Physics 566: Quantum Optics I

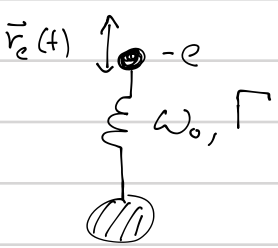
Lecture 3: Lorentz Oscillator Model

An important theme in quantum optics is the quantum mechanical interaction of atoms and photons. In order to have a deeper understanding of this, it is important to first have a firm foundation in the purely classical description, which actually gets us quite far and provides a good deal of intuition on the basic phenomenology. For this purpose, we turn to the Lorentz oscillator model of absorption and emission.

Lorentz oscillator

We treat simple atom as an electron harmonically bound to a fixed nucleus

$$\underset{\sim}{\vec{E}}(\vec{r},t) = \vec{E}_0 e^{i\phi(\vec{r},t)} e^{-i\omega t}$$



The binding has a natural frequency ω_0, and is exponentially damped at rate Γ (we will return to the question of the underlying mechanism of Γ). An external electromagnetic wave exerts forces on the charges. We take the nucleus to be heavy and essentially fixed. The light electron respond

$$\text{Newton's Law: } \quad m\ddot{\vec{r}} + m\Gamma\dot{\vec{r}} = \underbrace{-m\omega_0^2\vec{r}}_{\text{Restoring force}} \underbrace{-e\left(\vec{E}(t) + \frac{\dot{\vec{r}}}{c}\times\vec{B}(t)\right)}_{\text{Lorentz force}}$$

We take the electron to be non-relativistic and thus magnetic force is negligible

$$\Rightarrow \ddot{\vec{r}} + \Gamma\dot{\vec{r}} + \omega_0^2\vec{r} = \frac{-e}{m}\vec{E}_0\cos(\omega t - \phi(\vec{R},t))$$

Where ϕ is the phase at the position of the center of mass

Note: We have made the electric dipole approximation in taking the value of $\vec{E}(\vec{r},t)$ only at the center of mass and ignoring the variation of $\vec{E}$ across the size of the atom. This is valid when $\underset{\uparrow \text{ Wavelength}}{\lambda} \gg \underset{\nwarrow \text{ Bohr radius}}{a_0}$.

The interaction potential $V_{int}(\vec{r}_e, t) = -e\vec{r}_e \cdot \vec{E}(\vec{R},t) = -\vec{d}\cdot\vec{E}(\vec{R},t)$

After a transient time $\mathcal{O}(\Gamma^{-1})$, the driven oscillator reaches steady-state. In steady state the electron oscillates at the frequency of the applied force, not the natural resonance frequency. We can thus take as our ansatz for the general steady state solution for the atomic position

Stead state: $\vec{r}_e(t) = \mathrm{Re}(\tilde{\vec{r}}_e e^{-i\omega t})$

$$\Rightarrow (-\omega^2 - i\omega\Gamma + \omega_0^2)\tilde{\vec{r}}_e = -\frac{e}{m}\vec{E}_0 e^{i\phi}$$

$$\Rightarrow \tilde{\vec{r}}_e = \left[\frac{-e/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}\right]\vec{E}_0 e^{i\phi}$$

In steady state, the field induces an oscillating electric dipole

$$\vec{d} = \mathrm{Re}(-e\tilde{\vec{r}}_e e^{-i\omega t}) = \mathrm{Re}\left(\tilde{\alpha}(\omega)\vec{E}_0 e^{-i\omega t + i\phi}\right)$$

Where $\tilde{\alpha}(\omega) = \dfrac{e^2/m}{\omega_0^2 - \omega^2 - i\omega\Gamma}$ is the dynamic polarizability.

Note, the real dipole moment:

$$\vec{d}(t) = \mathrm{Re}(\tilde{\alpha}(\omega))\vec{E}_0 \cos(\omega t - \phi) + \mathrm{Im}(\tilde{\alpha}(\omega))\vec{E}_0 \sin(\omega t - \phi)$$

Thus, the real part of $\tilde{\alpha}$ describes the amount of the induced dipole that oscillates in phase with the drive and the imaginary part of $\tilde{\alpha}(\omega)$ describes the amount that oscillates in quadrature.

