Quantum Optics - Physics 566
Coherent States - Quasiclassical Light

Are photons real?

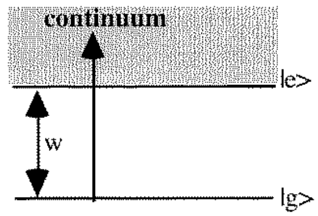
The nature of electromagnetic radiation and its interaction with matter has played an important historical role in the development of quantum mechanics. From Planck's introduction of the quantum in the description of the black-body spectrum and Einstein's description of the photoelectric effect, the photon played a central role. Today we have a very successful theory of quantum electrodynamics (QED) which adequately describes all observed interaction of photons and electrons. Given the current state of success of our models we may pose the question, in what situations is a quantum description of the electromagnetic field necessary? So far we have seen that complete description of spontaneous emission required the existence of a quantized field, though even the rate of emission can be understood classically as radiation reaction. The vacuum field played a role only in initiating the initiating the random dipole. Is this the only phenomenon which demands this sophisticated theory?

At first glance one may say that the result of the photoelectric effect already demands a nonclassical theory of light. After all Einstein invented the concept to explain the previous anomalous results of the experiments. Amongst them are

- The emission of a photo-electrons has a threshold energy which depends only on the frequency of the light and is independent of its intensity.
- The energy of the photoelectrons increase linearly with the frequency of the light once it is above the threshold value.
- A photoelectron is observed almost instantaneously after the sample is illuminated no matter how small the light intensity (the classical power flux).

The "corpuscular" photon hypothesis naturally explains these results, for which Einstein was granted the Nobel prize. But, was it really necessary? In the recent decades it was realized (see R. Loudon, Rep. Progr. Phys. **43**, 913 (1980); W. E. Lamb and M. O. Scully in "Polarization Matière et Rayonnement", ed. Société Française de Physique, Presses Universitaires de France, Paris (1969)), that in fact a *semiclassical* theory of the photoelectric effect, in which the light is treated classically but the matter is treated quantum mechanically according to the Schrödinger equation completely explains these results.

Consider for example a photoelectric detector modeled by a collection of atoms possessing a ground state |g> and a continuum of ionized states |e>, separated by an energy interval W.



If a monochromatic electromagnetic field is incident on these atoms, the rate of photo-ionization can be found using a classical field by Fermi's Golden rule

$$\frac{d}{dt}P_{e\to g} = \frac{\pi}{2\hbar}\int dE_e \left|\langle e|d|g\rangle\right|^2 E_0^2\, \delta(E_e - E_g - \hbar\omega)\, D(E_e),$$

where $D(E_e)$ is the density of final electron energies, an E_0 is the classical electric field amplitude. All of the observed phenomena are explained by this expression. The existence of an energy threshold relies on the fact that the density of excited states vanishes if $E_e < E_g - \hbar\omega$, and thus the kinetic energy of the electron is $\hbar\omega - W$. The rate of photo-ionization is proportional to the classical intensity $I \sim E_0^2$. Thus, the probability of photo-ionizing in a very short time interval Δt is

$$P_{e\to g} = \eta\, I\, \Delta t$$

where η is a constant proportional to the absorption cross section for ionization. The probability of detecting a photoelectron is nonvanishing for an arbitrarily small Δt. Thus, we have established that the observed features of the photoelectric effect are explained by quantizing the matter of the photodetector and treating the electromagnetic field *classically.* We should note that at the time of Einstein hypothesis, no quantum theory of the atom had been established. Einstein's genius allowed him to make the leap in faith that has withstood the test of time. Indeed, the result that there is a nonvanishing probability of photoionizing for an arbitrarily short time interval of interaction is still in sharp contrast to the classical theory, requiring $\hbar\omega$ of energy to be removed from the field, where the classical theory would predict that only a small fraction of that energy would have reached the atom. This type of nonlocal energy conservation would have been very distasteful to Einstein.

During the 1960s L. Mandel took this semiclassical theory further in order to explain the statistics of photoelectrons (see L. Mandel, Progress in Optics, **2**, 181 (1963), ed. E. Wolf). We seek the probability of photo-ionizing n atoms in a time T. According to the semiclassical theory, the probability of detecting 1 photon in a very short time interval between t and t +Δt is $P(n=1; t, t+\Delta t) = \eta I \Delta t$, and thus the probability of detecting no photons is $P(n=0; t, t+\Delta t) = 1 - \eta I \Delta t \approx \exp(-\eta I \Delta t)$. Furthermore, he made the fundamental assumption that the detection of photons in two separate time intervals are **statistically independent**. These are the classic properties of a Poison distribution. Thus, the probability of detecting n photons in the time interval T is

$$P(n,T) = \frac{\bar{n}(T)^n}{n!} e^{-\bar{n}(T)},$$

where $\bar{n}(T) = \eta I T$ is the average number of photons detected in a time T for a constant intensity I. NOTE, the assumptions that go into this formula imply that *both* the arrival time of photons and the absorption of the photon are Poison processes. This was hypothesized in Mandel's landmark paper, L. Mandel, Proc. Phys. Soc. (London) **72**, 1037 (1958).

