Physics 566 - Lecture 7

Two-Level Atom Damped Response

Given the master equation for a two-level atom introduced in Lect. 7, we can study the steady state response in the presence of driving and relaxation.

Recall the time evolution of the expected values of the electric dipole moment driven by a monochromatic field $\vec{E}_L(\vec{r},t) = \vec{e}\, E_0 \cos\omega_L t$

$$\langle \vec{d}\cdot\vec{e}\rangle(t) = d_{eg}\left(u(t)\cos\omega_L t - v(t)\sin\omega_L t\right) = d_{eg}\,\mathrm{Re}\left[(u-iv)\,e^{-i\omega_L t}\right]$$

$u(t)$: In phase with $\vec{E}_L \Rightarrow$ Index of refraction

$-v(t)$ In quadrature with $\vec{E}_L \Rightarrow$ Asorption/emission

$$= \mathrm{Tr}(\hat{\rho}\,|g\rangle\langle e|) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_-) = \mathrm{Tr}\left(\hat{\rho}\,\frac{\hat{\sigma}_x - i\hat{\sigma}_y}{2}\right) = \frac{1}{2}(u-iv)$$

$$\Rightarrow \langle \vec{d}\cdot\vec{e}\rangle = \mathrm{Re}\left[2\,d_{eg}\,\rho_{eg}\,e^{-i\omega_L t}\right]$$

We now use the master equation to find ρ_{ge} in steady state, and thus the steady-state dipole response. In steady state, $\dot{\rho}_{\alpha\beta} = 0$

$$\Rightarrow \left(+i\Delta - \frac{\Gamma}{2}\right)\rho_{eg}^{s.s.} + i\frac{\Omega}{2}\left(\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.}\right) = 0$$

$$\Rightarrow \rho_{eg}^{s.s.} = \frac{-\Omega/2}{\Delta + i\Gamma/2}\left(\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.}\right)$$

Suppose the conditions were "weak driving," so in steady state, $\rho_{ee}^{s.s.} \ll \rho_{gg}^{s.s.} \approx 1$

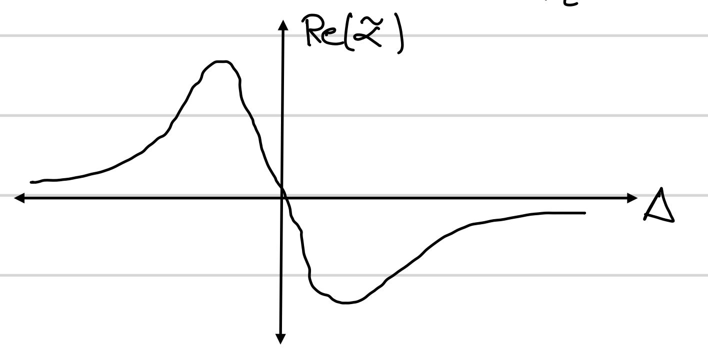
$$\Rightarrow \rho_{eg}^{s.s.} \approx \frac{\Omega/2}{\Delta + i\Gamma/2} = \frac{-d_{eg}/2\hbar}{\Delta + i\Gamma/2} E_0 \; : \quad \text{Linear response}$$

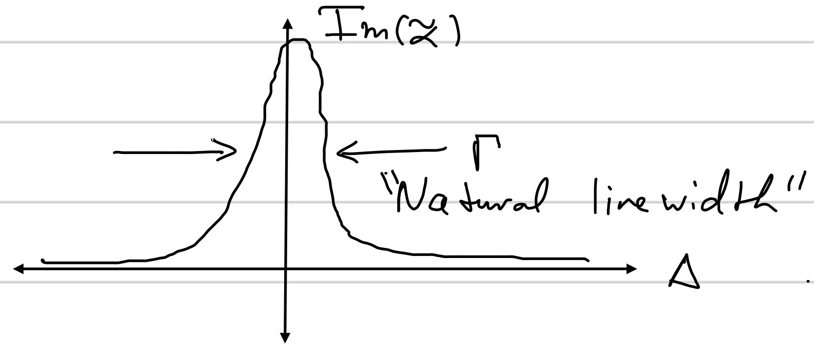
$$\Rightarrow \langle \hat{\vec{d}}\cdot\vec{e}\rangle_{s.s.} = \mathrm{Re}\left[2\,d_{eg}\,\rho_{eg}^{s.s}\,e^{-i\omega_L t}\right] = \mathrm{Re}\left[\frac{-|d_{eg}|^2/\hbar}{\Delta + i\Gamma/2} E_0\, e^{-i\omega_L t}\right] = \mathrm{Re}\left(\tilde{\alpha}\, E_0 e^{-i\omega_L t}\right)$$

