Physics 566 - Quantum Optics I

Lecture 15: Quantum Optical Coherence

The core subject of quantum optics is coherence – the capacity for system to exhibit interference. In particular we are interested in quantum coherence – i.e., interference between alternative quantum processes – associated with electromagnetic fields. For the majority of the course so far, we have focussed on atomic coherence, i.e., coherent superposition of atomic energy levels for which the atomic reponse to electromagnetic fields is nonclassical, e.g. Rabi oscillations and electromagnetically induced transparency.

We now want to turn our attention to the electromagnetic field itself. This is a subtle business. The electromagnetic fields are described classically as waves, so there is a sense in which coherence in electromagnetism is a classical phenomenon. But electromagnetic fields are also described by particles, so there is a sense in which coherence is a quantum phenomenon associated with the interference of paths the particle takes. We saw this early in the course. The interference in a Mach-Zender interferometer could alternatively be described by interfering classical waves or by interference of probability amplitudes associated with two indistinguishable paths a photon can take on its way to a detector. So, although there is a quantum explanation underlying the observed interference fringes, the phenomenon is "essentially classical" in nature, in that the classical theory of electromagnetic waves gives the proper prediction of the observations. Once we include the semiclassical description of photon detection (quantum absorbers), we need not quantize the field to describe Mach-Zender-type interference.

Our goal, thus, is to study quantum optical coherence, and to understand the conditions under which this is irreducibly quantum mechanical in nature, and when the classical theory can explain the phenomenon. This distinction allows us to distinguish classical light vs. nonclassical light.

Review: Classical Statistical Optics

To distinguish "classical light" from "nonclassical light" we first review the classical theory, studied at the beginning of the semester. In particular it is important to understand classical statistical optics, whereby the complex wave amplitude $\tilde{E}(\vec{r},t)$ is a random variable due to our incomplete knowledge of the source that produced the field. We write the field as decomposition into modes

$$\tilde{E}(\vec{r},t) = \sum_{\vec{k}} \underbrace{C_{\vec{k}}}_{\text{Take Fourier coefficients}} e^{i(\vec{k}\cdot\vec{x}-\omega_k t)}$$

(for simplicity, we take the field to be polarized)

normalized relative to character E-field of photon $C_{\vec{k}} \equiv \sqrt{\frac{8\pi\hbar\omega_k}{V}}\, \alpha_{\vec{k}}$

The classical "state of field" is determined by probability distribution we assign to the mode amplitudes: $P(\{\alpha_{\vec{k}}\}, t)$. We typically make the following assumptions:

(1) The statistics are "stationary," i.e. $P(\{\alpha_{\vec{k}}\})$ is independent of t.

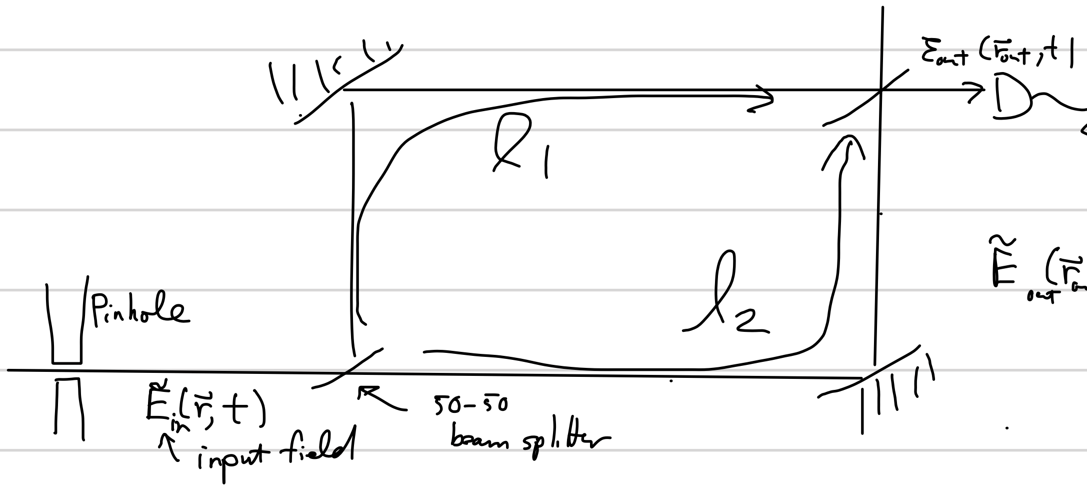
(This doesn't mean that nothing changes as function of time, just that the statistics are constant)

(2) The dynamics are "ergodic" $\Rightarrow$ Sampling the field at different times is equivalent to sampling from the probability distribution $P(\{C_k\}) \Rightarrow$ expectation values are equivalent to time averages.

$$\langle \tilde{\mathcal{F}}(\tilde{E}(\vec{r},t))\rangle = \int d^2\{\alpha_{\vec{k}}\}\, P(\{\alpha_{\vec{k}}\})\, \tilde{\mathcal{F}}(\tilde{E}(\vec{r},t)) \equiv \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, \tilde{\mathcal{F}}(\tilde{E}(\vec{r},t)) = \overline{\tilde{\mathcal{F}}(\tilde{E}(\vec{r},t))}$$

The ergodic assumption is a good approximation for natural light sources.

Consider now the Mach-Zender interferometer:



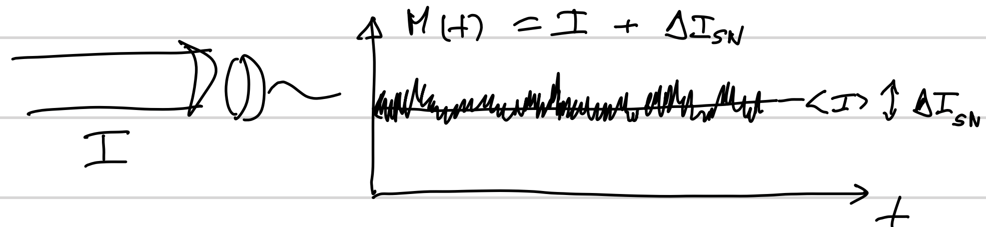
$$\langle I_{out}\rangle = \langle \tilde{E}^*_{out}(\vec{r},t)\, \tilde{E}_{out}(\vec{r},t)\rangle$$

$$\tilde{E}_{out}(\vec{r}_{out},t) = \frac{1}{2}\left(\tilde{E}_{in}(\vec{r},t_1) + \tilde{E}_{in}(\vec{r},t_2)\right)$$

(up to overall phase)

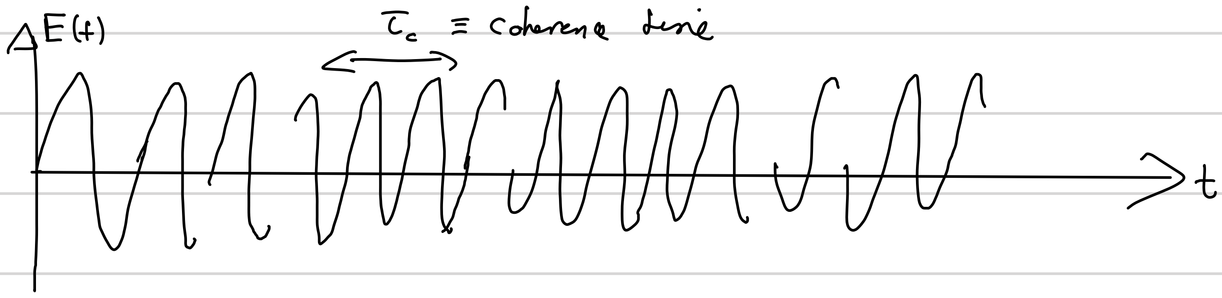
$$t_1 = t - \frac{l_1}{c}, \quad t_2 = t - \frac{l_2}{c}$$

