Physics 566 - Quantum Optics I
Lecture 13 - The Jaynes Cummings Model

Atom interacting with the quantized field

The fundamental Hamiltonian describing the atom, the quantum electromagnetic field, and their interaction is the "minimal coupling" Hamiltonian. In the Schrödinger picture,

$$\hat{H} = \underbrace{\sum_j \frac{1}{2m_j}\left(\vec{p}_j - \frac{e}{c}\hat{\vec{A}}_\perp(\hat{\vec{r}}_j)\right)^2 + V_0(\hat{\vec{r}}_j)}_{\hat{H}_A + \hat{H}_{AF} \text{ (Atom + Atom-field Hamiltonian)}} + \underset{\hat{H}_F \text{ (Field Hamiltonian)}}{\sum_{k,\mu}\hbar\omega_k\left(\hat{a}^\dagger_{k\mu}\hat{a}_{k\mu} + \frac{1}{2}\right)}$$

Here $\{\hat{\vec{r}}_j, \hat{\vec{p}}_j\}$ are the electron's canonical coordinates relative to a fixed nucleus, V_0 is the electrostatic interaction between electrons and binding to the nucleus, $\hat{\vec{A}}_\perp$ is the quantum vector potential for the transverse field in the Coulomb gauge. As in our discussion of the semiclassical theory, when the motion of the electrons is nonrelativistic, we can make a multipole expansion of the charge/field interaction. Said equivalently, we focus on wavelengths of the EM waves $\lambda \gg a_0$ (the Bohr Radius). After such n expansion, we can express the Hamiltonian as

$$\hat{H} = \hat{H}_A + \hat{H}_F + \hat{H}_{AF}$$

$$\hat{H}_A = \sum_j\left(\frac{\hat{p}_j^2}{2m_j} + V_0(\hat{\vec{r}}_j)\right) = \underbrace{\sum E_i |u_i\rangle\langle u_i|}_{\text{Atom's energy levels (should also include unbound states)}} : \text{ Atomic Hamiltonian}$$

$$\hat{H}_F = \sum_{k,\mu}\hbar\omega_k\left(\hat{a}^\dagger_{k\mu}\hat{a}_{k\mu} + \frac{1}{2}\right) : \text{ Free field Hamiltonian}$$

$$\hat{H}_{AF} = -\hat{\vec{d}}\cdot\hat{\vec{E}}(\vec{R}) : \text{ Dipole interaction Hamiltonian}$$

$$\hat{\vec{d}} = -e\sum_j \hat{\vec{r}}_j \text{ (atomic electric dipole)}, \quad \hat{\vec{E}}(\vec{R}) = \text{Quantum electric field @ atomic center.}$$

Of particular interest is the case of the "two-level atom," where there is strong interaction between modes of field a the transition between two levels $|g\rangle$ and $|e\rangle$. That is, if the system starts in $|g\rangle$ or $|e\rangle$ it stays (with high probability) in the subspace

Two-level atom interacting with quantized field:

$$\hat{H} \approx E_g |g\rangle\langle g| + E_e |e\rangle\langle e| + \sum_{\vec{k},\mu} \hbar\omega_k \hat{a}^\dagger_{\vec{k}\mu} \hat{a}_{\vec{k}\mu} - \left(\vec{d}_{eg} |e\rangle\langle g| + \vec{d}_{ge} |g\rangle\langle e|\right) \cdot \hat{\vec{E}}(\vec{R})$$

$$= \frac{\hbar\omega_{eg}}{2} \hat{\sigma}_z + \sum_{\vec{k},\mu} \hbar\omega_k \hat{a}^\dagger_{\vec{k},\mu} \hat{a}_{\vec{k},\mu} - \left(\vec{d}_{eg} \hat{\sigma}_+ + \vec{d}_{ge} \hat{\sigma}_-\right) \cdot \hat{\vec{E}}(\vec{R})$$

Note: I dropped the zero point energy, which has no effect on dynamics, and used the usual Pauli pseudo-spin representation, with the atomic energy zero $\frac{1}{2}$-way between $|g\rangle$ + $|e\rangle$.

$$\text{Now} \quad \hat{\vec{E}}(\vec{R}) = \underbrace{\hat{\vec{E}}^{(+)}(\vec{R})}_{\text{Positive freq component}} + \underbrace{\hat{\vec{E}}^{(-)}(\vec{R})}_{\text{Negative freq. component}}$$

$$\hat{\vec{E}}^{(+)}(\vec{R}) = \sum_{k,\mu} \sqrt{2\pi\hbar\omega_k}\, \underbrace{\vec{u}_{\vec{k},\mu}(\vec{R})}_{\text{mode function @ atomic position}} \hat{a}_{\vec{k},\mu} = \hat{\vec{E}}^{(-)\dagger}(\vec{R})$$

$$\int d^3r \; \vec{u}^*_{\vec{k}',\mu'}(\vec{r}) \cdot \vec{u}_{\vec{k},\mu}(\vec{r}) = \delta_{\vec{k}\vec{k}'} \delta_{\mu'\mu} \quad \text{(orthonormal modes)}$$

$$\Rightarrow \hat{H}_{AF} = \frac{\hbar\omega_{eg}}{2} \hat{\sigma}_z + \sum_{\vec{k},\mu} \hbar\omega_k \hat{a}^\dagger_{k,\mu} \hat{a}_{k,\mu} +$$

$$+ \underbrace{\sum_{\vec{k},\mu} \hbar g_{\vec{k},\mu} \hat{a}_{\vec{k}\mu} \hat{\sigma}_+ + \hbar g^*_{\vec{k}\mu} \hat{a}^\dagger_{\vec{k}\mu} \hat{\sigma}_-}_{\text{"co-rotating" terms}} + \underbrace{\sum_{\vec{k},\mu} \hbar g_{\vec{k},\mu} \hat{a}_{\vec{k},\mu} \hat{\sigma}_- + \hbar g^*_{\vec{k},\mu} \hat{a}^\dagger_{\vec{k},\mu} \hat{\sigma}_+}_{\text{"counter-rotating" terms}}$$

where $\hbar g_{\vec{k},\lambda} \equiv -\vec{d}_{eg} \cdot \vec{u}_{\vec{k},\lambda}(\vec{R}) \sqrt{2\pi\hbar\omega_k}$ is the atom-photon coupling energy