Atom dynamical linear polarizability

$$\tilde{\alpha} = \frac{-|d_{eg}|^2}{\hbar\left(\Delta + i\frac{\Gamma}{2}\right)} = \frac{|d_{eg}|^2}{\hbar}\left(\frac{-\Delta + i\Gamma/2}{\Delta^2 + \frac{\Gamma^2}{4}}\right) \quad \text{Complex Lorentzian}$$

$$\tilde{\alpha} = f\,\tilde{\alpha}_{class}: \quad f = \frac{2m\omega_{eg}}{\hbar}|\langle e|\vec{r}|g\rangle|^2, \qquad \tilde{\alpha}_{class} = \frac{\frac{e^2}{2m\omega_{eg}}}{-\Delta - i\frac{\Gamma}{2}}$$





$$\boxed{\text{The dipole response of a two-level atom to weak driving near resonance s.t. } \rho_{ee} \ll \rho_{gg} \approx 1 \text{ is equivalent to the response of a linear harmonic oscillator!}}$$

Rate Equations

After coherences have relaxed to steady state, we can look at the dynamics solely of populations. These are known as the "rate equations" in laser physics.

$$\Rightarrow \dot{\rho}_{ee} = -\dot{\rho}_{gg} = -\Gamma\rho_{ee} + i\frac{\Omega}{2}(\rho_{eg} - \rho_{ge}) = -\Gamma\rho_{ee} + \Omega\,\mathrm{Im}(\rho_{ge})$$

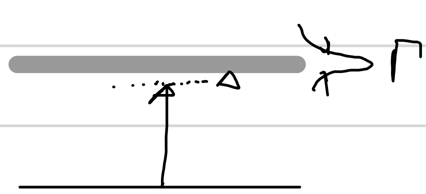
$$\Rightarrow \dot{\rho}_{ee} = -\Gamma\rho_{ee} - \frac{\left(\frac{\Omega^2}{4}\right)\Gamma}{\Delta^2 + \Gamma^2/4}(\rho_{ee} - \rho_{gg})$$

Interpretation: Spontaneous & Stimulated Emission/Absorption

Fermi's Golden Rule: $\Gamma_{abs} = \frac{2\pi}{\hbar}|\langle e|\hat{H}^{(+)}_{int}|g\rangle|^2\,\mathcal{D}(E_e = E_g + \hbar\omega_L)$

$$\hat{H}_{int} = -\hat{d}\cdot\vec{E}_L(t) = \underbrace{\frac{\hbar\Omega}{2}\hat{\sigma}_+ e^{-i\omega_L t}}_{\hat{H}^{(+)}_{int}} + \underbrace{\frac{\hbar\Omega}{2}\hat{\sigma}_- e^{+i\omega_L t}}_{\hat{H}^{(-)}_{int}}$$

Density of states



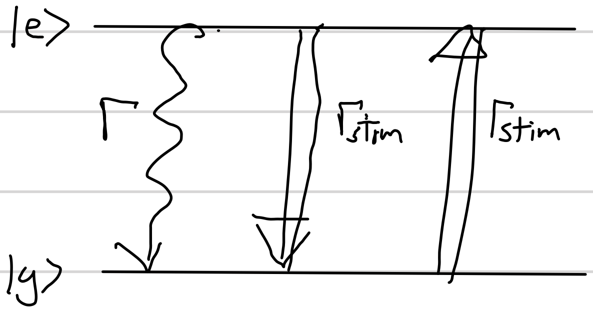
$$\mathcal{D}(E_e = E_g + \hbar\omega_L) = \frac{\hbar\Gamma/2\pi}{(E_e - E_g - \hbar\omega_L)^2 + \left(\frac{\hbar\Gamma}{2}\right)^2}$$