We assume a "square-law" detector which responds to the intensity. For a strong field with a steady intensity, e.g. a laser, the photo current will register as



the shot noise $\Delta I_{SN} \propto \sqrt{\langle I \rangle}$ and is uncorrelated in each time interval, so the time average $\overline{M(t)} = \overline{I}$ and in the ergodic assumption, $\overline{M}(t) = \langle I \rangle = \langle |\tilde{E}|^2 \rangle$

For a "natural" light source, as we studied, the intensity is not steady. For example, in collision broadened source, the field undergoes phase jumps



· Thus the electric field only has a finite "coherence" time before it loses phase memory. The output intensity of the interferometer exhibits this finite coherence.

$$\Rightarrow \langle I_{out} \rangle = \frac{1}{4}\left[\langle |\tilde{E}(t_1)|^2 \rangle + \langle |\tilde{E}(t_2)|^2 \rangle + 2\,\mathrm{Re}\underbrace{\langle \tilde{E}^*(t_1)\,\tilde{E}(t_2) \rangle}_{\text{temporal correlation function}} \right]$$

(Dropping position dependence because all $\tilde{E}$ @ same $\vec{r}$)

For stationary statistics $\langle |\tilde{E}(t_1)|^2 \rangle = \langle |\tilde{E}(t_2)|^2 \rangle = \langle I_{in} \rangle$, $\langle \tilde{E}^*(t_1)\tilde{E}(t_2) \rangle = \langle \tilde{E}^*(\tau)\tilde{E}(0) \rangle \equiv G^{(1)}(\tau)$, $\tau = t_1 - t_2$

$$\langle I_{out} \rangle = \frac{\langle I_{in} \rangle}{2}\left[1 + \frac{\mathrm{Re}\left(G^{(1)}(\tau)\right)}{\langle I_{in} \rangle} \right] \equiv \frac{\langle I_{in} \rangle}{2}\left(1 + \mathrm{Re}\left(g^{(1)}(\tau)\right) \right)$$

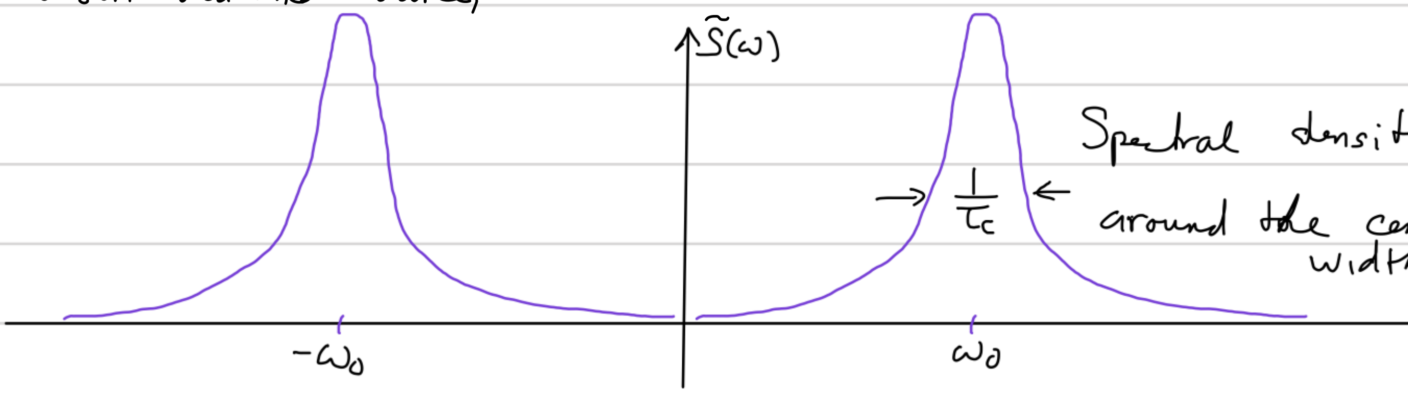
According to the Weiner-Khinchine theorem, studied earlier

$$G^{(1)}(\tau) = \langle \tilde{E}^*(\tau)\,\tilde{E}(0) \rangle = \int_0^\infty \frac{d\omega}{2\pi}\,\tilde{S}(\omega)\,e^{-i\omega t} = \text{Positive frequency component of the Fourier transform of the spectral density}$$

$$\langle \tilde{E}^*(\omega)\,\tilde{E}(\omega') \rangle = \tilde{S}(\omega)\,\delta(\omega - \omega')$$

Example: Collision broadened "natural light" $\tilde{S}(\omega) = \frac{\langle I_{in} \rangle}{2}\left[\frac{\frac{1}{\tau_c}}{(\omega-\omega_0)^2 + \left(\frac{1}{\tau_c}\right)^2} + \frac{\frac{1}{\tau_c}}{(\omega+\omega_0)^2 + \left(\frac{1}{\tau_c}\right)^2} \right]$

For collision broadened source,

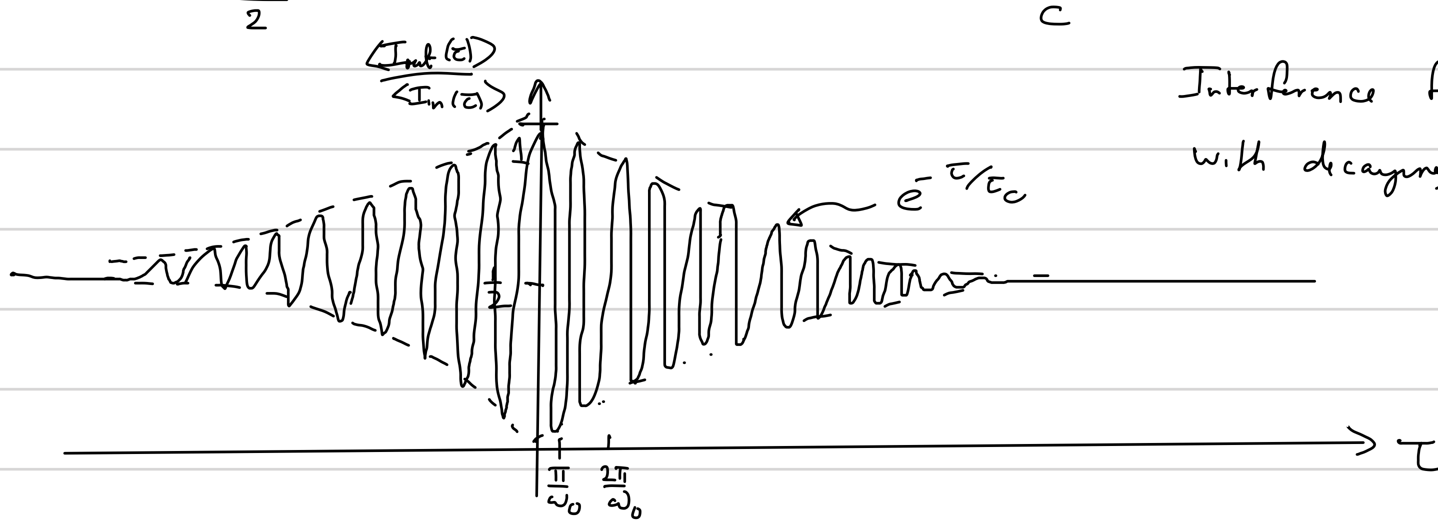


Spectral density is Lorentzian around the central freq, width $\cong \frac{1}{\tau_c}$

$$\tilde{S}(\omega) = \langle I_{in} \rangle \left[\frac{\frac{1}{\tau_c}}{(\omega-\omega_0)^2 + 1/\tau_c^2} + \frac{\frac{1}{\tau_c}}{(\omega+\omega_0)^2 + 1/\tau_c^2} \right]$$

$$\Rightarrow G^{(1)}(\tau) = \langle \tilde{E}^*(\tau)\tilde{E}(0) \rangle = g^{(1)}(\tau) \langle I_{in} \rangle, \qquad g^{(1)}(\tau) = e^{-\tau/\tau_c} e^{-i\omega_0 \tau}$$

$$\Rightarrow \langle I_{out}(\tau) \rangle = \frac{\langle I_{in} \rangle}{2} \left(1 + e^{-\tau/\tau_c} \cos \omega_0 \tau \right) \qquad \tau = \frac{l_1 - l_2}{c}$$