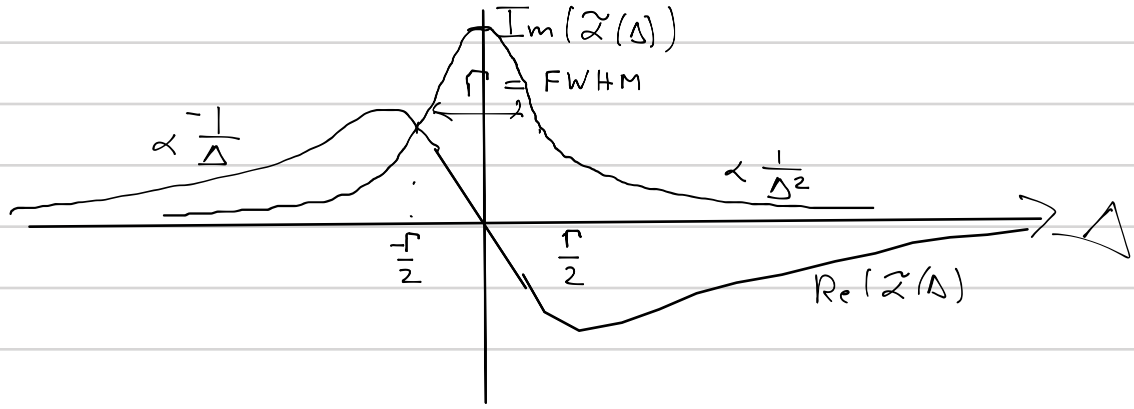
We often are interested in the atomic response near resonance. Define the detuning away from resonance $\Delta \equiv \omega - \omega_0$ ($\Delta < 0 \Rightarrow$ red detuning, $\Delta > 0 \Rightarrow$ blue detuning). Near resonance $\Rightarrow |\Delta| << \omega_0$; also we assume weak damping $\Gamma << \omega$ $\Rightarrow$

$$\tilde{\alpha}(\omega) = \frac{e^2/m}{(\omega_0+\omega)(\omega_0-\omega) - i\omega\Gamma} = \frac{e^2/m}{(2\omega_0+\Delta)(-\Delta) - i(\omega_0+\Delta)\Gamma}$$

$$\approx \frac{e^2/2m\omega_0}{-\Delta - i\frac{\Gamma}{2}} \qquad \text{Complex Lorentzian}$$

Near resonance:

$$\tilde{\alpha}(\Delta) \approx \frac{e^2}{2m\omega_0}\left[\underbrace{\frac{-\Delta}{\Delta^2+\Gamma^2}}_{\text{"Dispersive Lineshape"}} + i\,\underbrace{\frac{\Gamma/2}{\Delta^2+\frac{\Gamma^2}{4}}}_{\text{Lorentzian}}\right]$$



The Re($\tilde{\alpha}(\Delta)$) is known as a dispersive lineshape because of the connection to the index of refraction. Given an ensemble of Lorentz oscillators in a dilute gas of # density N, the induced polarization density $\vec{P} = N\vec{d}$

Wave eqn: $\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E} = 4\pi\frac{\partial^2\vec{P}}{\partial t^2} \Rightarrow \left(\nabla^2 + \frac{\omega^2}{c^2}\right)\underset{\sim}{E} = -4\pi\frac{\omega^2}{c^2}N\vec{d}$

$$\Rightarrow \left(\nabla^2 + (1+4\pi N\tilde{\alpha}(\omega))\frac{\omega^2}{c^2}\right)\underset{\sim}{E} = 0$$