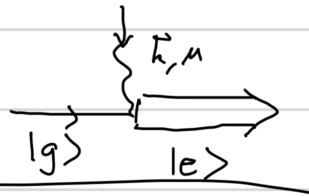
The notation "co-rotating" and "counter-rotating" corresponds to the notion we saw in the semiclassical model of Rabi oscillations. They correspond to "resonant" and "anti-resonant" respectively. To see this, go to the "interaction picture" where operator evolve in time according to the free evolution

$$\hat{a}_{\vec{k}\lambda}(t) = \hat{a}_{\vec{k}\lambda} e^{-i\omega_k t}, \qquad \hat{\sigma}_-(t) = \hat{\sigma}_- e^{-i\omega_{eg} t}$$

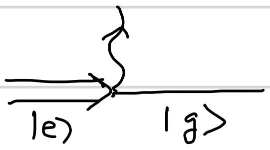
$$\Rightarrow \hat{H}^{(I)}_{AF} = \underbrace{\sum_{\vec{k},\mu} \left(\hbar g_{\vec{k},\lambda} \hat{a}_{\vec{k}\lambda} \hat{\sigma}_+ e^{-i(\omega_k - \omega_{eg})t} + h.c\right)}_{\text{Co-rotating (resonant)}} + \underbrace{\sum_{\vec{k},\lambda} \left(\hbar g_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda} \hat{\sigma}_- e^{-i(\omega_k + \omega_{eg})t} + h.c\right)}_{\text{Counter-rotating (antiresonant)}}$$

Feynman's picture: Elementary processes:

Absorption: $\hat{a}_{\vec{k},\mu}\hat{\sigma}_+$



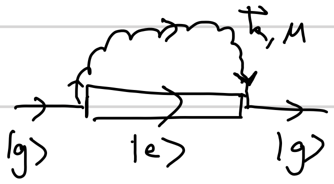
Emission: $\hat{a}^\dagger_{\vec{k},\mu}\hat{\sigma}_-$



Co-rotating terms

The resonant terms can conserve energy with one elementary process. The anti-resonant (counter-rotating terms) are necessarily "virtual processes" that don't conserve energy in one interaction. They are necessarily multiphoton (nonlinear)

E.g.



Emission and absorption of a virtual photon

This process leading to a shift in the ground state energy (Lamb shift) ⇒ "renormalization"

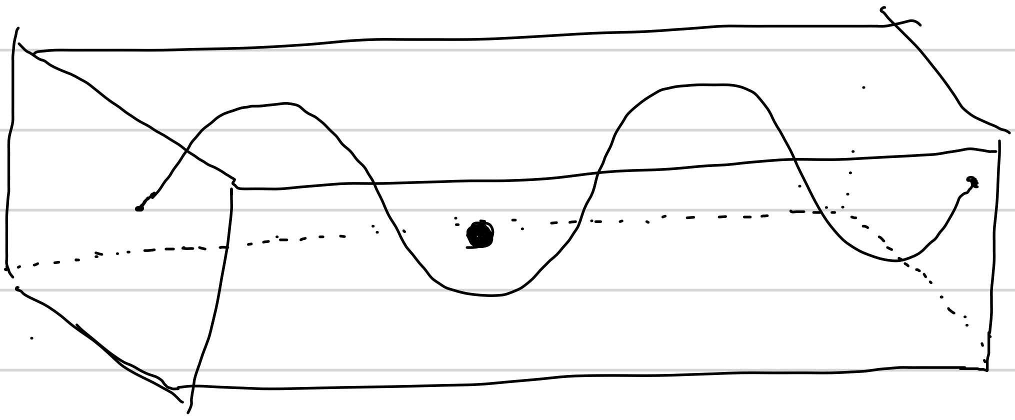
The counter-rotating terms are negligible compared to the resonant co-rotating terms.

Hamiltonian for two-level atom interaction with quantum field in the dipol and rotating wave approximations:

$$\boxed{\hat{H} = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \sum_{\vec{k},\mu}\hbar\omega_k\,\hat{a}^\dagger_{\vec{k},\mu}\hat{a}_{\vec{k},\mu} + \sum_{\vec{k},\mu}\hbar\left(g_{\vec{k},\mu}\hat{a}_{\vec{k},\mu}\hat{\sigma}_+ + g^*_{\vec{k},\mu}\hat{a}^\dagger_{\vec{k},\mu}\hat{\sigma}_-\right)}$$