The mean squared fluctuation in photon number detected in this time interval as determined by the Poison probability distribution is

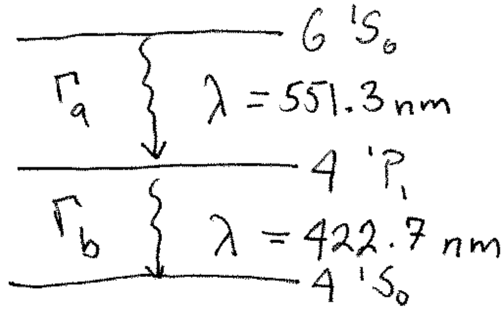
$$\Delta n(T)^2 = \langle n^2 \rangle - \langle n \rangle^2 = \sum_n n^2 P(n,T) - \left(\sum_n n P(n,T) \right)^2 = \bar{n}(T).$$

The rms fluctuation $\Delta n(T) = \sqrt{\bar{n}(T)}$ is known as **shot noise**. Because it represents the minimum uncertainty for a perfectly stable classical intensity due to the quantum randomness in the detection, this fluctuation is sometime known as the **standard quantum limit**. More generally, for fields with a fluctuating intensity,

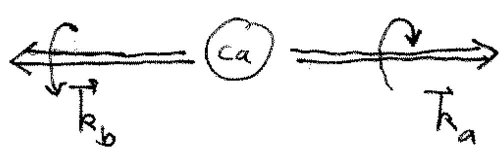
$$P(n,T) = \int dI \; p(I) \frac{(\eta I T)^n}{n!} e^{-\eta I T}: \text{ Mandel's formula.}$$

The total fluctuation $\Delta n(T)^2 \geq \bar{n}(T)$, with equality only for a perfectly stable intensity. Thus, in the semiclassical theory, shot noise is the minimum possible fluctuation, stemming from the randomness of quantum mechanical photo-ionization process (and the implicit assumption of random photon arrival times).

Given the success of the semiclassical theory to explain many observed phenomena, are there situations where it fails? When we carefully examine Mandel's theory, we see that they only assumption that may be violated is the statistical independence of photoionization. This must be true in the *semiclassical* theory, since the only statistical effect is the random time at which the atom absorbs the photon, which is statistically independent from the time any other atom absorbs, and no further correlations can be built into the classical field. However, there are situations which *cannot* be described by any classical theory. Consider for example the light produced in the "atomic cascade" of Ca spontaneously emitting through the transitions $6\,{}^1S_0 \rightarrow 4\,{}^1P_1 \rightarrow 4\,{}^1S_0$:

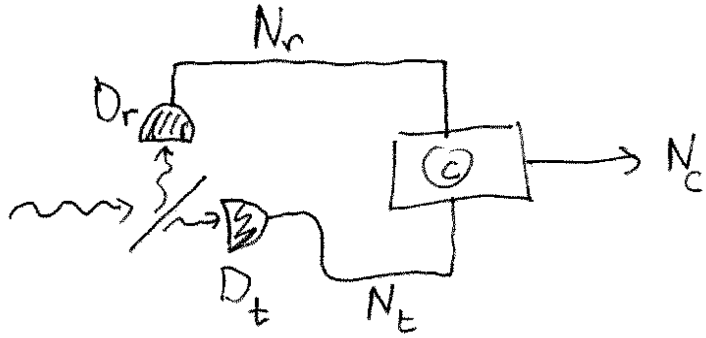


The intermediate $4\,{}^1P_1$ state is very short lived, with a lifetime τ=4.5 nsec. Thus, any photon emitted at 551.3 nm on the $6\,{}^1S_0 \rightarrow 4\,{}^1P_1$ transition must be correlated with the emission of a 42.7 nm photon on the $4\,{}^1P_1 \rightarrow 4\,{}^1S_0$ transition. In addition, because, the atom starts and ends in a state with total angular momentum J=0, these two photons must be anti-correlated in their helicity in order to conserve total angular momentum:



Are there observable phenomena associated with the light produced in the atomic cascade that cannot be accounted for by the semi-classical theory? Of course the answer is, yes, many.

One particular example strikes at the heart of the difference between the classical and quantum models of the electromagnetic field. In the classical theory, the field is a continuous wave of arbitrary amplitude, whereas in the quantum theory the field is composed of **indivisible** quanta, the photons. Consider a field incident on a perfect 50-50 partially reflecting, partially transmitting, beam splitter, in which photo-ionization detectors are placed at two output ports, and then correlated by a coincidence counter.



According to the classical theory, the intensity of an incident field, no matter how weak, will be equally divided at the beam splitter, and therefore, the average number of coincidence counts registered in any time window will be at least as big as the product of the number of counts registered by each detector individually,

$$N_c \geq N_r N_t .$$

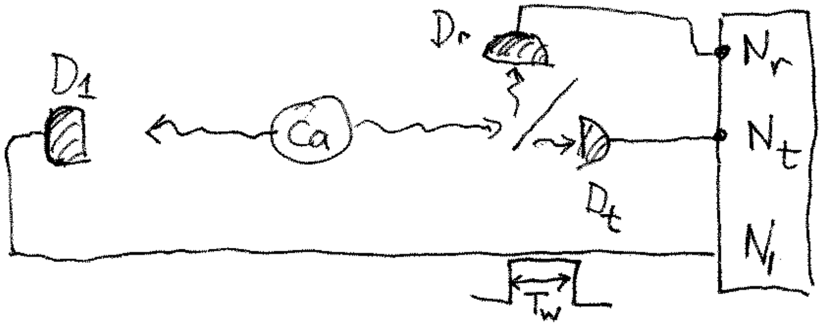
To see this, note that for a classical light source

$$\langle I_r I_t \rangle = \langle (\langle I_r \rangle + \delta I_r)(\langle I_t \rangle + \delta I_t) \rangle = \langle I_r \rangle \langle I_t \rangle + \langle \delta I_r \delta I_t \rangle \geq \langle I_r \rangle \langle I_t \rangle .$$

The coincidence counter measure intensity *correlations.*

This inequality is in sharp contrast to the predicts of a quantum theory. Suppose the radiation incident on the beam splitter consists of a **single photon**, and the experiment is repeated many times with an ensemble of identically prepared photons. Because the photon is an indivisible object, it cannot be detected at both output port simultaneously. Once the photon is detected, it must be found at either detector "r" or detector "t", but not both. Thus we expect the number coincidence counts to be strictly zero.