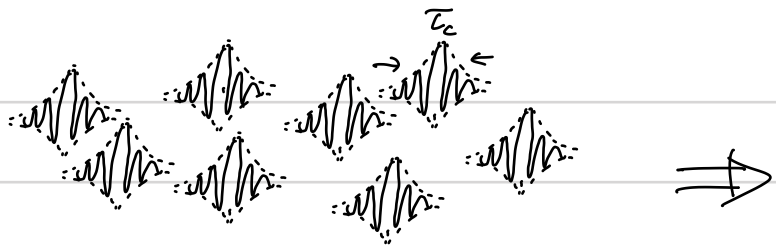
Interference fringes with decaying visibility.

The visibility of the fringes is defined

$$V(\tau) = \frac{I_{max}(\tau) - I_{max}(\tau)}{I_{max}(\tau) + I_{max}(\tau)} = |g(\tau)| = e^{-\tau/\tau_0}$$

Thus, by measuring the visibility of the fringes, one can determine the coherence time.

While the classical wave theory explains the interference fringes, we can also explain the observation in terms of indivual quanta (photons). The light beam can be loosely considered to be a beam of photons, that are "wave packets" each of which has a temporal shape determined by its spectral density



This, of course, is just a cartoon, and each photon is assumed to be in same transverse mode defined by the beam (e.g. a pinhole), and, as we will see, the number of photons in a given temporal mode will fluctuate in a way that is highly dependent on the type of light we are considering. These fluctuations, however, play no role in the Mach-Zender interferometer as we measure the mean intensity, or equivalently, the mean number of photons. From the quantum perspective, the mean photo-current is proportional to the mean number of photons incident on detector per second. From the semiclassical model, the mean number of photons recorded in a short time Δt (for fast photo-detectors)

$$\langle n_{out}(t) \rangle = \underset{\substack{\uparrow \\ \text{detector sensitivity}}}{\eta} \langle I_{out}(t) \rangle \Delta t = \eta \langle |\hat{E}_{out}(t)|^2 \rangle \Delta t$$

As we know from quantum mechanics, as emphasized by Dirac, each photon arrives at the photon detector with a probability amplitude that depends on its history. The probability of a photon arriving at the detector depends only on each photon interfering only with itself. This probability for the interferometer

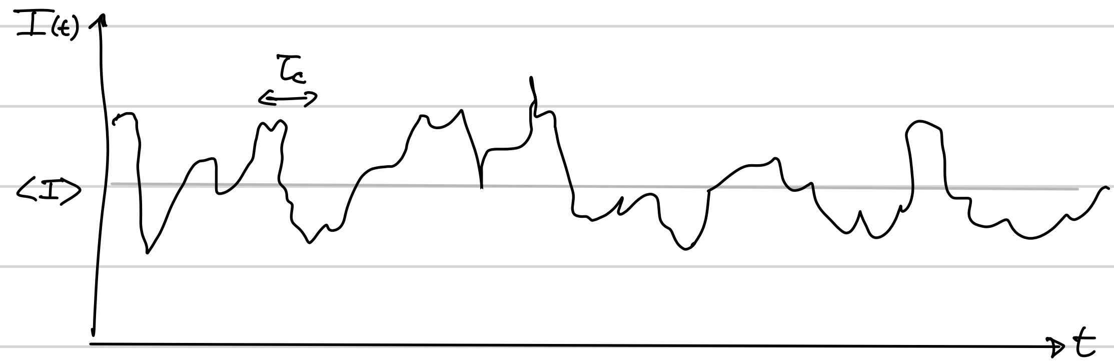
$$P_{out}(\tau) = \frac{\langle n_{out}(\tau) \rangle}{\langle n_{in} \rangle} = \eta \left(\frac{\langle (\tilde{E}^*_{in}(\tau) + \tilde{E}^*_{in}(0)) \, (\hat{E}_{in}(\tau) + \hat{E}_{in}(0)) \rangle}{\langle |\hat{E}_{in}|^2 \rangle} \right) = \frac{\eta}{2} \left(1 + \mathrm{Re}\, (g^{(1)}(\tau)) \right)$$

The electric field amplitude plays the role of the probability amplitude for detecting the photon. When the interferometer becomes too imbalanced, $\tau \gg \tau_c$, the photon wave packet no longer overlaps with itself at the output, and there is no inference. (Note: Here the photon wave packets are in mixed states corresponding to the statistical mixture of frequency components

Hanbury Brown & Twiss: The "Intensity Interferometer"

If the goal is to measure the coherence time of the field, the Mach-Zender interferometer will become unwieldy if τ_c is too long. In the 1950s two radio astronomers, Hanbury Brown (one person) and Twiss realized that one can measure the coherence time by an interferometer that correlated intensity rather than electric field (Actually, HBT were interested in the spatial correlation length, as they were interested in measuring the angular size of stars. But even there, the temporal correlation of signals is essential for the measurement, and fundamentally, this plays a central role in our understands of quantum optics.)

Recall, for chaotic light, the intensity fluctuates greatly over time



The intensity at any give time is a stochastic variable, with probability $p(I(t)) = \frac{1}{\langle I \rangle} e^{-\frac{I(t)}{\langle I \rangle}}$

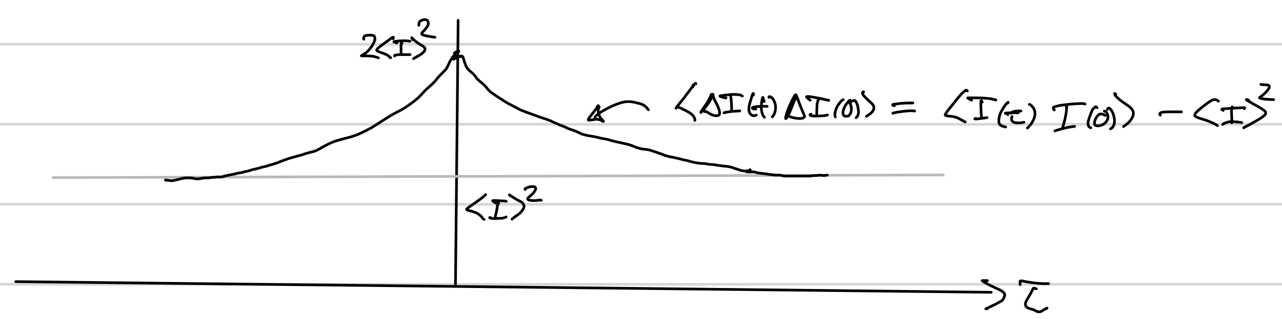
Recall this implies $\langle I^2 \rangle = 2\langle I \rangle^2 \Rightarrow \Delta I^2 = \langle I \rangle$ <u>intensity fluctuations</u>

The coherence time can thus be measured (if we have fast detectors) by looking at the <u>intensity</u> autocorrelation