Cavity QED + Jaynes-Cummings

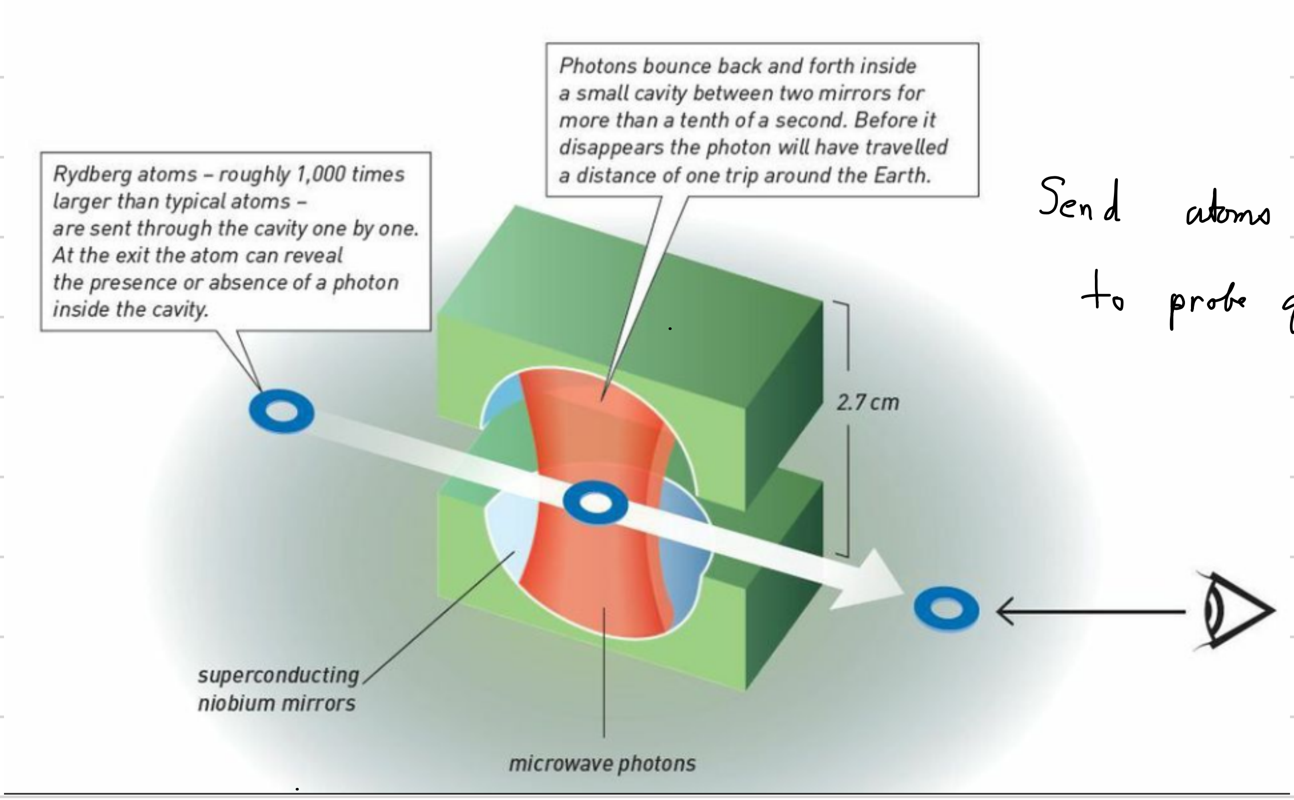
An important paradigm in quantum optics is the study of a two-level atom interacting with one mode of the electromagnetic field. Such a model is relevant in describing the interaction of a two-level atom with modes in an electromagnetic cavity, When a particular mode of of the cavity has frequency ω_c near a two-level resonance of the atom.



Cartoon: Mode of an electromagnetic cavity with an atom inside.
= Cavity QED

In practice, there are typically two paradigms in which we explore cavity QED: superconducting cavities and dielectric cavities. Superconducting cavities are lossy at optical frequencies, but the quality factor is extremely high at rf/microwave frequency. Thus, one paradigm for cavity QED is a superconducting microwave cavity coupling to an atomic dipole resonance ω_{eg} that has microwave frequency. Because the energy levels of an atom scale like $\frac{1}{n^2}$, this is typically for high principle quantum numbers ⇒ "Rydberg levels"

E.g. Serge Haroche (Nobel Prize, 2012): Rubidium atoms $|g\rangle = |n=50, \ell=49\rangle$
$|e\rangle = |n=51, \ell=50\rangle$ (circular Rydberg states): $\omega_{eg}/2\pi = 51$ GHz,



Photons bounce back and forth inside a small cavity between two mirrors for more than a tenth of a second. Before it disappears the photon will have travelled a distance of one trip around the Earth.

Rydberg atoms – roughly 1,000 times larger than typical atoms – are sent through the cavity one by one. At the exit the atom can reveal the presence or absence of a photon inside the cavity.

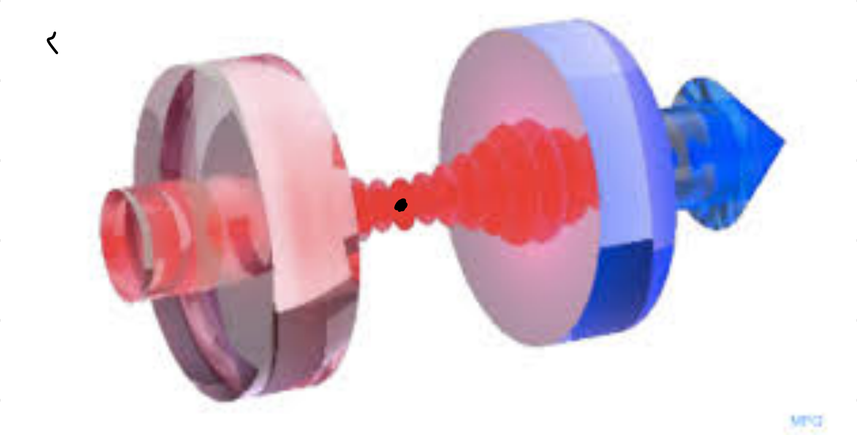
Send atoms through cavity to probe quantum field.

