Physics 566: Quantum Optics I
Problem Set 3
Due Thursday, September 21, 2017

Problem 1: Qubits encoded in photon polarization (25 points)

The two orthogonal polarization states of a photon define a qubit. Let us define the standard basis

$$\frac{\mathbf{e}_H + i\mathbf{e}_V}{\sqrt{2}} : \text{ right hand circular (positive helicity)} \Rightarrow \left|\uparrow_z\right\rangle$$

$$\frac{\mathbf{e}_H - i\mathbf{e}_V}{\sqrt{2}} : \text{ left hand circular (negative helicity)} \Rightarrow \left|\downarrow_z\right\rangle$$

where $(\mathbf{e}_H, \mathbf{e}_V)$ are linear polarizations along some defined "horizontal" and "vertical" axes.

The Bloch sphere description of the polarization is known as the "Poincaré sphere", with each point on the surface representing a possible elliptical polarization. The three Cartesian coordinates of the Bloch vector are also known as the "Stokes parameters".

(a) To what polarization vectors do you associate $\left|\uparrow_x\right\rangle, \left|\downarrow_x\right\rangle$ and $\left|\uparrow_y\right\rangle, \left|\downarrow_y\right\rangle$? What is the nature of the polarization, linear, circular, elliptical, and along what direction if linear?

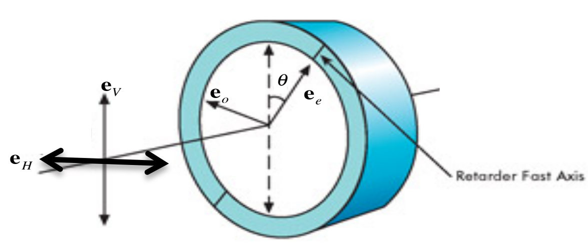
(b) Generalize: What is the polarization vector corresponding to an arbitrary state of the qubit, $\left|\uparrow_n\right\rangle$? Give the ellipticity and semi-major/minor axes of the ellipse in terms of the direction (θ,ϕ) of the state $\left|\uparrow_n\right\rangle$ on the Poincaré sphere?

(c) Sketch the Poincaré sphere, denoting the polarization states at the north and south pole, and at a few points along the equator as well as a few great circle a with constant latitude/longitude.

A wave plate is an optical element with birefringence, i.e., the index of refraction is in different along two orthogonal axes, "ordinary" and "extraordinary" m n_o, n_e. The result is that the phase shift imparted to the light depends on the polarization of the light with eigenvectors

$$\mathbf{e}_o \Rightarrow e^{i\frac{\omega}{c}n_o L}\mathbf{e}_o, \quad \mathbf{e}_e \Rightarrow e^{i\frac{\omega}{c}n_e L}\mathbf{e}_e$$

where L is the thickness of crystal. By orienting the crystal at an angle θ with respect to the H,V axes, one can transform the polarization state.



(d) Write the transformation of the polarization state by the wave plate as an SU(2) matrix acting on the Poincaré sphere. The matrix should be specified by two parameters, θ and $\Delta\phi = 2\pi \frac{(n_e - n_0)L}{\lambda}$, expressed in any representation you prefer.

(e) A quarter-wave plate has $L = \frac{\lambda}{4(n_e - n_o)}$; a half-wave plate has $L = \frac{\lambda}{2(n_e - n_o)}$.

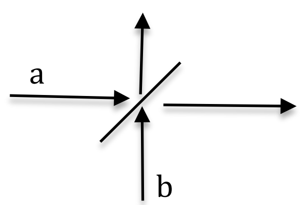
How should the quarter-wave plate be oriented to transform horizontal polarization to circular polarization? What is the rotation on the Poincaré sphere?
How should a half-wave plate be oriented to transform horizontal polarization to vertical polarization? What is the rotation on the Poincaré sphere?

(f) Extra credit (5 points). Show that an arbitrary SU(2) transformation on the Poincaré sphere can be constructed using two quarter-wave plates and one half-wave plates.

Problem 2: SU(2) Interferometers (30 points)

There is a formal equivalence between a Mach-Zender-type optical interferometer and a so-called Ramsey interferometer for any two-level quantum system that we will study soon. We also call this an SU(2) interferometer.

(a) Consider the following optical transformation: A symmetric beam splitter with transmission amplitude t and reflection amplitude r.