$$= \frac{\Gamma/2\pi\hbar}{\Delta^2 + \Gamma^2/4} \quad \text{Atomic lineshape}$$

$$\boxed{\begin{aligned} &\Gamma_{abs} = \frac{\Omega^2/4}{\Delta^2 + \Gamma^2/4}\Gamma = \Gamma_{\text{stimulated-emission}} \equiv \Gamma_{stim} \\ &\Gamma_{stim} = \frac{s}{2}\Gamma, \quad s = \frac{\Omega^2/2}{\Delta^2 + \Gamma^2/4} \equiv \text{Saturation parameter} \end{aligned}}$$

We thus arrive at rate equations for the populations $N_e \equiv \rho_{ee}$, $N_g \equiv \rho_{gg}$

$$\boxed{\begin{aligned} \dot{N}_e &= -(\Gamma + \Gamma_{stim}) N_e + \Gamma_{stim} N_g \\ \dot{N}_g &= -\Gamma_{stim} N_g + (\Gamma + \Gamma_{stim}) N_e \end{aligned}}$$



$$\dot{N}_e = -\dot{N}_g$$

$$N_e + N_g = 1$$

Steady state $\Rightarrow$ detailed balance $\dot{N}_e = \dot{N}_g = 0$

$$\dot{N}_e^{S.S.} = 0 = -(\Gamma + \Gamma_{stim}) N_e^{S.S.} + \Gamma_{stim}(1 - N_e^{S.S.})$$

$$\Rightarrow \boxed{\begin{aligned} N_e^{S.S.} &= \frac{\Gamma_{stim}}{\Gamma + 2\Gamma_{stim}} = \frac{s/2}{1+s}, \quad N_g = \frac{1 + s/2}{1+s} \\ s &= \frac{2\Gamma_{stim}}{\Gamma} = \frac{\Omega^2/2}{\Delta^2 + \frac{\Gamma^2}{4}} \quad \text{Saturation parameter} \end{aligned}}$$

For a given detuning & oscillator strength, the saturation parameter determines the intensity $|E|^2 \propto \Omega^2$ at which we pump substantial population into the excited state.

$$\boxed{\begin{aligned} &\text{When } s \ll 1, \quad N_e \approx \tfrac{s}{2} \ll 1, \quad N_g \approx 1 - \tfrac{s}{2} \approx 1 \quad (\text{Linear regime}) \\ &\text{When } s \to \infty, \quad N_e \approx N_g \approx \tfrac{1}{2}, \text{ the transition is "saturated"} \end{aligned}}$$

<u>Resonant Behavior: $\Delta = 0$</u>

Saturation parameter $s(\Delta=0) \equiv s_0 = \frac{2\Omega^2}{\Gamma^2} \Rightarrow s_0 = 1$ when $\Omega = \Gamma/\sqrt{2}$

On resonance, saturate the transition when $\Omega \sim \Gamma$

Since $\Omega = -\frac{d_{eg}E}{\hbar} \Rightarrow \Omega^2 \propto E^2 \propto I$ (intensity of laser)

$$\Rightarrow S_0 = \frac{2\Omega^2}{\Gamma^2} = \frac{I}{I_{sat}}$$; $I_{sat} \equiv$ "saturation intensity" = Intensity at which $S_0 = 1$

$$I_{sat} = \frac{I}{S_0} = \frac{I}{2\Omega^2/\Gamma^2} = \frac{\hbar^2\Gamma^2}{2d_{eg}^2}\frac{I}{E_0^2}$$

Aside: $\Gamma = \frac{4}{3}\frac{d_{eg}^2}{\hbar}\left(\frac{\omega_{eg}}{c}\right)^3$, $I = \frac{c}{8\pi}E_0^2$

$$\Rightarrow \boxed{I_{sat} = \hbar\omega_{eg}\frac{1}{6\pi\left(\frac{c}{\omega_{eg}}\right)^2}\frac{\Gamma}{2} = \frac{\hbar\omega_{eg}}{\sigma_0}\frac{\Gamma}{2} = \frac{\text{energy}}{\text{time}\cdot\text{Area}}, \quad \sigma_0 = 6\pi\lambdabar^2 = \text{absorption/scattering cross section}}$$