Haroche cavity: Best mirrors ever produced. Photon lifetime in cavity $\tau_{cav} = 0.13$ seconds.

$$\text{Cavity } Q = \omega_c \tau_{cav} = 4.2 \times 10^{10}$$

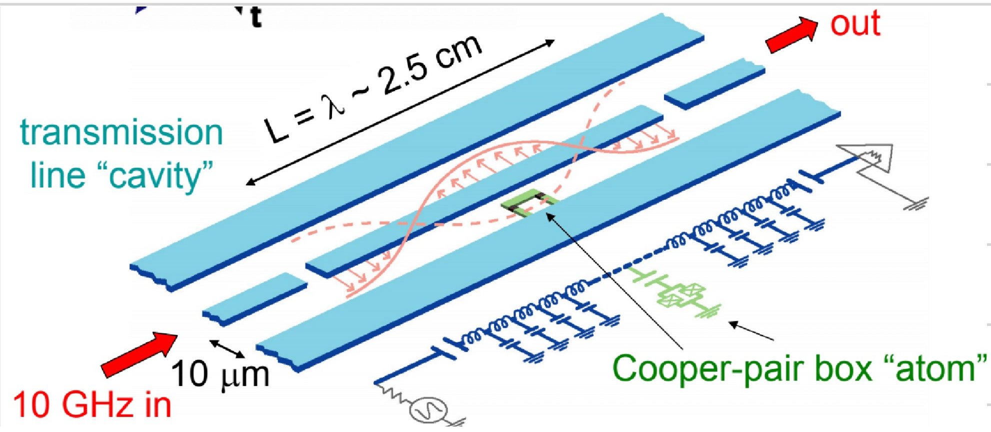
⇒ Photon bounces 1.5 billions times before lost (travels 40,000 km = earth circumference).

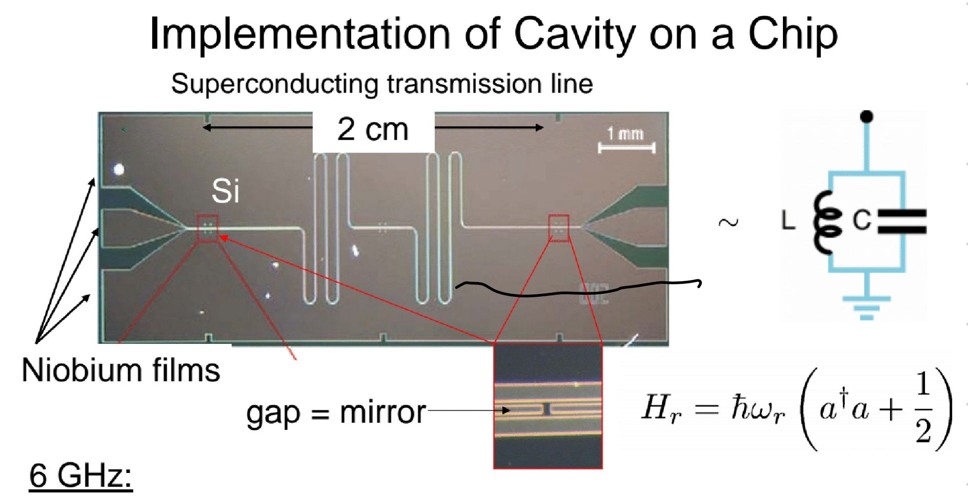
To probe an optical mode, one considers a high finesses Fabry-Perot cavity with an atom trapped inside. A cavity resance in tuned near to an optical transition ω_{eg}, typically $|g\rangle$ = atom ground state, $|e\rangle$ = first excited state of an alkali atom (Cs, Rb).



Here we probe the atom photon coupling by looking at the transmitted light.

In the last decade a new a powerful realization of cavity QED has come on the scene — "Circuit QED" with superconducting circuits (Girvin, Yale). Here as with Rydberg atoms, the photons are microwaves ~ 10 GHz. The "qubit" are two macroscopic states of current across a superconducting Josephson junction. The microwaves can be in a strip line cavity, compatible with "on chip" technology.



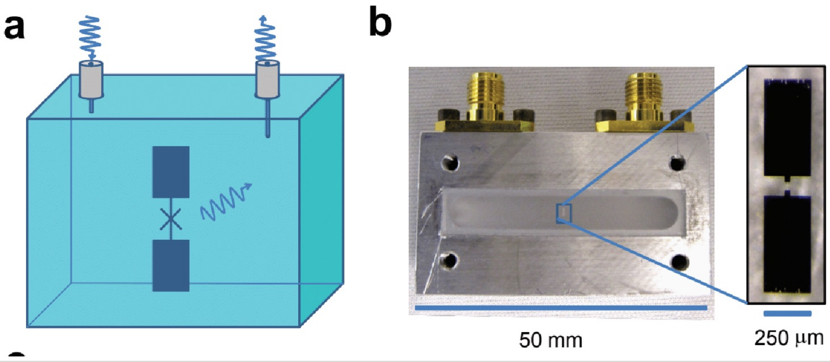


Implementation of Cavity on a Chip

$$H_r = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)$$

6 GHz:

Most recently, this is extended into 3D superconducting cavities with large dipole antennas.



The advantage of this is that the dipole moment can be HUGE, leading to many orders greater coupling constants g, when compared to atoms.

From our general expression for the atom-field coupling in the dipole + RWA, for a single mode, freq ω_c

$$\hat{H} = \hbar\omega_{eg}|e\rangle\langle e| + \hbar\omega_c\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma})$$

$$\boxed{\hat{H} = \frac{\hbar\omega_{eg}}{2}\hat{\mathbb{1}} + \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \hbar\omega_c\hat{a}^\dagger\hat{a} + \hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-)}$$

Jaynes-Cammings Hamiltonian

Where $\hbar g = -\langle e|\vec{d}|g\rangle \cdot \vec{u}_c(\vec{R})\sqrt{2\pi\hbar\omega_c}$ is the coupling constant for the cavity mode taken to be real.