An experiment to test this result was performed by A. Aspect, P. Grangier, and G. Roger at the Institut d'Optique in Orsay, France (J. Optics (Paris), **20**, 119 (1989)). Using the photons produced in the atomic cascade of Ca described above, they were able to prepare an ensemble of single photon states by using one the photons as a "trigger" for coincidence detection of the other:



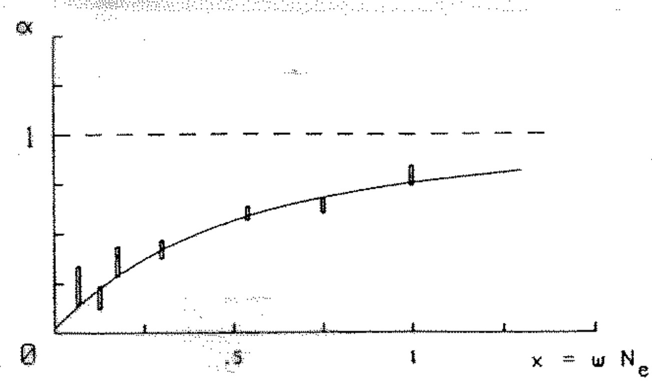
A beam splitter and a photodetector are placed equidistant from the radiating Ca source. By detecting one of these photons directly, we know that there must be exactly one photon incident on the beam splitter to within the 4.5 nsec lifetime of the intermediate state. Thus, we can use detection of the first photon at detector "1" as a trigger that opens a time window for coincidence counts within the uncertainty of the "simultaneity" of the photons' birth T_{window}=2×4.5 nsec=9 nsec. One compares the number of coincidence counts N_c to the product of the number recorded by the detectors at the two output ports of the beam splitter, normalized by the number of pair produced as measured by the number of counts at detector #1. Then, it is easy to show that according to the semiclassical theory, these counts must satisfy the inequality,

$$N_c \geq \frac{N_r N_t}{N_1}, \text{ or in terms of counting rates } \dot{N}_c \geq \frac{\dot{N}_r \dot{N}_t}{\dot{N}_1}.$$

where the equality is satisfied if the source has a perfectly stable classical intensity. On the other hand, if the field at the beam splitter consists of an ensemble of specially prepared single photons, such as in the atomic cascade, then we expect,

$$N_c = 0,$$

in maximum violation of the semiclassical theory. Shown below is a plot of $\alpha \equiv (N_1 N_c)/(N_r N_t)$, as a function of the product of T_{window} times the pumping rate into the 6^1S_0 state.



We see a violation of the semiclassical theory which requires α≥1. The deviation from perfect anticorrelation, α=0, when the pumping rate or T_{window} is large arises become coincidence counts can arise from two photons "born" from twins at different times.

Thus, we see our first example of a "nonclassical" state of light, which cannot be explained by any semiclassical theory. The characterization of nonclassical light is a subtle business. For example, suppose we take "classical" state of light and attenuate with a very strong filter so that the intensity is very weak. By adjusting the attenuation so that the average number of photons detected in a time interval T is much smaller than one, according to Mandel's semiclassical theory, the probability of detecting two or more photons in that time interval will be much much smaller according to the Poison distribution

$$P(2,T) = \frac{\bar{n}(T)^2}{2} e^{-2} << \bar{n}(T), \quad \text{if } \bar{n}(T) << 1.$$

Thus, one might expect that number of coincidence counts to violate the classical inequality if such a weak source is incident on a beam splitter. Aspect *et al.* performed this experiment by attenuating 8 ns pulses light produced by a photodiode so that the Poison distribution describing the number of photon detected in the duration of one pulse was on the order $\bar{n}(T) \sim 0.01$. Using a 9 ns time window for coincidence counts (triggered by the known arrival time of the pulse), they found the following table of data

N_1 (s^{-1})	N_{2r} (s^{-1})	N_{2t} (s^{-1})	T (s)	$N_c T$	$\frac{N_{2r} \cdot N_{2t}}{N_1} T$
4 760	3,02	3,76	31 200	82	74,5
8 880	5,58	7,28	31 200	153	143
12 130	7,90	10,2	25 200	157	167
20 400	14,1	20,0	25 200	341	349
35 750	26,4	33,1	12 800	329	313
50 800	44,3	48,6	18 800	840	798
67 600	69,6	72.5	12 800	925	955

We see that for all intensities of incident pulses, the semiclassical inequality $\alpha \geq 1$ is satisfied, with $\alpha \sim 1$ for the larger intensities, where the source is "shot noise limited". Thus, simply attenuating a classical light source produces a state of light whose properties can be fully explained by the semiclassical theory. In this sense, strongly attenuated light is not what one usually calls "nonclassical light". This is in contrast to the light produced in the atomic cascade, whose correlations lead to phenomena which could not be described by the semiclassical theory.

Our goal in this lecture, and those to follow is to more quantitatively characterize the properties if the electromagnetic field, and in doing so distinguish those phenomena which are unique to the quantum field theory. If this theory is valid, then it should contain the classical theory as one of its limits. Accordingly, in this lecture we will examine the classical limit $\hbar \to 0$ in an attempt to better understand the connection between these two theories. This correspondence is best described by a particular state of the quantized radiation field known as the quasi-classical or **coherent state**.