Plane wave sol $\Rightarrow \underset{\sim}{\vec{E}}(\vec{r}) = e^{i\tilde{n}(\omega)\frac{\omega}{c}\hat{k}\cdot\vec{r}}$ $\qquad \hat{k}$ = direction of propagation

$$\tilde{n}(\omega) = \sqrt{1+4\pi N\tilde{\alpha}(\omega)} \cong 1 + 2\pi N\tilde{\alpha}(\omega) \quad \text{(Complex index of refraction)}$$

$$\Rightarrow \underset{\sim}{\vec{E}}(z) = \vec{E}_0\,\underbrace{e^{-2\pi N\,\mathrm{Im}(\tilde{\alpha})\frac{\omega}{c}z}}_{\text{Attenuation}}\;\underbrace{e^{i[1+2\pi N\,\mathrm{Re}(\tilde{\alpha})]\frac{\omega}{c}z}}_{\text{dispersion}} \quad (\text{taking } \hat{k} \text{ in z-direction})$$

Note: Intensity attenuation $I(z) = |\underset{\sim}{E}(z)|^2 = I_0 e^{-az}$ Beer's Law

$$a = 4\pi N\,\mathrm{Im}(\tilde{\alpha})\frac{\omega}{c} = N\sigma_{abs} \xleftarrow{} \text{absorption cross-section}$$

The attenuation of the field arises because energy is absorbed by the atoms. To see this consider the rate at which the fields do work on the atoms

$$\frac{d}{dt}W = \dot{\vec{r}}\cdot\vec{F}_{field} = -e\dot{\vec{r}}\cdot\vec{E}(\vec{R},t) = \dot{\vec{d}}\cdot\vec{E}$$

$$\frac{d}{dt}W = -\omega\, \mathrm{Re}(\tilde{\alpha}(\omega)) \sin(\omega t-\phi)\, E_0^2 \cos(\omega t-\phi) + \omega\, \mathrm{Im}(\tilde{\alpha}(\omega)) \cos(\omega t-\phi)\, E_0^2 \cos(\omega t-\phi)$$

Time averaging over the rapid oscillations

$$\overline{\frac{dW}{dt}} = \omega\, \mathrm{Im}(\tilde{\alpha}(\omega)) \frac{E_0^2}{2} \Rightarrow$$ $\mathrm{Im}(\tilde{\alpha}(\omega))$ describes energy absorbed by the atom from the field

This also follows from Maxwell's Eqns: $\vec{J}\cdot\vec{E}$ is the rate at which field do work on currents. $\vec{J} \propto \dot{\vec{d}} \Rightarrow$ for average energy transfer, $\vec{d}$ must oscillated as $\sin(\omega t-\phi)$ so $\dot{\vec{d}}$ oscillates like $\cos(\omega t-\phi)$.

Source of dissipation?

We see that the energy absorbed by the atom is proportional to the imaginary part of $\tilde{\alpha}(\omega)$ and this is proportional to Γ, the oscillator decay rate. This makes sense, because the absorption and decay of the oscillator are both dissipative process, i.e. irreversible. The problem of dissipation and irreversibility in quantum optics is an important topic that we will study throughout the course.

What is the source of Γ in the Lorentz oscillator model? An important contribution to Γ is collisional phase shifts, discussed in Lecture 2. The collisions interrupt the phase, and make the oscillator have a component in quadrature with the driving field, leading to absorption on average. The energy taken from the field goes to heat put into gas in thermal equilibrium

We can also see that collisions as damping the oscillations on average. In a simple model, the phase of the oscillator diffuses. The probability that the oscillator has phase ϕ at time t, assume each collision kicks the phase by ϕ_1, is $P(\phi,t) = \frac{1}{\sqrt{2\pi\,\Delta\phi^2(t)}}\, e^{-\frac{\phi^2}{2\Delta\phi^2(t)}}$, $\Delta\phi^2 = \phi_1^2 \frac{t}{\tau_0}$ ← collision time

$$\Rightarrow \vec{r}(t) = \vec{r}_0 \int_{-\infty}^{\infty} d\phi\, P(\phi,t)\, e^{-i(\omega t-\phi)} = \vec{r}_0\, e^{-\phi_1^2 \frac{t}{\tau_0}}\, e^{-i\omega t}$$

$$\Rightarrow \Gamma_{\text{collision}} = \frac{\phi_1^2}{\tau_0}$$

Natural linewidth

Suppose we have a single Lorentz oscillator fixed in space. There are no collisions and no "friction" in the binding spring. Do oscillation proceed undamped forever?