The characteristics of the atom-photon system for a single electromagnetic mode, governed by this Hamiltonian was first studied in a seminal paper, "Comparison of Quantum and Semiclassical Radiation Theories," by E.T. Jaynes + F.W. Cummings, Proc. IEEE 51 89 (1963). This is one of the most important paradigms in quantum optics and is now known as the Jaynes-Cumming model.

The Jaynes-Cummings Ladder (Dressed States)

To understand the dynamics governed by the Jaynes-Cummings model, consider first the eigenstates of the Jaynes-Cummings Hamiltonian. These are the fully quantum version of the atom-laser "dressed states we studied in the semiclassical model. In condensed-matter physics, the quasi-particle describing photons + dipole material response are known as "polaritons."

The Hilbert Space for the joint atom + photon system: $\mathcal{H} = h_{Atom} \otimes h_{Field}$

$h_{atom} = \mathbb{C}^2$, spanned by $\{|g\rangle, |e\rangle\}$,

h_{Field} = Single mode Fock span, spanned by $\{|n\rangle \mid n = 0, 1, 2, 3, \cdots\}$

$$\mathcal{H} \text{ spanned by } \underbrace{\{|g\rangle \otimes |n\rangle,\ |e\rangle \otimes |n\rangle\}}_{\equiv \text{ "bare states"}} \equiv \{|g,n\rangle, |e,n\rangle \mid n = 0, 1, 2, \dots\}$$

Symmetry: There is a important conserved quantity in the system

Consider: $\hat{N}_T = \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_- = \hat{a}^\dagger \hat{a} + |e\rangle\langle e|$: Total # of excitation in field + atom

$$[\hat{H}, \hat{N}_T] = [\hat{H}_A + \hat{H}_F + \hat{H}_{AF}, \hat{N}_T] = [\hat{H}_{AF}, \hat{N}_T] = [\hbar g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-), \hat{a}^\dagger \hat{a} + \hat{\sigma}_+ \hat{\sigma}_-]$$

$$= \hbar g \left(\underbrace{[\hat{a}, \hat{a}^\dagger \hat{a}]}_{\hat{a}} \hat{\sigma}_+ + \hat{a} \underbrace{[\hat{\sigma}_+, \hat{\sigma}_+ \hat{\sigma}_-]}_{-\hat{\sigma}_+} + \underbrace{[\hat{a}^\dagger, \hat{a}^\dagger \hat{a}]}_{-\hat{a}^\dagger} \hat{\sigma}_- + \hat{a}^\dagger \underbrace{[\hat{\sigma}_-, \hat{\sigma}_+ \hat{\sigma}_-]}_{\hat{\sigma}_-} \right)$$

$\Rightarrow [\hat{H}, \hat{N}_T] = 0 \Rightarrow$ the total excitation in the system is conserved.

This is a reflection of the RWA, which imposes an additional symmetry on the system. Thus N_T is a "good quantum number". The Hamilton matrix is thus Block-Diagonal in the 2x2 subspaces $\{|g,n\rangle, |e,n-1\rangle\} \Rightarrow N_T = n$ (for $n = 0$ only one state)

$N_T = 0$: $|g,0\rangle$, $N_T = 1$: $\{|g,1\rangle, |e,0\rangle\}$, $N_T = 2$: $\{|g,2\rangle, |e,1\rangle\}$, etc.

$$\hat{H} = \begin{bmatrix} H_0 & & & \\ & H_1 & & \\ & & H_2 & \\ & & & \ddots \end{bmatrix}$$

$H_n = \hat{H}$ is basis $\{|g,n\rangle, |e,n-1\rangle\}$

$$\langle g,n|\hat{H}|gn\rangle = n\hbar\omega_c\,, \quad \langle e,n-1|\hat{H}|e,n-1\rangle = \hbar\omega_{eg} + (n-1)\hbar\omega_c = -\hbar\Delta + n\hbar\omega_c$$

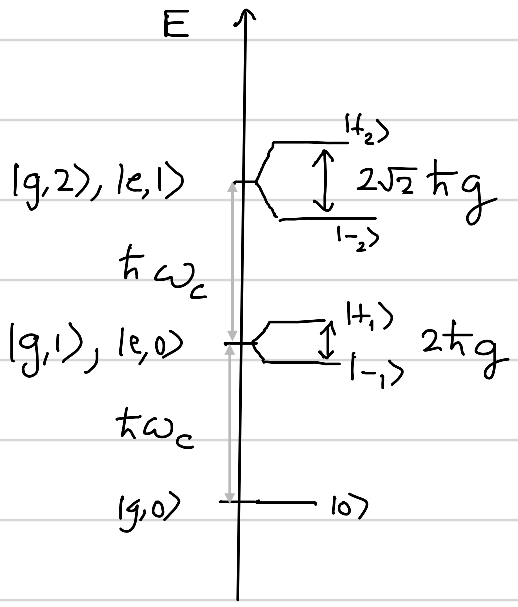
$$\langle e,n-1|\hat{H}|g,n\rangle = \hbar g\langle n-1|\hat{a}|n\rangle = \hbar\sqrt{n}\,g$$

$$H_n = \hbar\begin{bmatrix} -\Delta+n\omega_c & \sqrt{n}\,g \\ \sqrt{n}\,g & n\omega_c \end{bmatrix}\begin{matrix}|e,n-1\rangle \\ |g,n\rangle\end{matrix} \qquad \mathrm{Tr}(H_n) = (2n-1)\,\omega_c$$