Coherent states of the Simple Harmonic Oscillator

In lecture #6 we saw that the classical field can be described by a collection of simple harmonic oscillators (SHO), one for each mode of the field. The quantum field is then obtained by associating each of these oscillators with its quantum counterpart that satisfy the canonical commutation relations. Thus, by studying the properties of the quantum SHO we can understand those of the quantized electromagnetic field.

Recall the properties of the SHO discussed in detail in lecture #6. The Hamiltonian is

$$H = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 = \hbar\omega\left(Q^2 + P^2\right) = \hbar\omega\left(\alpha^* \alpha\right),$$

where $Q = q/Q_0$ and $P = p/P_0$ are dimensionless phase space coordinates with $\frac{P_0^2}{2m} = \frac{1}{2} m\omega^2 Q_0^2 = \hbar\omega$, and the complex phase space amplitude defined by

$$\alpha \equiv \frac{Q + iP}{\sqrt{2}}.$$

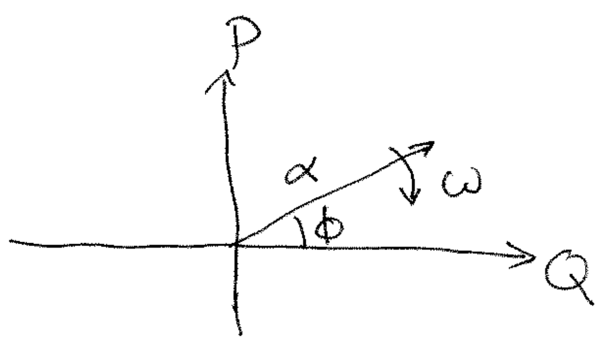
The Hamilton equations of motion have the solutions

$$\begin{aligned}
Q(t) &= Q(0)\cos\omega t + P(0)\sin\omega t = A\cos(\phi - \omega t) \\
P(t) &= P(0)\cos\omega t - Q(0)\sin\omega t = A\sin(\phi - \omega t) \\
\alpha(t) &= \frac{Q(t) + iP(t)}{\sqrt{2}} = \alpha(0)e^{-i\omega t} = Ae^{i\phi}e^{-i\omega t},
\end{aligned}$$

where the amplitude and phase of the oscillation are determined by the initial conditions

$$\sqrt{2}A = \sqrt{Q(0)^2 + P(0)^2}, \quad \phi = \tan^{-1}(Q(0)/P(0)).$$

These equations of motions are conveniently displayed by a phasor diagram in phase space (the complex α plane)



Thus the magnitude square of the complex amplitude, $\alpha^*\alpha$, is a conserved quantity, whereas the complex amplitude itself is not.

The quantum oscillator follows by the association

$Q \to \hat{Q},\ P \to \hat{P},\ \alpha \to \hat{a} = \frac{\hat{Q} + i\hat{P}}{\sqrt{2}},$ $[\hat{Q},\hat{P}] = i$ [redacted] $[\hat{a},\hat{a}^{\dagger}] = 1.$

Stationary states of the Hamiltonian are eigenstates of the number operator $\hat{N} \equiv \hat{a}^{\dagger}\hat{a}$,

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle, \text{ where } \hat{N}|n\rangle = n|n\rangle,\ \hat{a}|n\rangle = \sqrt{n}|n-1\rangle,\ \hat{a}^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle.$$

The state $|0\rangle$ is the ground state defined by $\hat{a}|0\rangle = 0|0\rangle$. This states has zero average occupation number $\langle n\rangle = \langle 0|\hat{N}|0\rangle = 0$. It is a minimum uncertainty state

$\Delta Q \Delta P =$ [redacted] $\frac{1}{2}$

with equal uncertainty in both Q and P, $\Delta Q = \Delta P = \frac{1}{\sqrt{2}}$.

We may now ask, which states of the harmonic oscillator most closely the resemble the classical counterparts? By this we mean that the expectation value of any observable follows the classical trajectory with minimum quantum uncertainty. One quick response might be to apply Bohr's correspondence principle, and consider stationary states with a large occupation number $n \to \infty$. However, in any number state we have

$$\langle n|\hat{Q}|n\rangle = \langle n|\hat{P}|n\rangle = 0.$$

Thus, any stationary state is distinctly nonclassical.

In order to find the quasi-classical states we may take a hint from our quantization procedure. The quantum SHO was defined by associating the classical variables with quantum operators. Thus, a natural choice is to define a state which is an eigenstate of the classical variables. Since it is impossible to find a simultaneous eigenstate of $\hat{Q}$ and $\hat{P}$, the best compromise to find "phase-space" eigenstates, that is eigenstates of the complex amplitude operator $\hat{a}$,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle.$$

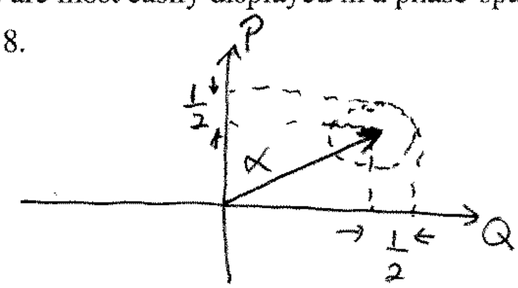
These states are known as **coherent states** for a historical reason which will become clear later on. (Note: the name coherent state should not be confused with the term "pure state"). In this state, expectation values of observables are replaced by their classical values,

$$\langle\alpha|\hat{Q}|\alpha\rangle = \langle\alpha|\left(\frac{\hat{a}+\hat{a}^{\dagger}}{2}\right)|\alpha\rangle = \frac{\alpha+\alpha^{*}}{\sqrt{2}} = Q,$$
$$\langle\alpha|\hat{P}|\alpha\rangle = \langle\alpha|\left(\frac{\hat{a}-\hat{a}^{\dagger}}{2i}\right)|\alpha\rangle = \frac{\alpha-\alpha^{*}}{\sqrt{2}i} = P,$$
$$\langle\alpha|\hat{N}|\alpha\rangle = \langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle = \alpha^{*}\alpha.$$