The saturation parameter is a characteristic of the atom. For near unit oscillator strength, $\Gamma \sim \frac{\omega_{eg}^2}{c} r_{class} \Rightarrow I_{sat} \sim \hbar\frac{\omega_{eg}^4}{c^3} r_c \sim 1\ \frac{mW}{cm^2}$ for an optical transition

Off-resonance:

$$\boxed{s = \frac{S_0}{1+\frac{4\Delta^2}{\Gamma^2}} = \frac{I/I_{sat}}{1+\frac{4\Delta^2}{\Gamma^2}} \quad \left(s \propto \frac{I}{\Delta^2} \text{ for } \Delta \gg \Gamma\right)}$$

Absorption cross-section and photon scattering

Recall the definition of the cross-section:

Given an incident intensity I, the absorbed power is

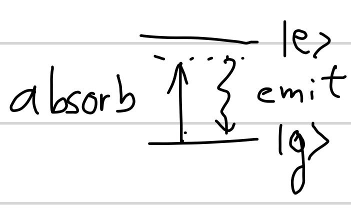
$$P_{abs} = I\sigma_{abs} = (\text{Rate of absorption}) \times \hbar\omega$$

Consider the case of weak excitation: $s \ll 1$

Rate of absorption $= N_g\Gamma_{stim} \approx 1\,\frac{s\Gamma}{2} \Rightarrow I\sigma_{abs} = \frac{s}{2}\Gamma\hbar\omega = \left(\frac{I/I_{sat}}{1+4\Delta^2/\Gamma^2}\right)\frac{\Gamma}{2}\hbar\omega$

On resonance $I\sigma_0 = \frac{I}{I_{sat}}\frac{\Gamma}{2}\hbar\omega \Rightarrow I_{sat} = \frac{\hbar\omega}{\sigma_0}\frac{\Gamma}{2}$ as before

For an atom, all absorbed light is eventually reëmitted $\Rightarrow$ scattering



For low saturation : Photon scattering rate : $\gamma_s = \frac{s}{2}\Gamma = \frac{I}{\hbar\omega}\sigma_{abs} = \frac{I}{\hbar\omega}\frac{\sigma_0}{1+\frac{4\Delta^2}{\Gamma^2}}$

Saturation and Power Broadening

Let us return to the steady-state coherences and the atomic polarizability $\tilde{\alpha}$. The lineshape we found was for weak excitation, i.e., $s \ll 1$.

More generally, $$\rho_{eg}^{s.s.} = \frac{-\Omega/2}{\Delta + i\Gamma/2}\left(\rho_{ee}^{s.s.} - \rho_{gg}^{s.s.}\right)$$

Steady state Bloch vector component

$$w^{s.s.} = \rho_{ee}^{s.s.} - \rho_{gg}^{s.s.} = -\frac{1}{1+s}$$

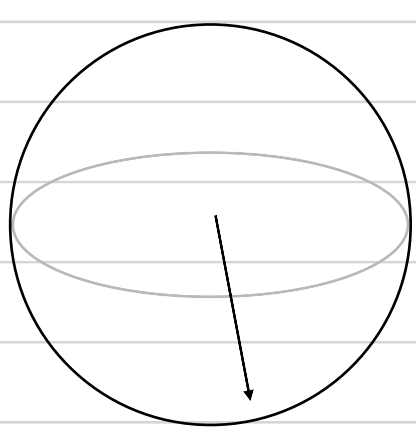
$$u^{s.s.} = 2\,\mathrm{Re}\left(\rho_{eg}^{s.s.}\right) = \frac{\Omega\Delta}{\Delta^2 + \Gamma^2/4}\left(\frac{1}{1+s}\right) = \frac{2\Delta}{\Omega}\,\frac{s}{1+s}$$

$$v^{s.s.} = -2\,\mathrm{Im}\left(\rho_{eg}^{s.s.}\right) = \frac{-\Omega\Gamma/2}{\Delta^2 + \Gamma^2/4}\left(\frac{1}{1+s}\right) = \frac{\Gamma}{-\Omega}\,\frac{s}{1+s}$$