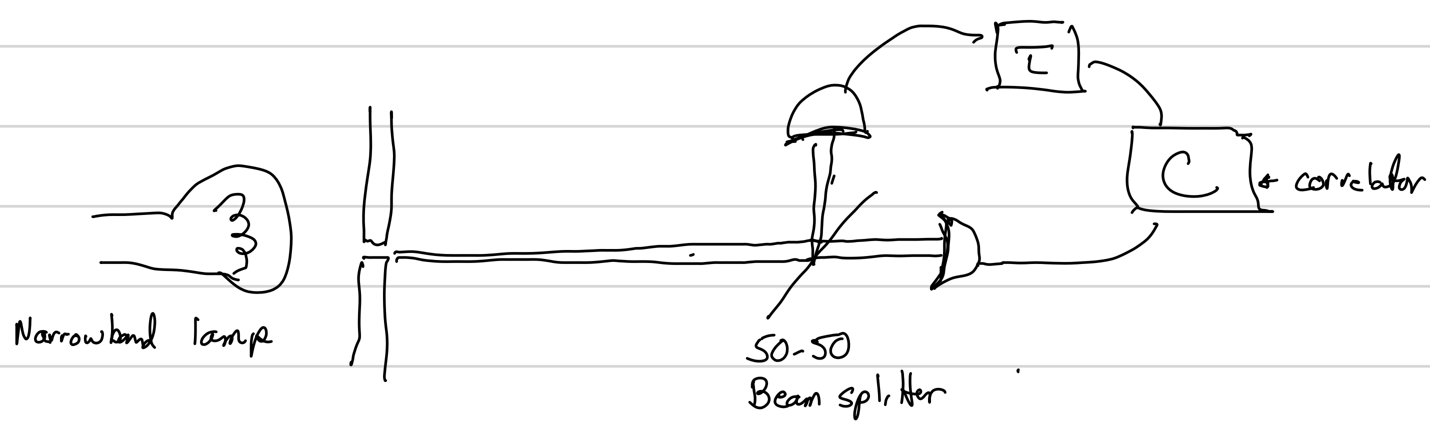
$$\langle I(\tau) I(0) \rangle = \langle |\tilde{E}(\tau)|^2 \, |\tilde{E}(0)|^2 \rangle$$

When $\tau \gg \tau_c$ we expect the intensities to be uncorrelated $\langle I(\tau) I(0) \rangle \approx \langle I(\tau) \rangle \langle I(0) \rangle = \langle I \rangle^2$

When $\tau = 0$ $\langle I(\tau=0)\, I(0) \rangle = \langle I^2 \rangle = 2\langle I^2 \rangle$. Thus we expect



In 1956 HBT set out to measure this in what is now know as the HBT configuration



The correlator essentially measures the time average of the product of the two photo currents coming from the square law detectors, with a delay τ between them. The signal from each detector $S(t) = \eta I(t) +$ shot noise (assuming fast detectors). Thus, the HBT correlation $\langle S(t+\tau)\, S(t)\rangle = \eta^2 \langle I(t+\tau) I(t)\rangle = \eta^2 \langle I(\tau)\, I(0)\rangle$ as the mean shot noise $= 0$ with no correlation of shot noise to $I(t)$. E.M. Purcell explained the HBT signal using classical statistical wave theory. Recall, for a chaotic field, the complex amplitude is a random variable, with probability of a Gaussian distribution.

$$P(\alpha_k) = \frac{1}{\pi N} e^{\frac{-|\alpha_k|^2}{\pi N}}$$

Thus the probability is distribution of $\tilde{E}(\vec{r},t)$ is also Gaussian, with mean zero

Consider $\langle I(\tau) I(0)\rangle = \langle |\tilde{E}(\tau)|^2 |\tilde{E}(0)|^2\rangle = \langle \tilde{E}^*(\tau)\, \tilde{E}(\tau)\, \tilde{E}^*(0)\, \tilde{E}(0)\rangle$

Recall for a multivariate Gaussian, the moments factorized into the sum of all two-point correlation functions (Wick's theorem)

$$\Rightarrow \langle I(\tau) I(0)\rangle = \underbrace{\langle \tilde{E}^*(\tau)\tilde{E}(\tau)\rangle}_{\langle I\rangle} \underbrace{\langle \tilde{E}^*(0)\tilde{E}(0)\rangle}_{\langle I\rangle} + \underbrace{\langle \tilde{E}^*(\tau)\tilde{E}(0)\rangle}_{G^{(1)}(\tau)} \underbrace{\langle \tilde{E}(\tau)\tilde{E}^*(0)\rangle}_{(G^{(1)}(\tau))^*} + \underbrace{\cancel{\langle \tilde{E}^*(\tau)\tilde{E}^*(0)\rangle \langle \tilde{E}(\tau)\tilde{E}(0)\rangle}}_{0 \text{ fast oscillation}}$$

$$\Rightarrow \boxed{\langle I(\tau) I(0)\rangle = \langle I\rangle^2 \left(1 + \frac{|G^{(1)}(\tau)|^2}{\langle I\rangle^2}\right) = \langle I\rangle^2 \left(1 + |g^{(1)}(\tau)|^2\right) = \langle I\rangle^2 \left(1 - e^{-2\tau/\tau_c}\right) \text{ (collision broadening)}}$$

For fast enough detectors, the intensity-intensity correlation function measures the <u>coherence time</u>, without balancing an interferometer.

But wait a minute! The fringes in the Mach-Zender interferometer were an <u>interference effect</u>. The decay of the fringes could be understood in terms of photons, each interfering with itself; the coherence time was the delay time whereby the photon wave packet no longer overlapped with itself. In the HBT configuration, we have two different photons detected at each of the detectors. This seemed to contradict Dirac's famous dictum that "a photon only interferes with itself." How do we understand the HBT effect in terms of photons? The HBT caused great controversy.

Quantum Theory of photon counting: Glauber Correlation Functions

The classical stochastic wave theory explains the HBT effect. But the question of the quantum explanation remains. The electromagnetic field is has wave-particle duality. How does one explain the HBT effect from the point of view of photon paths?

To answer the question we turn to the fully quantum theory, as developed by Roy Glauber, that led to the modern theory of quantum optics, for which he was awarded the Nobel Prize in 2005. Let us return to the photoelectric effect, but now in the fully quantum theory, including the quantized electromagnetic field.
Consider a one atom detector. The interaction Hamiltonian is taken as the dipole interaction

$$\hat{H} = -\hat{\vec{d}}\cdot\hat{\vec{E}}(\vec{r},t) = -\hat{\vec{d}}\cdot\hat{\vec{E}}^{(+)}(\vec{r},t) - \hat{\vec{d}}\cdot\hat{\vec{E}}^{(-)}(\vec{r},t)$$

We seek the transition probability $|g\rangle \Rightarrow |e\rangle$, where $|e\rangle$ is in the continuum for the atom. The electron of the photoionized atom is ultimately measured; the state of the field after photo-ionization is not measured.
Let $|\Psi^{AF}_{initial}\rangle = |g\rangle \otimes |\psi^{in}_F\rangle$, $|\Psi^{AF}_{final}\rangle = |e\rangle \otimes |\psi^{out}_F\rangle$. By Fermi's Golden Rule, the transition probability to photo-ionize the atom is proportional to

$$p^{(1)} \propto \sum_{\psi^{out}_F} |\langle e|d|g\rangle|^2 |\langle\psi^{out}_{field}|\hat{E}^{(+)}(\vec{r},t)|\psi^{in}_{field}\rangle|^2$$

$$\Rightarrow p^{(1)} \propto \sum_{\psi^{out}_F} \langle\psi^{in}_{field}|\hat{E}^{(-)}(\vec{r},t)|\psi^{out}_{field}\rangle\langle\psi^{out}_{field}|\hat{E}^{(+)}(\vec{r},t)|\psi^{in}_{field}\rangle$$