They are not stationary states of the Hamiltonian since $[\hat{a},\hat{H}] \neq 0$. The coherent states are minimum uncertainty states, with equal uncertainty in Q and P,

$$2\Delta Q^2 = 2\langle\alpha|\hat{Q}^2|\alpha\rangle - 2\langle\alpha|\hat{Q}|\alpha\rangle^2 = \langle\alpha|(\hat{a}^{\dagger}+\hat{a})^2|\alpha\rangle - (\alpha+\alpha^{*})^2 = 1$$
$$2\Delta P^2 = 2\langle\alpha|\hat{P}^2|\alpha\rangle - 2\langle\alpha|\hat{P}|\alpha\rangle^2 = -\langle\alpha|(\hat{a}^{\dagger}-\hat{a})^2|\alpha\rangle + (\alpha-\alpha^{*})^2 = 1$$
$$\Rightarrow \Delta Q\Delta P = \frac{1}{2}$$

The ground state $|0\rangle$ is the only example of a number state which is also a coherent state. The properties of a coherent states are most easily displayed in a phase-space diagram analogous to the classical diagram on page 8.



The mean values are characterized by the solid phasor, and dotted circle represents the quantum uncertainties in the complex amplitude.

<u>Number and phase uncertainties</u>

Although the coherent states are not eigenstates of the number operator, the number states represent a complete basis for the SHO. The coherent states can be expressed as a superposition

$$|\alpha\rangle = \sum_n c_n |n\rangle.$$

Using the eigenstate definition of the coherent state,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle = \sum_n c_n \hat{a}|n\rangle = \sum_n c_n \sqrt{n}|n-1\rangle.$$

Projecting both sides of the equation with some particular number state $|m\rangle$, we arrive at the recursion relation,

$$\alpha\langle m|\alpha\rangle = \alpha c_m = c_{m+1}\sqrt{m+1} \Rightarrow c_{m+1} = \frac{\alpha}{\sqrt{m+1}} c_m .$$

Thus, $c_n = \frac{\alpha^n}{\sqrt{n!}} c_0$. We can determine the constant c_0 by normalization,

$$\langle\alpha|\alpha\rangle = \sum_n |c_n|^2 = \sum_n \frac{|\alpha|^{2n}}{n!}|c_0|^2 = e^{|\alpha|^2}|c_0|^2 = 1$$
$$\Rightarrow |c_0|^2 = e^{-|\alpha|^2} \Rightarrow c_0 = e^{-|\alpha|^2/2} \text{ (choose to be real)}.$$

We now have the representation of the coherent state in terms of the number states,

The probability distribution of occupation number n is given by the absolute square of the expansion coefficients,

$$P_n = |c_n|^2 = e^{-|\alpha|^2}\frac{\left(|\alpha|^2\right)^n}{n!} = e^{-\langle n\rangle}\frac{\langle n\rangle^n}{n!},$$

where we have used the fact that $\langle n\rangle = |\alpha|^2$. Thus we arrive at a remarkable result. The probability of occupation number n is distributed according to a Poison distribution as defined on page 3. The fluctuation in occupation number is given by the "standard quantum limit"

$$\Delta N^2 = \langle\alpha|\hat{N}^2|\alpha\rangle - \langle\alpha|\hat{N}|\alpha\rangle^2 = |\alpha|^2 = \langle n\rangle.$$

Thus, a coherent state is in accordance with "shot noise" given by the semiclassical model.

Phase uncertainty of the coherent state

Since these states have an uncertainty in occupation number, we may ask whether there is a canonical conjugate observable to this variable. The natural choice follows from the classical description. On page 8 we defined the complex amplitude in terms of a polar decomposition into amplitude and phase,

$$\alpha = Ae^{i\phi}.$$

Since the number operator is the quantum analog of A^2, the natural choice of the canonically conjugate variable is the phase ϕ. Classically, these are the so called "action-angle" phase-space variables. Quantum mechanically, a brute force quantization in terms of action angle variables in not possible as we shall see.

In the theory of Hilbert spaces it always possible to make a "polar" decomposition of an arbitrary operator, analogous to the polar decomposition of a complex number. In particular, we can decompose the annihilation and creation operators as

$$\hat{a} = \hat{N}^{1/2}\, \widehat{e^{i\phi}}, \quad \hat{a}^\dagger = \left(\widehat{e^{i\phi}}\right)^\dagger \hat{N}^{1/2}$$

where

$$\hat{N}^{1/2} \equiv \sum_n n^{1/2} |n\rangle\langle n|, \quad \widehat{e^{i\phi}} \equiv \sum_n |n\rangle\langle n+1|.$$

Note that the "hat" was placed over the whole operator $e^{i\phi}$, rather than ϕ itself. This is because the operator $e^{i\phi}$ is not unitary, and thus does not represent the exponentiation of a Hermitian phase operator. It is easy to show that

$$\left[\widehat{e^{i\phi}}, \left(\widehat{e^{i\phi}}\right)^\dagger\right] = |0\rangle\langle 0|.$$