Energy conservation say no because the oscillating charge radiates E/M energy, and the radiated field carry energy. This is the most fundamental source of damping, radiation damping, which gives rise to the natural linewidth. (see Jackson, Classical Electrodynamics Ch. 16)

Recall, Larmor's formula Power radiated by an accelerating electron

$$P(t) = \frac{2}{3}\frac{e^2}{c^3}|\ddot{\vec{r}}|^2$$

This energy must come from the kinetic energy of the oscillating electron. The effect of a field on the charge that is the source of that field has a long and difficult history. In classical electrodynamics, this is known as the theory of "radiation reaction" and was formulated at the beginning of the 20th century by Abraham and the Lorentz in his classical theory of the electron. We review it here, following the treatment in Jackson's 3rd edition of Classical Electrodynamics, chapter 16

First, let's get a feel for the scales in the problem. The electron radiates with constant acceleration a for a time T, the energy radiated is

$$E_{rad} = \frac{2}{3}\frac{e^2a^2}{c^3}T$$

The radiated field will have nonnegligable effect on the motion when this energy is on the order of the kinetic energy gained by the electron during this acceleration. This determines a characteristic time scale τ_{rad} according to

$$\frac{e^2a^2}{c^3}\tau_{rad} = m(a\tau_{rad})^2 \Rightarrow \tau_{rad} = \frac{e^2}{mc^3} = 6.66\times10^{-24} \text{ sec}$$

Thus, we expect if the time of acceleration $T \gg \tau_{rad}$, radiation reaction is a perturbation on the equations of motion. We can interpret τ_{rad} according to the Lorentz classical theory of the electron. In this theory the electron is assume to be a ball of charge, of radius r_{class}, such that the potential energy of the distribution accounts for the rest energy

$$\frac{e^2}{r_{class}} = mc^2 \Rightarrow r_{class} = \frac{e^2}{mc^2} = 2.8 \times 10^{-15}\, m$$

$$\tau_{rad} = \frac{r_{class}}{c} = \text{time it takes light to propagate across the classical electron}$$

In this time the classical field and classical electron are correlated.

Consider now the equation of motion of the electron under the influence of an externally applied force as well as radiation reaction.

$$m\ddot{\vec{r}} = \vec{F}_{ext} + \vec{F}_{rad}$$

By energy conservation, we demand that the work done by the radiation-reaction force is equal to the energy decrease radiated into the field,

$$\int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{v}\, dt = -\int_{t_1}^{t_2} \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}} \cdot \dot{\vec{v}}\, dt$$

Using integration by parts $$= +\int_{t_1}^{t_2} \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} \cdot \vec{v}\, dt - \frac{2}{3}\frac{e^2}{c^3}\dot{\vec{v}} \cdot \vec{v}\Big|_{t_1}^{t_2}$$

If $\dot{\vec{v}} \cdot \vec{v} = 0$ @ t_1 + t_2 and/or the motion is periodic,

$$\Rightarrow \int_{t_1}^{t_2} \left(\vec{F}_{rad} - \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}}\right) \cdot \vec{v}\, dt = 0 \Rightarrow \vec{F}_{rad} = \frac{2}{3}\frac{e^2}{c^3}\ddot{\vec{v}} = \frac{2}{3} m \tau_{rad} \dddot{\vec{r}}$$

$$\Rightarrow m\ddot{\vec{r}} = \frac{2}{3} m \tau_{rad} \dddot{\vec{r}} + \vec{F}_{rad}$$

This is known as the Abraham-Lorentz equation. It is not well behaved diff 'eqn, having pathological solutions. For example, suppose there is no external force. There are two possible solutions:

$$\ddot{\vec{r}}(t) = \begin{cases} 0 \\ a\, e^{\frac{3}{2} t/\tau_{rad}} \leftarrow \text{acausal !} \end{cases}$$

The second solution is run away self acceleration, and should be rejected.