(I ignore the polarization of the field under the assumption we drive a given dipole transition)

The sum over final field state can then be extended to a sum over all states over the field since additional states not connected by the photo-ionization process will have zero matrix element $\langle\psi^{out}_{field}|\hat{E}^{(+)}(\vec{r},t)|\psi^{in}_{field}\rangle$

$$\Rightarrow p^{(1)} \propto \langle \hat{E}^{(+)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r},t)\rangle$$

This is the quantum version of the semiclassical theory of photon counting:

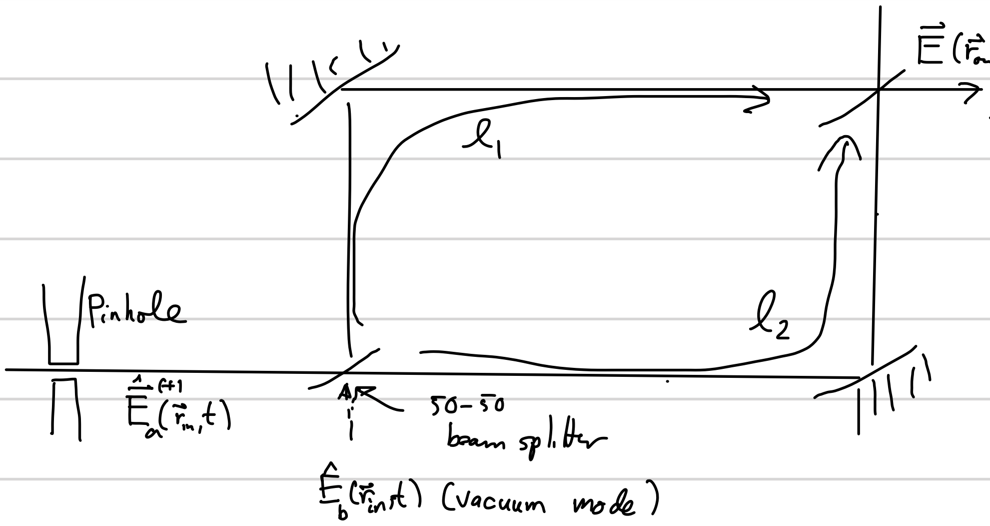
Probability to count a photon in short time Δt:

Semiclassical: $p^{(1)} = \eta\langle I(\vec{r},t)\rangle\Delta t = \eta\int d[\mathcal{E}]\,|\mathcal{E}(\vec{r},t)|^2\,\Delta t$

Quantum: $p^{(1)} = \eta\,\underbrace{\langle\hat{I}(\vec{r},t)\rangle_{\hat{\rho}}}_{\text{expected value over incident field}}\Delta t$: $\hat{I}(\vec{r},t) = \hat{E}^{(-)}(\vec{r},t)\,\hat{E}^{(+)}(\vec{r},t)$ (intensity operator)

<u>Quantum theory of "first-order" interference</u>

Consider again the Mach-Zender interferometer. The configuration is exactly the same as we saw before:



$$\langle \hat{I}_{out} \rangle = \langle \hat{E}_a^{(-)}(\vec{r}_{out},t)\, \hat{E}_a^{(+)}(\vec{r}_{out},t) \rangle$$

$$t_1 = t - \frac{l_1}{c}, \qquad t_2 = t - \frac{l_2}{c}$$

$$\hat{E}^{(+)}(\vec{r}_{out},t) = \frac{1}{2}\left(\hat{E}_a^{(+)}(\vec{r}_{in}, t-t_1) + \hat{E}_a^{(+)}(\vec{r}_{in}, t-t_2)\right) + \frac{1}{2}\left(\hat{E}_b^{(+)}(\vec{r}_{in}, t-t_1) - \hat{E}_b^{(+)}(\vec{r}_{in}, t-t_2)\right)$$

$$\Rightarrow \langle \hat{I}_{out} \rangle = \frac{1}{4}\Big[\langle \hat{E}_{in}^{(-)}(t-t_1)\,\hat{E}_{in}^{(+)}(t-t_1)\rangle + \langle \hat{E}_{in}^{(+)}(t-t_2)\,\hat{E}_{in}^{(-)}(t-t_2)\rangle + \langle \hat{E}_{in}^{(-)}(t-t_2)\,\hat{E}_{in}^{(+)}(t-t_1)\rangle + \langle \hat{E}_{in}^{(+)}(t-t_1)\,\hat{E}_{in}^{(-)}(t-t_2)\rangle\Big]$$

$$= \frac{1}{2}\left[\langle \hat{I}_{in} \rangle + \mathrm{Re}\,\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\rangle\right]$$

(Where $\hat{E}_{in}^{(+)}(t) \equiv \vec{E}_a(\vec{r}_{in},t)$)

where under the assumption of stationary statistics: $\langle \hat{I}_{in} \rangle = \langle \hat{E}^{(-)}(t_1)\,\hat{E}^{(+)}(t_1)\rangle = \langle \hat{E}^{(-)}(t_2)\,\hat{E}^{(+)}(t_2)\rangle$

$$\langle \hat{E}^{(-)}(t_1)\,\hat{E}^{(+)}(t_2)\rangle = \langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\rangle, \qquad \tau = t_2 - t_1$$

The expected value of the intensity measured at the output port of the interferometer takes exactly the same form as the classical expression. We recover exactly the classical expression when the state of a field in a <u>coherent state</u>.

$$\hat{E}^{(+)}(\vec{r},t)\,|\{\alpha_k\}\rangle = \mathcal{E}(\vec{r},t)\,|\{\alpha_k\}\rangle$$

Quasiclassical field $\mathcal{E}(\vec{r},t) = \sum_k \sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \alpha_k\, e^{i(\vec{k}\cdot\vec{r} - \omega_k t)}$

A coherent state with stationary statistics is monochromatic $\mathcal{E} = \mathcal{E}_0\, e^{i(\vec{k}\cdot\vec{r} - \omega t)}$ (single mode)

$$\Rightarrow \langle \hat{I}_{out} \rangle = \frac{1}{2}\Big[\underbrace{|\mathcal{E}_0|^2}_{\Rightarrow \langle \hat{I}_0 \rangle} + |\mathcal{E}_0|^2 \cos\omega_0\tau\Big] = \frac{\langle \hat{I}_0 \rangle}{2}\left(1 + \cos\omega_0\tau\right)$$

: Perfectly coherent field

Within the quantum theory, we can interpret this as interference, photon by photon. Though we have written the field in terms of modes of volume with periodic boundary conditions, for beams of light, it is convenient to consider traveling wave modes. Let consider a quasimonochromatic beam of light, with central frequency ω_0, propagating in the z-direction, with transverse mode treated as a tophat with mode area A. Restricted to states of this sort, the electric field operator can be approximated as (Next Page)

$\hat{E}^{(+)}(z,t) = \sqrt{\frac{2\pi\hbar\omega_0}{A}}\ \hat{\Psi}(z,t)\ e^{i(k_0 z - \omega_0 t)}$ (At positions within the beam area A)

Here $\hat{\Psi}(z,t)$ is the "slowly varying envelope" of the pulse $\left\|\frac{\partial\hat{\Psi}}{\partial z}\right\| \ll k_0\|\hat{\Psi}\|$, $\left\|\frac{\partial\hat{\Psi}}{\partial t}\right\| \ll \omega_0\|\hat{\Psi}\|$

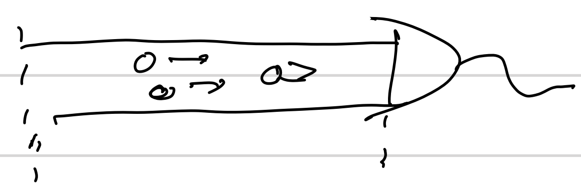
For the free field, $\left(\frac{\partial}{\partial z} - \frac{1}{c}\frac{\partial}{\partial t}\right)\hat{\Psi}(z,t) = 0 \Rightarrow \hat{\Psi}(z,t) = \hat{\Psi}(z=0, t - z/c) = \hat{\Psi}(z-ct, 0)$

the envelope satisfy equal-time commutator $[\hat{\Psi}(z,t), \hat{\Psi}^{+}(z',t)] = \delta(z-z')$

It also follows for the free field $[\hat{\Psi}(z,t), \hat{\Psi}^{+}(z,t')] = \delta(c(t-t')) = \frac{1}{c}\delta(t-t')$

The operator $\hat{\Psi}^{+}(z,t)\hat{\Psi}(z,t)$ is the photon number density / length @ z,t.

