Physics 566: Quantum Optics I
Problem Set 1
Due Thursday, September 7, 2017

Problem 1: Gaussian probability distributions (20 points)

Consider a Gaussian for one random variable $p(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-\frac{\left(x-\langle x\rangle\right)^2}{2\sigma^2}}$.

(a) Show that the characteristic function is $\chi(k)=e^{ik\langle x\rangle-\frac{k^2\sigma^2}{2}}$

(b) Show that $\left\langle\left(x-\langle x\rangle\right)^n\right\rangle=\begin{cases}0, & n\ odd\\ (n-1)!!\sigma^n, & n\ \ even\end{cases}$

Now consider a mutinomial Gaussian for a the random variables $\mathbf{x}=(x_1,x_2,...,x_N)$,
$p(\mathbf{x})=\frac{1}{\sqrt{(2\pi)\det C}}\exp\left[-\frac{1}{2}(\mathbf{x}-\mathbf{a})^T\cdot C^{-1}\cdot(\mathbf{x}-\mathbf{a})\right]$, where C is the "covariance matrix," an *NXN* symmetric matrix with nonnegative eigenvalues.

(c) Show that $\langle\mathbf{x}\rangle=\mathbf{a}$, $\left\langle\Delta x_i\Delta x_j\right\rangle=\left\langle\left(x_i-\langle x_i\rangle\right)\left(x_j-\langle x_j\rangle\right)\right\rangle=C_{ij}$

(d) A Gaussian is completely specified by its mean can covariance matrix. For simplicity let take $\langle\mathbf{x}\rangle=0$. Show that all the moments of the multivariate Gaussian are defined in terms of the two point-correlations

$$\left\langle x_1x_2\cdots x_{2n}\right\rangle=\sum\prod\left\langle x_ix_j\right\rangle=\sum\prod C_{ij}$$

$$\left\langle x_1x_2\cdots x_{2n-1}\right\rangle=0$$

where the notation $\sum\prod$ means summing over all distinct ways of partitioning x_1, …, x_n into pairs x_i,x_j and each summand is the product of the n pairs. For example

$$\left\langle x_1x_2x_3x_4\right\rangle=\left\langle x_1x_2\right\rangle\left\langle x_3x_4\right\rangle+\left\langle x_1x_3\right\rangle\left\langle x_2x_4\right\rangle+\left\langle x_1x_4\right\rangle\left\langle x_2x_3\right\rangle$$

Problem 2: Wiener-Khinchin Theorem (25 Points)
Consider a real function $f(t)$ (this could be a deterministic or random process). Defining the Fourier transform in our usual way, $\tilde{f}(\omega)=\int_{-\infty}^{\infty}f(t)e^{+i\omega t}\,dt$; this exists if $f(t)$ is square integrable. In this case according to Parceval's theorem $\int_{-\infty}^{\infty}f^2(t)dt=\int_{-\infty}^{\infty}\left|\tilde{f}(\omega)\right|^2\frac{d\omega}{2\pi}$ and $\left|\tilde{f}(\omega)\right|^2$ is known as the spectral density.

(a) Let $C(\tau)=\int_{-\infty}^{\infty} f(t)f(t+\tau)dt$ (autocorrelation function). Show that for $f(t)$ real

$$\left|\tilde{f}(\omega)\right|^2=\int_{-\infty}^{\infty} C(\tau)e^{i\omega\tau}\,d\tau\,,\quad C(\tau)=\int_{-\infty}^{\infty}\left|\tilde{f}(\omega)\right|^2 e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

That is, the spectrum density is the Fourier transform of the autocorrelation function and vice versa. This a form of the *Wiener-Khinchin Theorem*.

For a formally stationary process, $\tilde{f}(\omega)$ does not exist. In that case we have to be a little more careful. One defines the time average power $P=\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt=\int_{-\infty}^{\infty} S(\omega)\frac{d\omega}{2\pi}$, where $S(\omega)$ is the power spectral density. If follow that $S(\omega)=\lim_{T\to\infty}\frac{1}{T}\left|\int_{-T/2}^{T/2} f(t)e^{i\omega t}\,dt\right|^2$

(b) Show that the general form of the Wiener-Khinchin Theorem is

$$S(\omega)=\int G(\tau)e^{+i\omega\tau}d\tau\text{ , where }G(\tau)\equiv\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)f(t+\tau)dt=\int S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

(c) Now let $f(t)$ be an *ergodic and stationary* random process. Show that

$\left\langle \tilde{f}^*(\omega)\tilde{f}(\omega')\right\rangle=2\pi S(\omega)\delta(\omega-\omega')$, where angle brackets is the ensemble average.

Next, note that for a real function $\tilde{f}(-\omega)=\tilde{f}^*(\omega)$, Thus

$$f(t)=\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t}=\underbrace{\int_0^{\infty}\frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t}}_{f^{(+)}(t)}+\underbrace{\int_0^{\infty}\frac{d\omega}{2\pi}\tilde{f}^*(\omega)e^{+i\omega t}}_{f^{-}(t)}$$

$f^{(\pm)}(t)$ is known as the "positive/negative frequency component, and" for a real function $f^{(+)}(t)=\left[f^{-}(t)\right]^*$. Note, we often define the complex "analytic signal" $\tilde{f}_c(t)=2f^{(+)}(t)$. Then $f(t)=\mathrm{Re}\left[\tilde{f}_c(t)\right]$.

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(d) Consider the complex correlation function that determines temporal coherence in a standard interferometer for an ergodic, stationary process, $\Gamma(\tau)=\left\langle E^{(-)}(0)E^{(+)}(\tau)\right\rangle$. Show that

$$\mathrm{Re}\left[\Gamma(\tau)\right]=\frac{1}{2}\int_{-\infty}^{\infty} S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi},\quad S(\omega)=2\int_{-\infty}^{\infty}\mathrm{Re}\left[\Gamma(\tau)\right]e^{+i\omega\tau}\,d\tau$$

(e) What is the power spectrum of natural light arising from a collision broadened source? Sketch the output intensity from a Mach-Zender interferometer.