An alternative to the Abraham Lorentz equation. An alternative approach is to treat radiation reaction as a perturbation, and thus to lowest order, the acceleration is dominated by the external force. Then,

$$\int_{t_1}^{t_2} \vec{F}_{rad} \cdot \vec{v}\, dt = -\frac{2}{3}\int_{t_1}^{t_2} \frac{e^2}{c^3}\dot{\vec{v}} \cdot \frac{\vec{F}_{ext}}{m}\, dt = +\frac{2}{3}\tau \int_{t_1}^{t_2} \frac{d\vec{F}_{ext}}{dt} \cdot \vec{v} dt$$

$$\Rightarrow \vec{F}_{rad} = \frac{2}{3}\tau \frac{d\vec{F}_{ext}}{dt} = \frac{2}{3}\tau\left(\frac{\partial \vec{F}_{ext}}{\partial t} + \vec{v}\cdot\vec{\nabla}\vec{F}_{ext}\right)$$

This equation avoid the pathologies the the Abraham-Lorentz equation and should hold in the perturbative regime

We can now calculate the effect of radiation damping according to the classical theory. Consider equation for a bound electron under the force of the spring $F_{ext} = -m\omega_0^2 \vec{r}$ and radiation reaction

$$\Rightarrow m\ddot{\vec{r}} = -m\omega_0^2\vec{r} - \frac{2}{3}m\tau\dot{\vec{r}}$$

$$\Rightarrow \ddot{\vec{r}} + \Gamma_{rad}\dot{\vec{r}} + \omega_0^2\vec{r} = 0$$

The radiation damping rate, $\boxed{\Gamma_{rad} = \frac{2}{3}\omega_0^2 \tau_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0^2 = \frac{2}{3}(k_0 r_{class})\omega_0}$

As we will find, the quantum theory of electromagnetism yields a radiative decay rate from an excited to ground state known as the Einstein-A Coefficient

$$\Gamma_{quant} = \frac{4}{3}\frac{e^2}{\hbar}\underbrace{|\langle e|\hat{\vec{r}}|g\rangle|^2}_{\text{electron position matrix element}}\frac{\omega_0^3}{c^3} \leftarrow \omega_0 = \frac{E_e - E_g}{\hbar} \quad \text{(Bohr frequency)}$$

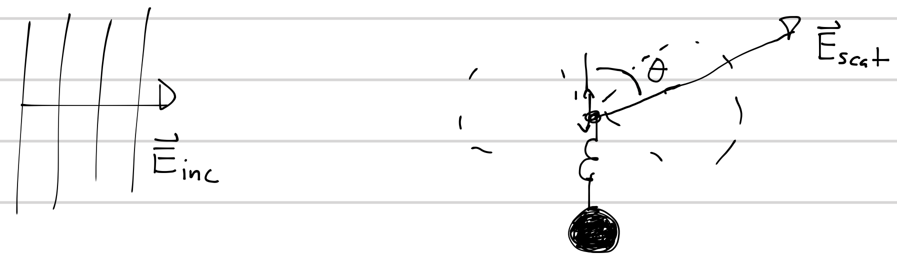
The absorption linewidth of real atomic resonances is thus not exactly given by the classical theory. Lorentz corrected this with fudge factors known as the oscillator strength,

$$\boxed{f = \frac{\Gamma_{quant}}{\Gamma_{class}} = \frac{2m\omega_0^2}{\hbar}|\langle e|\vec{r}|g\rangle|^2 = \frac{|\langle e|\vec{r}|g\rangle|^2}{\Delta X^2_{SHO}}}$$

The oscillator strength is the square of the ratio of the electron position matrix element to the rms width of the quantum SHO for the electron mass in & binding frequency ω_0.

Wave scattering, Energy Shift and Resonance Fluorescence

Consider now the problem of an incident E/M wave driving the bound electron. The oscillating charge will radiate into all directions according the dipole pattern. This is the problem of scattering of the incident wave.