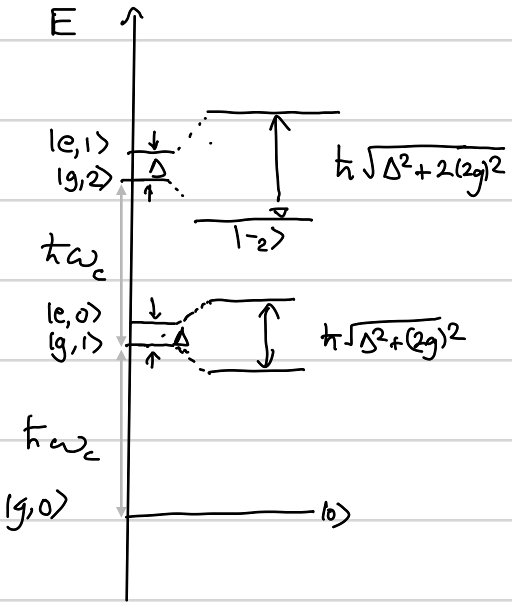
$$\Rightarrow H_n = \hbar\left(-\frac{\Delta}{2}+n\omega_c\right)\underbrace{\begin{bmatrix}1&0\\0&1\end{bmatrix}}_{\hat{\mathbb{1}}} - \frac{\hbar\Delta}{2}\underbrace{\begin{bmatrix}1&0\\0&-1\end{bmatrix}}_{\hat{\sigma}_3} + \hbar\sqrt{n}\,g\underbrace{\begin{bmatrix}0&1\\1&0\end{bmatrix}}_{\hat{\sigma}_1}$$

H_n is a "Rabi Hamiltonian" $\hat{H}_n = -\frac{\hbar\Delta}{2}\hat{\sigma}_3 + \frac{\hbar\Omega_n}{2}\hat{\sigma}_1$, for the "qubit" $|\uparrow\rangle = |g,n\rangle$, $|\downarrow\rangle = |e,n-1\rangle$ with Rabi frequency $\Omega_n = 2\sqrt{n}\,g$

Eigenvalues: $E_{\pm(n)} = n\hbar\omega_c - \frac{\hbar\Delta}{2} \pm \frac{\hbar}{2}\sqrt{\Delta^2+n(2g)^2}$, $\underbrace{|\pm(n)\rangle = \cos\frac{\theta_n}{2}|g,n\rangle \pm \sin\frac{\theta_n}{2}|e,n-1\rangle}_{\text{Dressed States}}$ $\tan\theta_n = \frac{2g\sqrt{n}}{-\Delta}$



$\Delta=0$



$\Delta<0$

This energy spectrum is known as the "Jaynes Cummings Ladder"

The bare state spectrum, with coupling $g=0$ has an equal spaced set of energy levels. the dressed states has a nonlinear (unequal spaced) spectrum.

Vacuum Rabi Oscillations

Suppose at time $t=0$, the atom is in the excited state $|e\rangle$ with zero photons in the cavity (vacuum $|0\rangle$). The cavity resonance is chosen equal to the atomic resonance, $\omega_c = \omega_{eg} \Rightarrow \Delta = 0$. How does the system evolve?

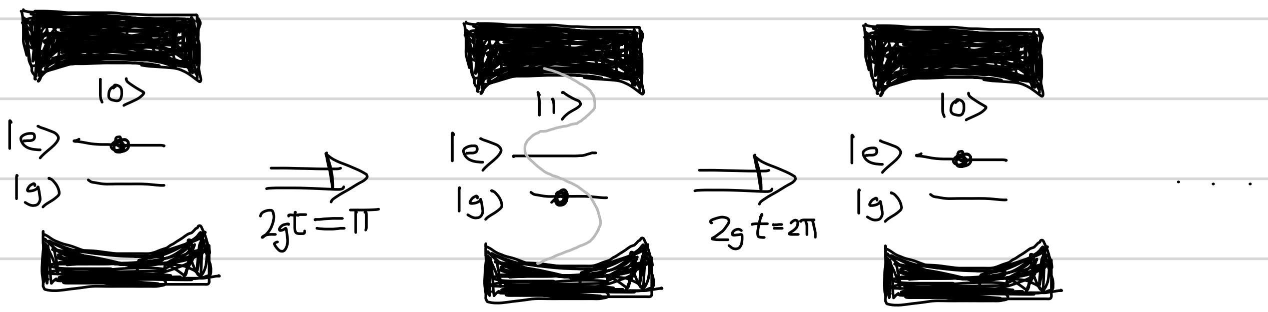
The initial state $|\Psi(0)\rangle = |e,0\rangle$ is not an eigenstate of the Hamiltonian. Thus, the joint atom/field system evolves as a function of time. Because $|e,0\rangle$ has exactly one excitation, it is coupled through the Jaynes-Cummings Hamiltonian only to $|g,1\rangle$.