Thus, the problem with defining a Hermitian phase operator arises from the fact that there is a lower bound on the occupation number (the ground state). One can however define eigenstates of $e^{i\phi}$,

$$\left|e^{i\phi}\right\rangle = \sum_{n=0}^{\infty} e^{in\phi} |n\rangle \Rightarrow \widehat{e^{i\phi}} \left|e^{i\phi}\right\rangle = e^{i\phi} \left|e^{i\phi}\right\rangle.$$

The phase probability distribution of an arbitrary state is the given by,

$$P(\phi) = \frac{1}{2\pi} \left|\left\langle e^{i\phi} \middle| \psi \right\rangle\right|^2.$$

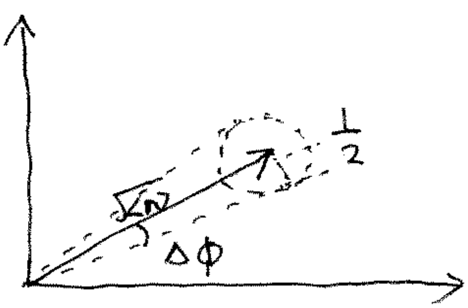
For a number state, P(ϕ)=1/2π, representing a state with a completely uncertain phase. For a coherent state,

$$P(\phi) = \frac{1}{2\pi}\left|\langle e^{i\phi}|\alpha\rangle\right|^2 = \frac{1}{2\pi}\left|\sum_{n=0}^{\infty} e^{-in\phi}\langle n|\alpha\rangle\right|^2 = \frac{e^{-|\alpha|^2}}{2\pi}\left|\sum_{n=0}^{\infty}\frac{(\alpha e^{-i\phi})^n}{\sqrt{n!}}\right|^2.$$

The terms in the sum are rapidly varying except near ϕ=Arg(α). Thus the average phase will be given by Arg(α) as expected. In addition one can show that the variance of the phase uncertainty is

$$\Delta\phi^2 = \frac{1}{4|\alpha|^2} = \frac{1}{4\langle n\rangle}.$$

These results make physical sense when viewed from our graphical representation.



Thus in order to create a state with small phase uncertainty requires a large mean excitation number of the coherent state (this is the classical limit). Though in general it is impossible to define a Hermitian phase operator, one can define an approximate one in the limit of large average excitation number since the problem arose from the ground state. In this case we have the approximate uncertainty relation

$$\Delta N^2 \Delta\phi^2 \geq \frac{1}{4}.$$

A coherent state is therefore a minimum number/phase uncertainty state with $\Delta N^2 = \langle n\rangle$, and $\Delta\phi^2 = 1/4\langle n\rangle$.

Phase space displacement operators and the time dependence of coherent states

A very useful way of handling the mathematics associated with coherent states is through the use of the unitary phase space "displacement" operator defined by,

$$D(\alpha) \equiv \exp(\alpha \hat{a}^\dagger - \alpha^* \hat{a}).$$

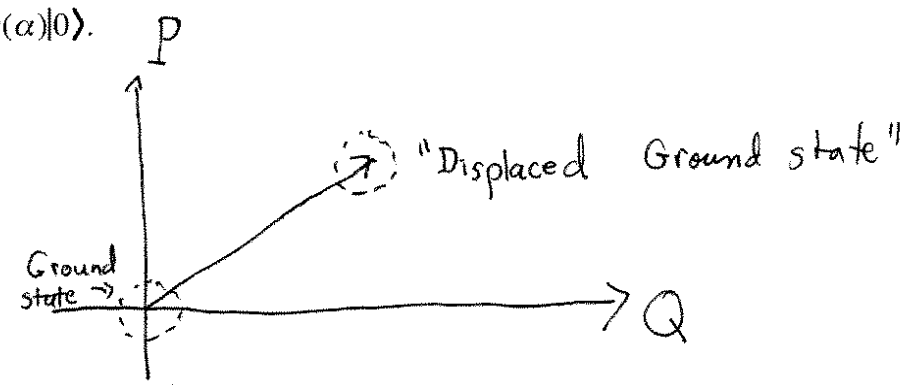
The physical meaning of this operator is clear when we write α in terms of its real and imaginary parts $\alpha = \frac{Q+iP}{\sqrt{2}}$,

$$D(\alpha) \equiv \exp\left\{\frac{(Q+iP)(\hat{Q}-i\hat{P})}{2} - \frac{(Q-iP)(\hat{Q}+i\hat{P})}{2}\right\} = \exp\left\{i\hat{P}Q - i\hat{Q}P\right\}.$$

Recall that $\exp\left\{i\hat{P}Q\right\}$ is the unitary translation operator in position space, and $\exp\left\{-i\hat{Q}P\right\}$ is the translation operator is momentum space. These operators do not commute. Thus the displacement operator represents a symmetrized translation in phase.

The coherent state is then equal to a unitary transformation on the ground state,

$$|\alpha\rangle = D(\alpha)|0\rangle.$$



To prove this, use the factor that D(α) is a phase-space displacement to show

$$D(\alpha)^\dagger \, \hat{a} \, D(\alpha) = \hat{a} + \alpha.$$

Then,

$$D(\alpha)^\dagger \hat{a}|\alpha\rangle = D(\alpha)^\dagger \, \hat{a} \, D(\alpha)|0\rangle = (\hat{a} + \alpha)|0\rangle = \alpha|0\rangle$$
$$\Rightarrow \hat{a}|\alpha\rangle = \alpha|\alpha\rangle$$

The displacement operator can also be used to easily obtain the decomposition in terms of number states. Using the operator theorem,

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-[\hat{A},\hat{B}]/2}, \text{ when } [\hat{A},[\hat{A},\hat{B}]] = [\hat{B},[\hat{A},\hat{B}]] = 0,$$

we get the so call "normal order" decomposition of D(α),

$$D(\alpha) = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{\alpha^*\hat{a}}.$$

Then

$$D(\alpha)|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{\alpha^*\hat{a}}|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger}|0\rangle$$
$$= e^{-|\alpha|^2/2} \sum_n \frac{\alpha^n \hat{a}^{\dagger n}}{n!}|0\rangle = \sum_n e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

as before.