The external force on the electron is now the binding plus the driving of the external electric field, so the equation of motion, including radiation reaction is

$$\ddot{\vec{r}} + \Gamma_{rad}\dot{\vec{r}} + \omega_0^2 \vec{r} = \frac{-e}{m}\left(\vec{E}(\vec{R},t) + \frac{2}{3}\tau\frac{\partial \vec{E}}{\partial t}\right)$$

Going to the complex representation: $\vec{r} = Re(\underset{\sim}{\vec{r}}\, e^{-i\omega t})$, $\vec{E} = Re(\underset{\sim}{\vec{E}}\, e^{-i\omega t})$

$$\Rightarrow \underset{\sim}{\vec{r}} = \frac{-e/m\,(1-\frac{2}{3}i\omega\tau)\underset{\sim}{\vec{E}}}{\omega_0^2 - \omega^2 - i\omega\Gamma_{rad}} \quad \Rightarrow \quad \tilde{\alpha} = \frac{\frac{e^2}{m}(1-\frac{2}{3}i\omega\tau)}{\omega_0^2-\omega^2 - i\omega\Gamma_{rad}}$$

The roots of the denominator determine the resonance frequency and absorption linewidth

$$\omega^2 + i\Gamma_{rad}\omega - \omega_0^2 = 0 \Rightarrow \omega_{\pm} = -i\frac{\Gamma_{rad}}{2} \pm \sqrt{\frac{\Gamma_{rad}^2}{4} + \omega_0^2}$$

$$\Rightarrow \omega_{\pm} \approx \pm(\omega_0 + \delta\omega_{rad}) - i\frac{\Gamma_{rad}}{2}$$

$$\text{where } \delta\omega_{rad} = \frac{\Gamma_{rad}^2}{8\omega_0} = \frac{1}{18}\omega_0^3\tau^2$$

The oscillator has natural damping rate Γ_{rad}. In addition there is a "radiative shift" in the resonance frequency $\delta\omega_{rad}$. In actuality, our simple model does not capture the true radiative shift in the resonance frequency. Firstly, our radiative reaction equation is good only to first order in τ. More importantly, in a real atom the vacuum fluctuations are a major contribution to perturbing the energy levels, leading to a much larger shift in resonance frequency

the quantum mechanical level shift: $\frac{\delta\omega_{qm}}{\omega_0} \sim \omega_0\tau \log\left(\frac{mc^2}{\hbar\omega_0}\right)$ Lamb Shift!

Keeping terms only to first order in τ, and looking at the near-resonance response the induced dipole moment

$$\underset{\sim}{\vec{d}}_{ind} = \frac{e^2/2m\omega_0}{-\Delta - i\frac{\Gamma_{rad}}{2}} \underset{\sim}{\vec{E}} = \tilde{\alpha}(\omega) \underset{\sim}{\vec{E}}$$

The induced dipole radiates dipole-radiation, whose electric field is

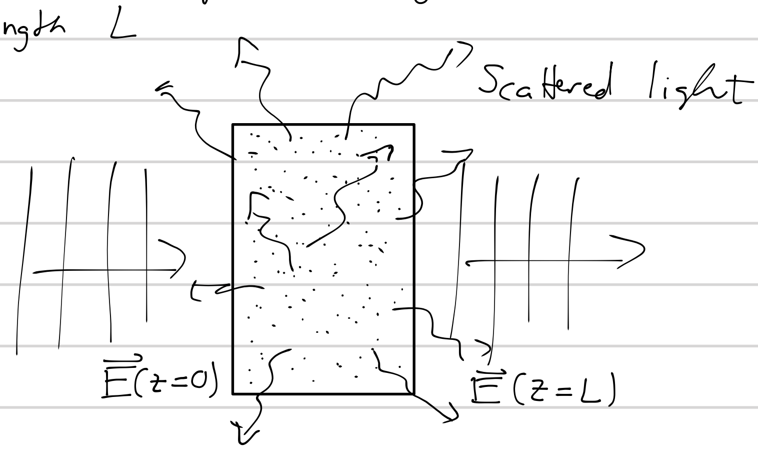
$$\vec{E}_{rad} = -k_0^2 \left(\hat{r} \times (\hat{r} \times \underset{\sim}{d}_{ind})\right) \frac{e^{i(k_0 r - \omega_0 t)}}{r}, \quad k_0 = \frac{\omega_0}{c}$$

