!} s_n(\gamma t). \tag{17.63}$$

We conclude from this expansion that the factorial moments of the distribution (17.61) are given by

$$\begin{aligned} \left\langle \frac{C!}{(C-n)!} \right\rangle &= (wt)^n s_n(\gamma t) \\ &= \langle C(C-1)\cdots(C-n+1)\rangle. \end{aligned} \tag{17.64}$$

For a Poisson distribution these moments would be simply $(wt)^n$. The first two of the moments (17.64) are

$$\langle C \rangle = wt \tag{17.65}$$

$$\langle C(C-1)\rangle = (wt)^2\left(1 + \frac{1}{\gamma t}\right). \tag{17.66}$$

The variance of the number of counts is thus

$$\langle C^2\rangle - \langle C\rangle^2 = \langle C\rangle\left\{1 + \frac{\langle C\rangle}{\gamma t}\right\} \tag{17.67}$$

The term $\langle C\rangle^2/\gamma t$ is the addition to the variance which is due to the fact that the photon arrival times are not statistically independent of one another.