The evolution is thus

$$|\Psi(t)\rangle = e^{-\frac{i}{\hbar}H_1 t}|e,0\rangle = e^{-ig\hat{\sigma}_1}|\uparrow\rangle = \cos(gt)|\uparrow\rangle - i\sin(gt)|\downarrow\rangle$$

$$\Rightarrow \boxed{|\Psi(t)\rangle = \cos(gt)|e,0\rangle - i\sin(gt)|g,1\rangle}$$

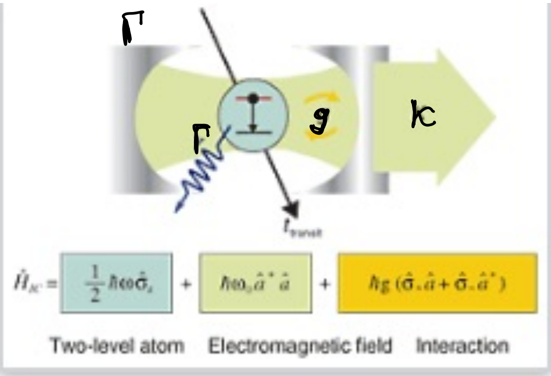
This is known as "Vacuum Rabi Oscillation" the field started initially in the vacuum. The atom coherently emits a photon into the cavity and then reabsorbs it in a periodic manner, with frequency $2g$, the "vacuum Rabi frequency" $\sim \sqrt{\frac{\hbar\omega_c}{V}}\, d_{eg}$



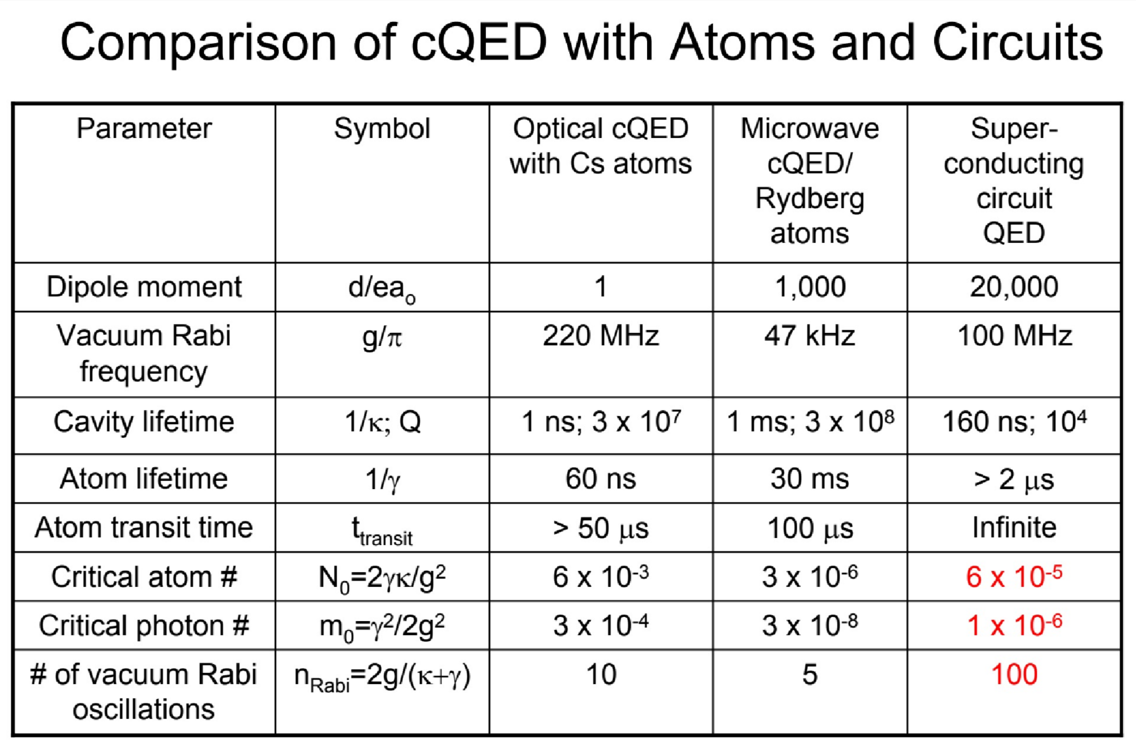
The coherent (reversible) excitation of energy between the atom and the quantum field is one of the Hallmark of quantum optics.

Cavity QED

The Jaynes-Cummings model is a closed quantum system description of the atom field coupling. In real systems, there is always decay. For example, in an optical system, the atom can spontaneous decay out the side, and a photon can leak out out the cavity (or be absorbed in the mirror)



For typically defines $C = \frac{4g^2}{\Gamma\kappa}$ = cooperativity. When $C >> 1$ one is said to be in the "strong coupling regime" where we can have many vacuum Rabi oscillations before the system decay.



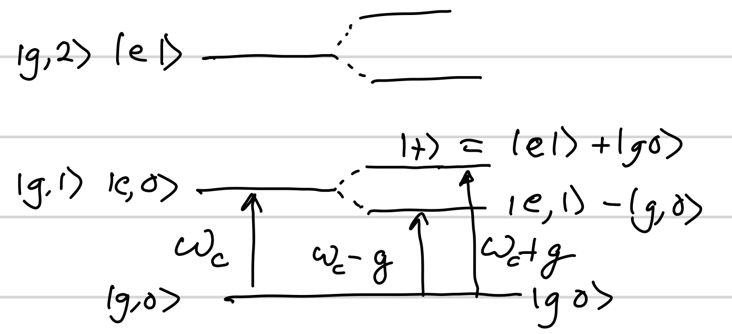
Comparison of cQED with Atoms and Circuits

Parameter	Symbol	Optical cQED with Cs atoms	Microwave cQED/ Rydberg atoms	Super-conducting circuit QED
Dipole moment	d/ea_o	1	1,000	20,000
Vacuum Rabi frequency	g/π	220 MHz	47 kHz	100 MHz
Cavity lifetime	$1/\kappa$; Q	1 ns; 3×10^7	1 ms; 3×10^8	160 ns; 10^4
Atom lifetime	$1/\gamma$	60 ns	30 ms	> 2 μs
Atom transit time	$t_{transit}$	> 50 μs	100 μs	Infinite
Critical atom #	$N_0=2\gamma\kappa/g^2$	6×10^{-3}	3×10^{-6}	6×10^{-5}
Critical photon #	$m_0=\gamma^2/2g^2$	3×10^{-4}	3×10^{-8}	1×10^{-6}
# of vacuum Rabi oscillations	$n_{Rabi}=2g/(\kappa+\gamma)$	10	5	100

Circuit cQED is the king of the strong coupling regime. And these numbers are old ~2013. The current 3D superconducting cavities and dipole antenna are orders of magnitude larger. With superducting circuits we achieve text-book-like realizations of the Jaynes-Cummings model.