Consider then such a light beam incident on a detector



incident plane: $Z = 0$ $\qquad$ $z = z_D$ Detector plane

Let $\hat{a}_D(t) = \frac{1}{\sqrt{c}}\hat{\Psi}(0, t - z_D/c) \Rightarrow \hat{a}^{+}_D(t)\hat{a}_D(t) = \frac{1}{c}\hat{\Psi}^{+}(0,t-z_D/c)\hat{\Psi}(0,t-z_D/c) \equiv \hat{n}(t)$

$= \text{Photon \# operator/time at detector}$

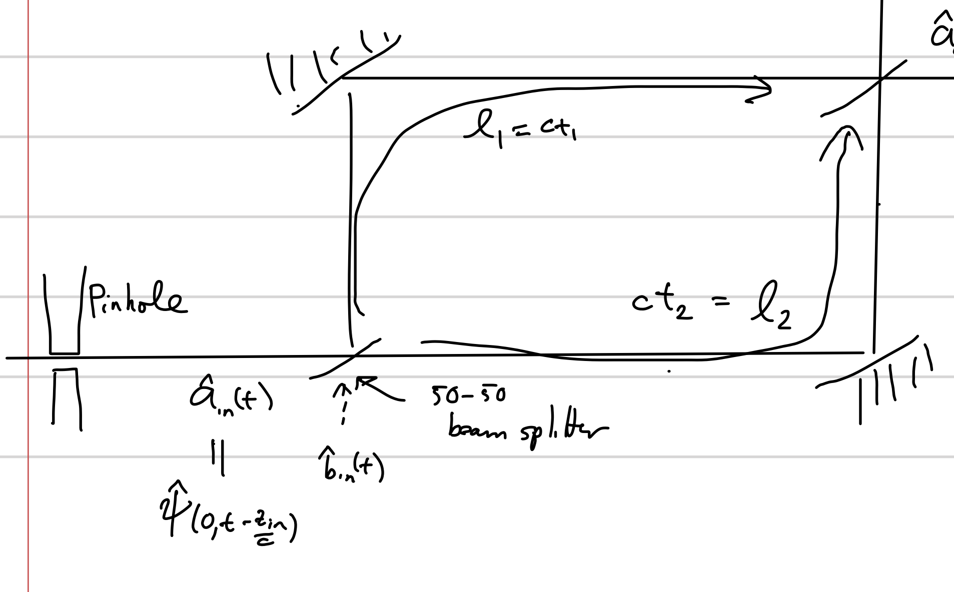
Note $[\hat{a}(t), \hat{a}^{+}(t')] = \delta(t-t')$, $\langle\hat{a}^{+}(t)\hat{a}(t)\rangle\Delta t$ = average # of photons detected in short Δt

In the frequency domain, $\hat{a}(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\hat{a}(\omega)\, e^{-i\omega t} = \sum_{\omega}' \sqrt{\frac{c}{L}}\,\hat{a}_\omega e^{-i\omega t}$ (for discretized modes)

$[\hat{a}(\omega), \hat{a}^{+}(\omega')] = \delta(\omega - \omega')$. $[\hat{a}_\omega, \hat{a}^{+}_{\omega'}] = \delta_{\omega\omega'}$

$\hat{a}^{+}(\omega)\hat{a}(\omega) = \hat{n}(\omega) = $ # of photons / frequency interval

Let us return to the Mach-Zender interferometer



$\hat{a}_{out}(t) = \hat{\Psi}(0, t - \frac{z_{out}}{c})$, $\hat{E}^{(+)}(z_{out}, t) = \hat{a}_{out}(t)\, e^{i(k_0 z_{out} - \omega_0 t)}$

$\langle\hat{n}_{out}\rangle = \langle\hat{E}^{(-)}(z_{out},t)\hat{E}^{(+)}(z_{out},t)\rangle = \langle\hat{a}^{+}_{out}(t)\,\hat{a}_{out}(t)\rangle$

$\hat{a}_{out}(t) = \frac{1}{2}\left(\hat{a}_{in}(t-t_1)e^{+i\omega_0 t_1} + \hat{a}_{in}(t-t_2)e^{+i\omega_0 t_2}\right)$

$+ \frac{1}{2}\left(\hat{b}_{in}(t-t_1)e^{i\omega_0 t_1} - \hat{b}_{in}(t-t_2)e^{i\omega_0 t_2}\right)$

these terms will not contribute if only vacuum in b_{in}-mode

$$\Rightarrow \langle \hat{n}_{out} \rangle = \frac{1}{4}\Big(\langle \hat{a}^{\dagger}_{in}(t-t_1)\hat{a}_{in}(t-t_1)\rangle + \langle \hat{a}^{\dagger}_{in}(t-t_2)\,\hat{a}_{in}(t-t_2)\rangle$$

$$+ \langle \hat{a}^{\dagger}_{in}(t-t_1)\,\hat{a}_{in}(t-t_2)\rangle e^{i\omega_0(t_1-t_2)} + \langle \hat{a}^{\dagger}_{in}(t-t_2)\,\hat{a}_{in}(t-t_1)\rangle\, e^{-i\omega_0(t_1-t_2)}\Big)$$

$$= \frac{1}{4}\Big(\langle \hat{n}(t-t_1)\rangle + \langle \hat{n}(t-t_2)\rangle\Big) + \frac{1}{2}\mathrm{Re}\Big(G^{(1)}(t-t_1;\, t-t_2)\; e^{i\omega_0(t_1-t_2)}\Big)$$

Where $G^{(1)}(t-t_1;\, t-t_2) = \langle \hat{a}^{\dagger}(t-t_1)\,\hat{a}(t-t_2)\rangle$

- For example, consider a <u>single photon wave packet</u>, for a pure state with pulse shape $\phi(t)$ describing the probability amplitude $|1_\phi\rangle = \hat{a}^{\dagger}[\phi]\,|0\rangle$ where
$\hat{a}^{\dagger}[\phi] = \int dt\, \phi(t)\, \hat{a}^{\dagger}(t) \Rightarrow \langle 1_\phi|\hat{a}^{\dagger}\hat{a}\,|1_\phi\rangle = 1$, $\langle 1_\phi|\hat{a}^{\dagger}(t-t_1)\,\hat{a}(t-t_2)|1_\phi\rangle = \phi^*(t-t_1)\,\phi(t-t_2)$

$$\Rightarrow \langle \hat{n}_{out}\rangle = \frac{1}{4}|\phi(t-t_1)|^2 + \frac{1}{4}|\phi(t-t_2)|^2 + \frac{1}{2}|\phi^*(t-t_1)\,\phi(t-t_2)|\,\cos(\omega_0(t_1-t_2)+\varphi)$$

If $|t_1-t_2|$ is much less than the width of the wave packet, each photon will interfere with itself.