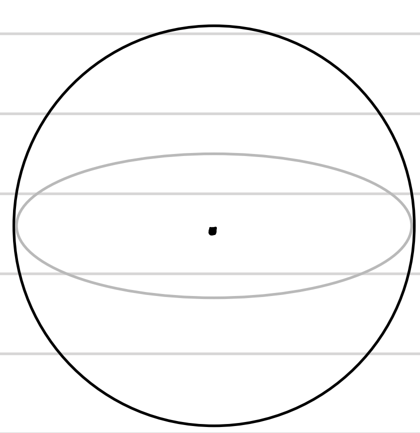
For $s \gg 1$ $\quad w^{s.s} \to 0 \Rightarrow u^{s.s.} \to 0,\ v^{s.s.} \to 0$

$\equiv \rho_{ee}^{s.s} = \rho_{gg}^{s.s.} = \frac{1}{2}\,,\ \rho_{eg}^{s.s.} = 0$

$\Rightarrow$ Maximally saturated state: $\hat{\rho} \doteq \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$: Maximally mixed state



$s \ll 1$

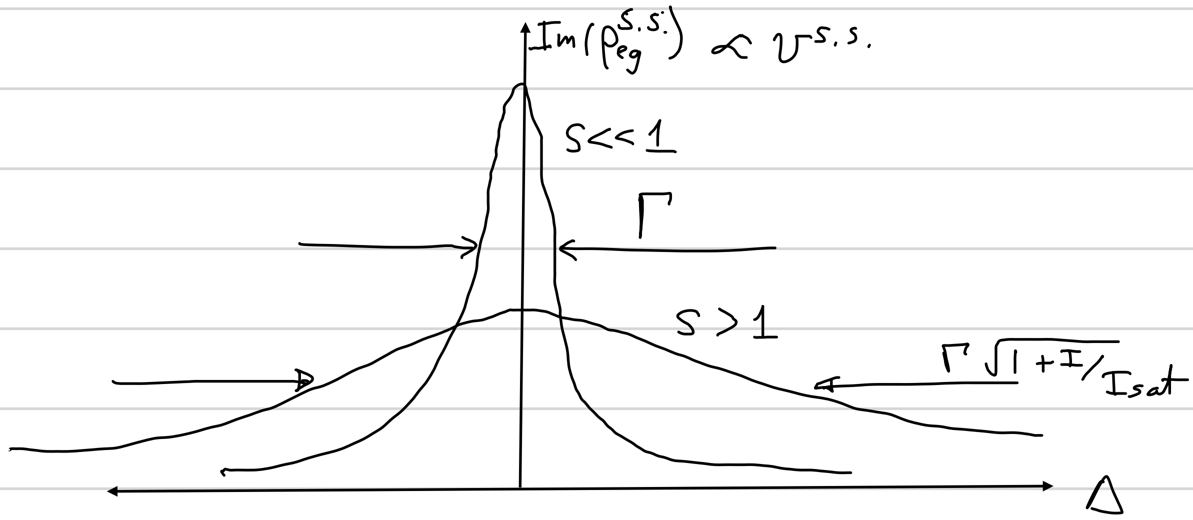


$s \to \infty$

Power Broadening

Note: $$\frac{s}{1+s} = \frac{\Omega^2/2\,[\Delta^2+\frac{\Gamma^2}{4}]^{-1}}{1+\Omega^2/2\,[\Delta^2+\frac{\Gamma^2}{4}]^{-1}} = \frac{\Omega^2/2}{\Delta^2+\frac{\Gamma^2}{4}+\frac{\Omega^2}{2}} = \frac{\Omega^2/2}{\Delta^2+\frac{\Gamma^2}{4}(1+I/I_{sat})}$$

$$\Rightarrow \quad v^{s.s.} = \frac{|\Omega|\,\Gamma/2}{\Delta^2+\frac{\Gamma^2}{4}\left(1+\frac{I}{I_{sat}}\right)}$$



The probability of exciting the atom is "broadened" when the intensity $I \gtrsim I_{sat}$. The effect is known as "power broadening."