$$= \tilde{\alpha}(\omega) k_0^2 \sin\theta \, E_0 \frac{e^{i(k_0 r - \omega_0 t)}}{r}$$

Note: The spectrum of the radiated field in monochromatic at frequency ω. That is, the scattering by the linear oscillator is elastic – The radiated spectrum is a delta function with the same freq as the drive. This scattered radiation when the atom is driven near resonance is known as "resonance fluorescence" The nature of resonance fluorescence has an important role in the history of quantum optics. We will return to it time and again in our study of the quantum nature of atom-photon interactions and the nonclassical nature of light itself.

Attenuation by Scattering

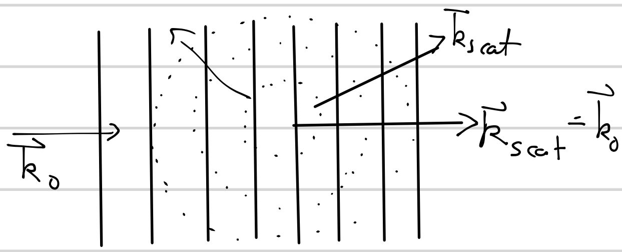
We have seen that when a plane wave propagates through a gas of atoms it will be attenuated, with exponentially decreasing intensity. After passing through a vapor cell of length L



$$I(L) = I_0 e^{-aL}, \quad a \propto \mathrm{Im}(\tilde{\alpha}(\omega)) \propto \Gamma_{rad}$$

In this case, the attenuation is due to scattering (radiation) of light into other directions.

The attenuation of a wave by scattering is a fundamental effect, and is codified in the optical theorem. Attenuation by scattering can be explained by destructive interference between the incident wave & the "forward scattered" wave, i.e. the wave scattered into the same spatial mode as the incident wave (here plane)



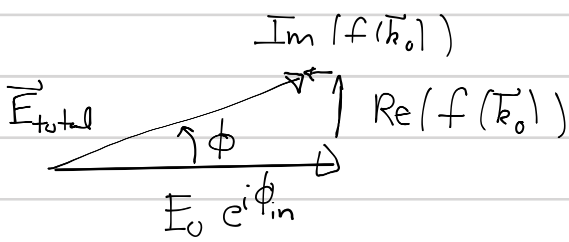
The field scattered field $E_{scat}(\vec{k}_{scat}) \propto f(\vec{k}_{scat}) \frac{e^{ikr}}{r}$

The imaginary part of the "scattering amplitude" into the forward direction, $f(\vec{k}_{scat} = \vec{k}_0)$ leads to destructive interference and attenuation. The field taken from the forward direction must be scattered into all other modes. This is the content of the optical theorem which states,

$$\boxed{\text{The total scattering cross-section } \sigma_{total} = \frac{4\pi}{k} \operatorname{Im}\left(f(\vec{k}_{scat} = \vec{k}_0)\right)}$$

(Note, I am neglecting the vector nature of the wave)

Note: the real part of the scattering amplitude leads to the index of refraction that is the transmitted field after passing through the gas is phase shifted with respect to the incident field. As a phasor



For the case of the plane wave the phase shift $\phi = \frac{\omega}{c} L(\operatorname{Re}(\tilde{n}) - 1)$
$$= 2\pi N \operatorname{Re}(\tilde{\alpha}) kL$$

Aside: There is a mysterious extra factor of "i" floating around since the real part of $\tilde{\alpha}(\omega)$ adds in quadrature, while the imaginary part adds in phase. This is a spatial propagation effect, since interference is in the "far field."