- For a <u>coherent state</u> Let $E^{(+)}(t) = \mathcal{E}(t)\, e^{-i\omega_0 t}$ be the classically field corresponding to the complex amplitude $\alpha(t) = \tilde{\alpha}(t)\, e^{-i\omega_0 t}$ slowly varying envelope $|\tilde{\alpha}(t)|^2 =$ average # photons / time

$$|\alpha(t)|^2 = |\tilde{\alpha}(t)|^2 = \frac{I(t)A}{\hbar\omega} = \text{Average rate of photon flux}$$

$$\int |\alpha(t)|^2 dt = \bar{n} = \text{average \# of photons in pulse}$$

Coherent State: $|\tilde{\alpha}\rangle \equiv e^{\int dt\,(\tilde{\alpha}(t)\,\hat{a}^{\dagger}(t) - \tilde{\alpha}^*(t)\,\hat{a}(t))}$, $\hat{a}(t)\,|\tilde{\alpha}\rangle = \tilde{\alpha}(t)\,|\tilde{\alpha}(t)\rangle$

function

$$\langle \hat{n}_{out}\rangle = \frac{1}{4}|\alpha(t-t_1)|^2 + \frac{1}{4}|\alpha(t-t_2)|^2 + \frac{1}{2}|\alpha^*(t-t_1)\,\alpha(t-t_2)|\,\cos(\omega_0(t_1-t_2)+\varphi)$$

We can normalize $\tilde{\alpha}(t) \equiv \sqrt{\bar{n}}\,\phi(t) \Rightarrow \phi(t) =$ single photon wave function

$$\langle \hat{n}_{out}\rangle = \frac{\sqrt{\bar{n}}}{4}|\phi(t-t_1)|^2 + \frac{\sqrt{\bar{n}}}{4}|\phi(t-t_2)|^2 + \frac{\sqrt{\bar{n}}}{2}|\phi^*(t-t_1)\,\phi(t-t_2)|\,\cos(\omega_0(t_1-t_2)+\varphi)$$

Each of the photons in the beam independently interferes with itself. The classical amplitude $\frac{\alpha(t)}{\sqrt{\bar{n}}}$ acts as the "wave function" of the photon.

• For chaotic light: The quantum state is a mixed state. For a given mode ω (here discretized), we have a statiscal mixture of different classical amplitudes. We can express this in the quantum theory as a statistical mixture of different coherent states:

$$\hat{\rho}_\omega = \int d^2\alpha_\omega \, P(\alpha_\omega) \, |\alpha_\omega\rangle\langle\alpha_\omega| \; : \quad \text{Glauber – Sudharshan P-representation}$$

$$P(\alpha_\omega) = \frac{1}{\pi \langle|\alpha_\omega|^2\rangle} e^{-\frac{|\alpha_\omega|^2}{\langle|\alpha_\omega|^2\rangle}} = \frac{1}{\pi\langle n_\omega\rangle} e^{-\frac{|\alpha_\omega|^2}{\langle n_\omega\rangle}} \quad \text{(Here considering discrete)}$$

where $\langle n_\omega\rangle = \mathrm{Tr}(\hat{\rho}_\omega \hat{a}^\dagger_\omega \hat{a}_\omega) = \int d^2\alpha_\omega \, |\alpha_\omega|^2 \, P(\alpha_\omega) = \langle|\alpha_\omega|^2\rangle$

For multi-mode: $\hat{\rho} = \bigotimes_\omega \hat{\rho}_\omega$, Note: For coherent state $|\tilde{\alpha}\rangle\langle\tilde{\alpha}| = \bigotimes_\omega |\alpha_\omega\rangle\langle\alpha_\omega|$

Chaotic light is stationary and ergodic.

$\langle \hat{a}^\dagger_\omega \hat{a}_{\omega'}\rangle = \langle n_\omega\rangle \delta_{\omega\omega'}$ $\qquad \frac{\hbar\omega\langle n_\omega\rangle}{AL}$ = spectral energy density

For continuum modes: $\langle \hat{a}(\omega)\hat{a}^\dagger(\omega')\rangle = \langle n(\omega)\rangle \delta(\omega-\omega')$

$$\langle \hat{n}_{in}\rangle = \langle \hat{a}^\dagger(t)\hat{a}(t)\rangle = \int \frac{d\omega\, d\omega'}{(2\pi)^2} \langle \hat{a}^\dagger(\omega)\hat{a}(\omega')\rangle e^{i(\omega-\omega')t} = \int \frac{d\omega}{2\pi} \langle n(\omega)\rangle$$

$$\langle \hat{n}_{out}\rangle = \frac{1}{2}\left(\langle \hat{n}_{in}\rangle + \mathrm{Re}\left(\langle \hat{a}^\dagger(\tau)\hat{a}(0)\rangle e^{i\omega_0\tau}\right)\right) = \frac{1}{2}\langle \hat{n}_{in}\rangle \left[1 + \mathrm{Re}\left(g^{(1)}(\tau)\, e^{i\omega_0\tau}\right)\right]$$

$$\langle \hat{a}^\dagger(\tau)\hat{a}(0)\rangle = \int \frac{d\omega\, d\omega'}{(2\pi)^2} \langle \hat{a}^\dagger(\omega)\hat{a}(\omega')\rangle e^{i\omega\tau} = \int \frac{d\omega}{2\pi} \langle n(\omega)\rangle e^{i\omega\tau}$$

The interferometer depends only on the spectral density of the light for each photon,

__General recovery of classical statistical fluctuations of partially coherent light:__

Mixed state: $\hat{\rho} = \int d^2\{\alpha_\omega\} \, P(\{\alpha_\omega\}) \, |\{\alpha_\omega\}\rangle\langle\{\alpha_\omega\}|$ (Statistical mixture of coherent states)

real, positive, probability distribution

$$\langle \hat{E}^{(-)}(t_1)\, \hat{E}^{(+)}(t_2)\rangle = \mathrm{Tr}(\hat{E}^{(-)}(t_2)\, \hat{E}^{(+)}(t_1)\, \hat{\rho}) = \underbrace{\int d^2\{\alpha_\omega\} \, P(\{\alpha_\omega\})\, \mathcal{E}^*(t_1)\, \mathcal{E}(t_2)}_{\text{Classical statistical average}}$$

Where $\langle \hat{n}_k \rangle = \langle |\alpha_k|^2 \rangle$ = averge # of photons in the mode

$$\Rightarrow \langle \hat{I}_{out} \rangle = \frac{1}{2}\left[\overline{|\mathcal{E}|^2} + Re\left(\overline{\mathcal{E}^*(t_1)\mathcal{E}(t_2)} \right) \right] = \frac{1}{2} I_0 \left(1 + |g^{(1)}(\tau)| \cos\omega_0 \tau \right)$$

coherence function = Fourier transform transform of spectral density

For natural light: $P(\{\alpha_\omega\}) = \prod \left[\frac{1}{\pi \langle |\alpha_\omega|^2 \rangle} e^{-\frac{|\alpha_\omega|^2}{\langle |\alpha_\omega|^2 \rangle}} \right] = \prod_k \frac{1}{\pi \langle \hat{n}_\omega \rangle} e^{-\frac{|\alpha_\omega|^2}{\langle \hat{n}_\omega \rangle}}$

Bose-Einstein distribution:

We have seen that "natural light" is represented by Gaussian fluctuations of the wave amplitude. We can also consider the representation of this mixed state in the number basis.
I will leave it as an exercise to show, for a given mode,

$$\boxed{\underbrace{\int d^2\alpha_\omega}_{\text{"continuous variable" (waves)}} \frac{1}{\pi \langle \hat{n}_\omega \rangle} e^{-\frac{|\alpha_k|^2}{\langle \hat{n}_\omega \rangle}} |\alpha_\omega\rangle\langle\alpha_\omega| = \underbrace{\sum_{n_k=0}^{\infty}}_{\text{discrete variable (particles)}} \frac{\langle \hat{n}_\omega \rangle^{n_\omega}}{(1 + \langle \hat{n}_\omega \rangle)^{n_\omega+1}} |n_\omega\rangle\langle n_\omega|}$$