The unitary operators $D(\alpha)$ form a group known as Heisenberg-Weyl group with the composition law,

$$D(\alpha)D(\beta) = e^{i\,\mathrm{Im}(\alpha\beta^*)} D(\alpha+\beta).$$

This groups leads to a rich variety of properties of the coherent states and the representation of operators in terms of these states.

Suppose we start the quantum oscillator in a coherent state $|\alpha\rangle$. Since this is not a stationary state of the Hamiltonian, it will evolve in time. Into what state will it evolve? The solution to the Schrödinger equation is

$$|\psi(t)\rangle = \exp\left\{-\frac{i}{\hbar}\hat{H}_0 t\right\}|\alpha\rangle = \hat{U}_0\hat{D}(\alpha)|0\rangle = \hat{U}_0\hat{D}(\alpha)\hat{U}_0^\dagger|0\rangle = \hat{U}_0 \exp(\alpha\hat{a}^\dagger - \alpha^*\hat{a})\hat{U}_0^\dagger|0\rangle$$
$$= \exp(\alpha\hat{U}_0\hat{a}^\dagger\hat{U}_0^\dagger - \alpha^*\hat{U}_0\hat{a}\hat{U}_0^\dagger)|0\rangle = \exp(\alpha e^{-i\omega t}\hat{a}^\dagger - \alpha^* e^{i\omega t}\hat{a})|0\rangle = |\alpha e^{-i\omega t}\rangle$$

Thus, a coherent sate remains a coherent state under the evolution of the oscillator's Hamiltonian, with the time dependent amplitude given by the classical equation, $\alpha e^{-i\omega t}$.

Coherent states of the electromagnetic field

In lecture #6 we learned that the quantized electromagnetic field can be viewed as an infinite collection of simple harmonic oscillators, one mode each normal mode. The positive frequency component of the electric field takes the form

$$\hat{\mathbf{E}}^{(+)}(\mathbf{x},t) = -i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\,\vec{\varepsilon}_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}.$$

A coherent state of the field is defined as an eigenstate of this operator,

$$\hat{\mathbf{E}}^{(+)}(\mathbf{x},t)\left|\mathbf{E}_{class}\right\rangle = \mathbf{E}_{class}(\mathbf{x},t)\left|\mathbf{E}_{class}\right\rangle.$$

Clearly this will be the case if each single mode of the oscillator is in a coherent state. Thus, the coherent state of the field is a product state of single mode coherent states,

$$\left|\mathbf{E}_{class}\right\rangle = \prod_{\mathbf{k},\lambda}\left|\alpha_{\mathbf{k},\lambda}\right\rangle \equiv \left|\left\{\alpha_{\mathbf{k},\lambda}\right\}\right\rangle,$$

where the c-number field is defined in terms of the single mode complex amplitudes,

$$\mathbf{E}_{class}(\mathbf{x},t) = -i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar\omega_{\mathbf{k}}}{V}}\,\vec{\varepsilon}_{\mathbf{k},\lambda}\,\alpha_{\mathbf{k},\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)}.$$

In a coherent state the expectation values of all field operators are replaced by the corresponding c-number value, with minimum quantum uncertainty.

Generation of coherent states of the field

Coherent states are generated by classical c-number currents in the same way as a classical c-number force drives the single mode oscillator. According to classical electromagnetic theory, the vector potential (in the Coulomb gauge) radiated by a current density **J**(**x**,t) is

$$\mathbf{A}(\mathbf{x},t) = \int d^3\mathbf{x}'\,dt'\;G(\mathbf{x}-\mathbf{x}',t-t')\,\mathbf{J}_{\perp}(\mathbf{x}',t'),$$

where

$$G(\mathbf{x}-\mathbf{x}',t-t') = \frac{\delta(t-t'-|\mathbf{x}-\mathbf{x}'|)}{c|\mathbf{x}-\mathbf{x}'|}$$

is the Green's function for the wave equation. The source field has the label $\perp$ to denote the transverse component which can be expressed as,

$$\mathbf{J}_\perp(\mathbf{x},t) = \sum_{\mathbf{k},\lambda}\left(j_{\mathbf{k},\lambda}(t)\,\vec{\varepsilon}_{\mathbf{k},\lambda}\,e^{i\mathbf{k}\cdot\mathbf{x}} + c.c.\right).$$

Substituting this expansion into the integral equation, after some algebra we find,

$$\mathbf{A}(\mathbf{x},t) = \sum_{\mathbf{k},\lambda}\left(\frac{-2\pi ic}{\omega_\mathbf{k}}\int dt'\, e^{-i\omega_\mathbf{k}(t-t')}\, j_{\mathbf{k},\lambda}(t')\,\vec{\varepsilon}_{\mathbf{k},\lambda}\,e^{i\mathbf{k}\cdot\mathbf{x}} + c.c.\right).$$

We now want to show that, quantum mechanically, a deterministic c-number current generates a coherent state such that the vector potential operator has the same expectation value, The interaction Hamiltonian has the form,

$$\hat{H}_{\text{int}} = -\frac{1}{c}\int d^3x\,\mathbf{J}_\perp(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x}).$$