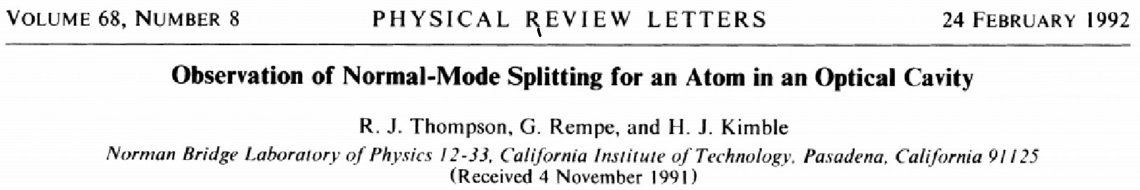
Vacuum Rabi Splitting

The dressed states of the closed quantum quantum system can be though of a "scattering resonances" for the open quantum system.



Inside the cavity there are two "hybrid modes" $|\pm\rangle = \frac{|e1\rangle \pm |g0\rangle}{\sqrt{2}}$ mixing cavity and atom degrees of freedom.

If we send a laser photon into the cavity, with no atom we have a cavity resonance at ω_c. With the atom inside the cavity, the resonance is split by $2g$. We $2g > \kappa, \Gamma$ we can resolved this splitting. This vacuum Rabi splitting was first observed in seminal experiments by H.J. Kimble, one of the fathers of cavity QED.



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Observation of Normal-Mode Splitting for an Atom in an Optical Cavity

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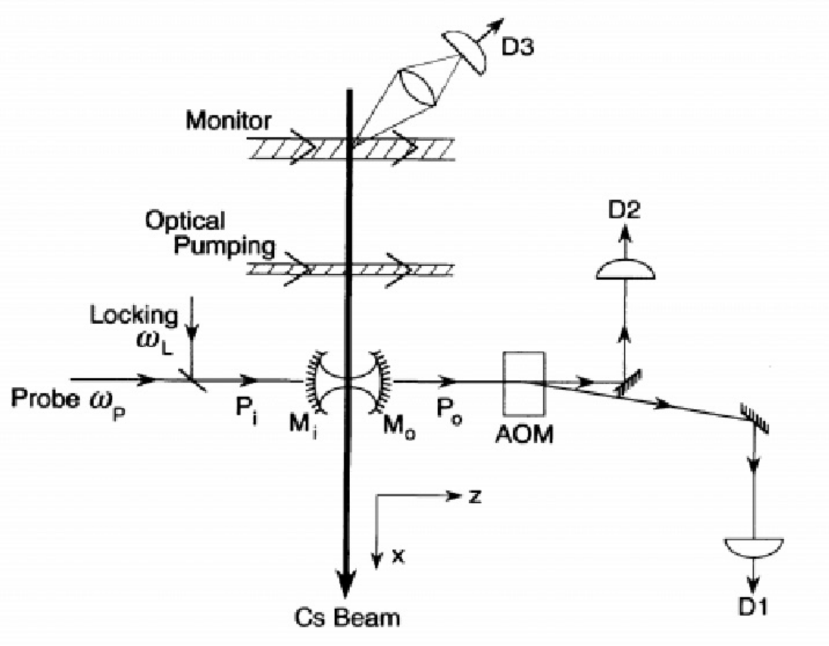

FIG. 1. Diagram of principal elements of the experiment.

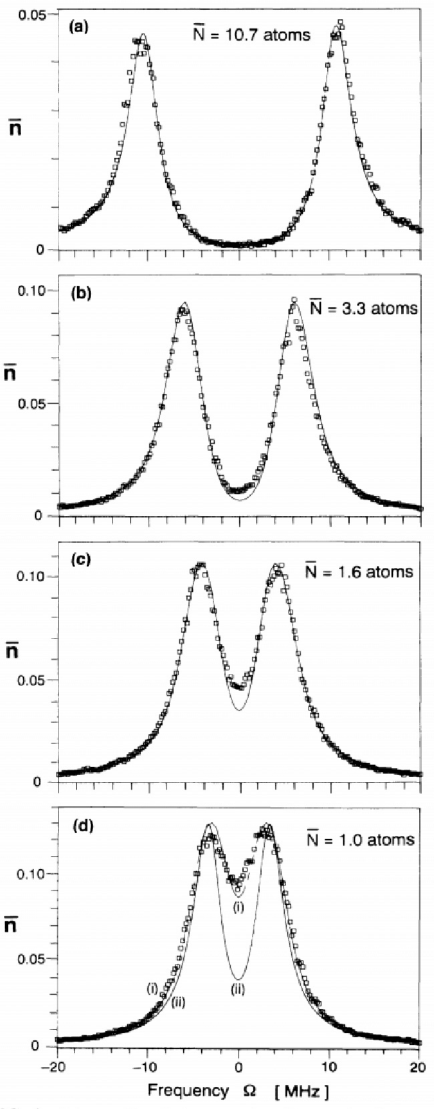

FIG. 2. Intracavity photon number $\bar{n}$ vs probe frequency Ω for four values of $\bar{N}$ and with $\omega_C \simeq \omega_A$. Curves in (a)-(c) and curve i in (d) are theoretical fits to the data including fluctuations in atomic number and position. Curve ii in (d) is from Eq. (1) for a single intracavity atom with optimum coupling g_0.