We can encode a qubit in the two orthogonal paths, "a" and "b", of a photon. We then define the standard basis

$$\left|\uparrow_z\right\rangle = \left|1_a, 0_b\right\rangle, \quad \left|\downarrow_z\right\rangle = \left|0_a, 1_b\right\rangle$$

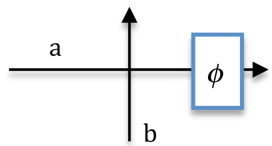
i.e., $\left|\uparrow_z\right\rangle$ is with one photon in path-a an no photons in path-b, and vice versa for $\left|\downarrow_z\right\rangle$. The transformation on the basis states is

$$\left|1_a, 0_b\right\rangle \Rightarrow t\left|1_a, 0_b\right\rangle + r\left|0_a, 1_b\right\rangle, \quad \left|0_a, 1_b\right\rangle \Rightarrow t\left|0_a, 1_b\right\rangle + r\left|1_a, 0_b\right\rangle$$

Show that the conditions for this map to be unitary are: $|t|^2+|r|^2=1$, $tr^*+t^*r=0$.

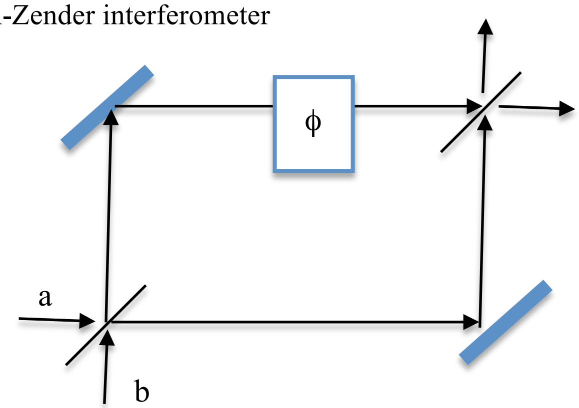
Write this map as an equivalent (up to a negligible phase) SU(2) rotation on the Bloch sphere.

(b) Show that the transformation in which mode-*a* gets a phase shift relative to mode-*b* is an SU(2) rotation. What is the axis and angle of the rotation?



(c) Show how to construct an arbitrary SU(2) rotation with phase shifters and a beam splitter.

Now consider a Mach-Zender interferometer



The 50-50 beam splitters are thin black lines and the mirrors are thick blue lines. We assume here that the optical path lengths of the two arms of the interferometer are equal. A phase shifter ϕ is placed in the upper arm.

(d) Show that up to a negligible overall phase, the sequence

beam-splitter → mirrors → phase shift →beam-splitter

is equivalent to the sequence of SU(2) transformations

$\pi/2$ -x-rotation → π -x-rotation → ϕ -z-rotation → $\pi/2$ -x-rotation

(e) Given the initial state $|\downarrow_z\rangle$, sketch the evolution on the Bloch sphere corresponding to this sequence.

(f) Show that up to an overall phase (which is negligible), the transformation from the input is equivalent to the rotation $e^{-i\frac{\phi}{2}\sigma_y}$. Using this, calculate the probability to find the state $|1_a,0_b\rangle$ at the output port given that the state was in $|1_a,0_b\rangle$ at the input port.

Problem 3: Some Algebra with Density Operators (15 Points)

(a) Prove some properties of the "trace" operation, $Tr(\hat{A}) \equiv \sum_{i=1}^{d} \langle e_i | \hat{A} | e_i \rangle$, where $\{|e_i\rangle\}$ is an orthonormal basis on a finite dimensional Hilbert space:

(i) $Tr(\hat{A})$ is independent of basis; $Tr\left(\hat{A}|\psi\rangle\langle\phi|\right) = \langle\phi|\hat{A}|\psi\rangle$; (ii) $Tr\left(\hat{A}\hat{B}\hat{C}\right) = Tr\left(\hat{C}\hat{A}\hat{B}\right)$.

(b) For a qubit whose density matrix is $\hat{\rho} = \frac{1}{2}\left(\hat{1} + \mathbf{Q}\cdot\hat{\vec{\sigma}}\right)$, where **Q** is the Bloch vector, show that $Tr\left(\hat{\rho}^2\right) = \frac{1}{2}\left(1 + |\mathbf{Q}|^2\right)$, and thus for a pure state |**Q**|=1, and the maximally mixed state |**Q**|=0.

(c) For the Mach-Zender interferometer Problem (2c), find the probability, $P_{1_{a,out}}$, to detect the photon in $|1_a;0_b\rangle$ at the output ports given the inputs in the state defined by each of the following density matrices:

$$(i)\ \hat{\rho}_{in} = |1_a,0_b\rangle\langle 1_a,0_b|;\ (ii)\ \hat{\rho}_{in} = \frac{1}{2}|1_a,0_b\rangle\langle 1_a,0_b| + \frac{1}{2}|0_a,1_b\rangle\langle 0_a,1_b|;$$

$$(iii)\ \hat{\rho}_{in} = \frac{1}{3}|1_a,0_b\rangle\langle 1_a,0_b| + \frac{2}{3}|0_a,1_b\rangle\langle 0_a,1_b|.$$

Sketch $P_{1_{a,out}}$ as a function of ϕ for each case and comment on your results.

Hint: You need to find $\hat{\rho}_{out}$ for each case, then determine $P_{1_{a,out}}$.