For a c-number current, this interaction Hamiltonian is *linear* in the field operator. That is, we neglect any effect the radiating fields have of the source. In the interaction picture, substituting in the normal mode expansion for the field operator

$$\hat{H}^{(I)}_{\text{int}}(t) = -\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar V}{\omega_\mathbf{k}}}\left(j^*_{\mathbf{k},\lambda}(t)e^{-i\omega_\mathbf{k}t}\,\hat{a}_{\mathbf{k},\lambda}\,e^{i\mathbf{k}\cdot\mathbf{x}} + H.c.\right).$$

If we start in the vacuum state, the state after interacting with the current for some time t is

$$\begin{aligned}|\psi\rangle &= U_0(t)\,U^{(I)}(t)|0\rangle \\ &= \exp\left(i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi V}{\hbar\omega_\mathbf{k}}}\int_0^t dt' j^*_{\mathbf{k},\lambda}(t')e^{i\omega_k(t-t')}\,\hat{a}_{\mathbf{k},\lambda} - H.c.\right)|0\rangle \\ &= \prod_{\mathbf{k},\lambda}\exp\left(\alpha^*_{\mathbf{k},\lambda}\,\hat{a}_{\mathbf{k},\lambda} - H.c\right)|0\rangle = \left|\{\alpha_{\mathbf{k},\lambda}\}\right\rangle,\end{aligned}$$

where

$$\alpha_{\mathbf{k},\lambda} = i\sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi V}{\hbar\omega_{\mathbf{k}}}}\int_o^t dt' j_{\mathbf{k},\lambda}(t')e^{-i\omega_k(t-t')}.$$

Thus, as promised, the c-number deterministic current generates a coherent state with the same eigenvalue as the classical field,

$$\left\langle\left\{\alpha_{\mathbf{k},\lambda}\right\}\middle|\hat{\mathbf{A}}(\mathbf{x},t)\middle|\left\{\alpha_{\mathbf{k},\lambda}\right\}\right\rangle = \sum_{\mathbf{k},\lambda}\sqrt{\frac{2\pi\hbar c^2}{V\omega_{\mathbf{k}}}}\left(\alpha_{\mathbf{k},\lambda}\,\vec{\varepsilon}_{\mathbf{k},\lambda}\,e^{i(\mathbf{k}\cdot\mathbf{x}-\omega_{\mathbf{k}}t)} + c.c.\right)$$
$$= \mathbf{A}_{\text{classical}}(\mathbf{x},t)$$

Completeness, and representations in terms of coherent states

Given the expansion of the coherent state in terms of number states one can show that these sates form an (over)complete basis,

$$\int\frac{d^2\alpha}{\pi}|\alpha\rangle\langle\alpha| = 1.$$

We use the term over complete because these states are not orthogonal

$$\langle\alpha|\beta\rangle = \exp\left[-\frac{1}{2}\left(|\alpha|^2+|\beta|^2\right)+\alpha^*\beta\right] \Rightarrow |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2}.$$

Although these states are not orthogonal, these become approximately so when $|\alpha-\beta|^2 >> 1$.

Given the completeness of the coherent states, any operator in Hilbert space can be expanded in terms of them. For example, we may consider expansions of the density operator representing a general state of the system,

$$\rho = \int d^2\alpha\; P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|.$$

The c-number function $P(\alpha,\alpha^*)$ is known as the **Glauber-Sudarshan** P-representation. It closely resembles a classical distribution of for an ensemble in phase space. However, this analogy must be treated with some care. In general $P(\alpha,\alpha^*)$ can be negative or highly singular for quantum states with have not classical analog. For a pure coherent state with complex amplitude α_0, the P-representation is,

$$P(\alpha,\alpha^*) = \delta^{(2)}(\alpha-\alpha_0).$$

The nonclassical states will have singularities worse than a delta function.

Because the coherent states form an over complete basis, the representation of the density operator in terms of them is not unique. Two other representations are the Q-representation,

$$Q(\alpha, \alpha^*) \equiv \langle \alpha | \hat{\rho} | \alpha \rangle,$$

and the Wigner function,

$$W(\alpha, \alpha^*) \equiv \int d^2\beta \; Tr(\hat{\rho}\, \hat{D}(\beta))\; e^{\beta\alpha^* - \alpha\beta^*}.$$

The Wigner function is always nonsingular, but can be negative. The Q-function is always positive, but is usually not useful for calculating expectation values, which is the essential role of the density operator. All of these functions are inter-related according to the group properties of the displacement operators discussed above. We will see this in detail in later lectures.

Multimode coherent state representation:

If the current were classical in nature, but stochastic (e.g.. thermal fluctuations in the current), then the radiated field would be a classical mixture of coherent states depending of the power spectrum of the modes of current oscillation,

$$\hat{\rho}_{field} = \int d\{\alpha_{\mathbf{k},\lambda}\}\; P(\{\alpha_{\mathbf{k},\lambda}\})\; \left|\{\alpha_{\mathbf{k},\lambda}\}\right\rangle\left\langle\{\alpha_{\mathbf{k},\lambda}\}\right|,$$

where $P(\{\alpha_{\mathbf{k},\lambda}\}) \geq 0$.

We have thus found that a classical current will always generate a coherent state for the electromagnetic field, or a statistical mixture with a positive definite P-representation. The classical current implies that the interaction Hamiltonian is linear in the field operators, and thus quantum fluctuations play no role. In order to generate a nonclassical state of light, one must consider some *nonlinear process*, such as four-wave mixing, parametric down conversion, or a saturable absorber (e.g. near resonant atomic excitation). In the next lectures we will characterize what measurements distinguish these nonclassical states from the quasi-classical ones.