The probability distribution of photon excitations: $P(n_\omega) = \frac{\langle \hat{n}_\omega \rangle^{n_\omega}}{(1+\langle \hat{n}_\omega \rangle)^{n_\omega+1}}$ is the famous Bose-Einstein distribution associated with a state of identical bosons (here photons) with average number $\langle \hat{n}_k \rangle$. There is an "effective temperature" $\langle \hat{n}_\omega \rangle = \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}$

We often use the term "thermal state" to represent the natural state of light, such as collision broadened light from a gas of atoms. Thermal light exhibits "first order coherence" with a coherence time depending on the power spectrum $\langle \hat{I}_{out} \rangle = \frac{\langle \hat{n}_{in} \rangle}{2} \left(1 + |g^{(1)}(\tau)| \cos \omega_0 \tau \right)$

Proof:

Recall: $P(n_\omega) = \frac{1}{Z} e^{-\frac{n\hbar\omega}{k_B T}}$ $\quad Z = \sum_n e^{-\frac{n\hbar\omega}{k_B T}} = \frac{1}{1 + e^{-\hbar\omega/k_B T}} = \langle n_\omega \rangle + 1$

$e^{-\hbar\omega/k_B T} = \frac{\langle n_\omega \rangle}{\langle n_\omega \rangle + 1} \quad \Rightarrow \quad P(n_\omega) = \frac{\langle n_\omega \rangle^n}{(\langle n_\omega \rangle + 1)^{n+1}}$

The Bose-Einstein distribution as fluctions in number of quanta

$$\boxed{\langle \hat{n}_\omega^2 \rangle = 2\langle \hat{n}_\omega \rangle^2 + \langle \hat{n}_\omega \rangle \quad \Rightarrow \quad \Delta n_\omega^2 = \langle \hat{n}_\omega \rangle^2 + \langle \hat{n}_\omega \rangle}$$

This is compared to a coherent state $\langle \Delta n_\omega \rangle_{coh} = \langle \hat{n}_\omega \rangle$ (Poisson)

Einstein recognized the fluctuations in the number of bosons in the mode for a Bose distribution as having two contributions

$$\Delta n^2 = \underbrace{\langle \hat{n}\rangle^2}_{\text{wave}} + \underbrace{\langle \hat{n}\rangle}_{\text{particle}}$$

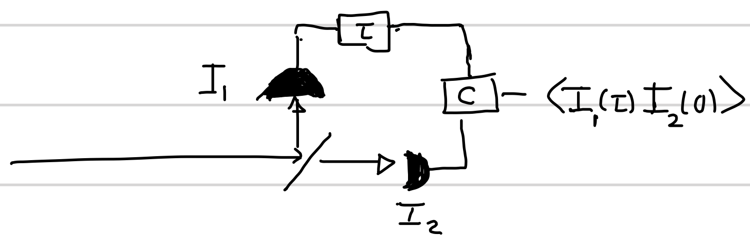
The particle contribution: Irreducible $\langle \hat{n}\rangle$ Poisson fluctuations

The wave contribution: $\Delta|\alpha|^{2\,2} = \langle|\alpha|^4\rangle - \langle|\alpha|^2\rangle^2$: for $P(\alpha) = \frac{1}{\pi\langle\hat{n}\rangle} e^{-\frac{|\alpha|^2}{\langle\hat{n}\rangle}}$

$\langle|\alpha|^2\rangle = \langle\hat{n}\rangle \quad \langle|\alpha|^4\rangle = 2\langle\hat{n}\rangle^2 \Rightarrow \Delta|\alpha|^{2\,2} = 2\langle\hat{n}\rangle$: Intensity fluctuations

Higher order correlations: Coincidence Counting

The Hanbury-Brown & Twiss effect differs fundamentally from the Mach-Zender-type first order interference effect in that it involves correlating intensities rather than field amplitudes, and thus involves the joint-probability for detecting more than one photon. For example, the temporal HBT effect



We can think about this as a photon counting experiment. The correlator, C, goes "click" if a photo-electron is ejected in detector-1 and one in detector-2, separated by time τ. This is a two-photon correlation. Thus, we must calculate the probabil. for two atoms (one in each detector) to absorb the photon. Following Glauber, consider two photons absorbed at positions $\vec{r}_1$ and $\vec{r}_2$ respectively. According to time-dependent perturbation theory, the transition probability

$$P^{(2)} = \sum_f \left|\langle \Psi_f, \{e_1, e_2\}| \hat{U}(t) |\Psi_i \{g_1, g_2\}\rangle\right|^2 \simeq \left|\langle \Psi_f, \{e_1, e_2\}| \left(\tfrac{-i}{\hbar}\right)^2 \int_0^t dt_1 \int_0^t dt_2\, \hat{H}_1^{(+)}(t_1)\, \hat{H}_2^{(+)}(t_2) |\Psi_i, \{g_1, g_2\}\rangle\right|^2$$

where Ψ_i and Ψ_f are initial and final states of the field. As before, we sum over the final states because we measure the electron, not the field after the detection. Here $\hat{H}^{(+)}_{int} = -\vec{d}_{eg} \cdot \hat{\vec{E}}^{(+)}(\vec{r}, t)\, |e\rangle\langle g|$, where the label 1 & 2 refer to absorbing atoms

Thus $p^{(2)} = \int_0^t dt_1 \int_0^t dt_1' \int_0^t dt_2 \int_0^t dt_2' \; S(t_1 - t_1')\, S(t_2 - t_2'') \sum_f \langle \psi_i | \hat{E}^{(-)}(\vec{r}_1, t_1')\, \hat{E}^{(-)}(\vec{r}_2, t_2') | \psi_f \rangle \langle \psi_f | \hat{E}^{(+)}(\vec{r}_1, t_1)\, \hat{E}^{(+)}(\vec{r}_2, t_2) | \psi_i \rangle$

Where $S(t-t')$ is the detector "response function" depending on the bandwidth (density of states). Under the approximation of instantaneous response $S(\tau) = \eta\, \delta(\tau)$.
Finally, we allow for the possibility of a mixed state input field ρ_{field}

$$p^{(2)} = \eta^2 \int_0^t dt_1 \int_0^t dt_2 \;\; G^{(2)}(\vec{r}_2 t_2, \vec{r}_1 t_1;\, \vec{r}_2 t_2, \vec{r}_1 t_1)$$

where $G^{(2)}(x_1, x_2, x_3, x_4) \equiv \langle \hat{E}^{(-)}(x_1)\, \hat{E}^{(-)}(x_2)\, \hat{E}^{(+)}(x_3)\, \hat{E}^{(+)}(x_4) \rangle$ $\quad x = (\vec{r}, t)$ space time.

is the second-order correlation function.

$\Rightarrow$ Rate of joint detection / sec^2 $= \eta^2\, G^{(2)}(\vec{r}_2 t_2, \vec{r}_1 t_1;\, \vec{r}_2 t_2, \vec{r}_1 t_1)$

normal ordering

$$G^{(2)}(\vec{r}_2 t_2, \vec{r}_1 t_1;\, \vec{r}_2 t_2, \vec{r}_1 t_1) = \langle : \hat{I}(\vec{r}_2 t_2)\, \hat{I}(\vec{r}_1 t_1) : \rangle = \langle : \hat{E}^{(-)}(\vec{r}_2 t_2)\, \hat{E}^{(+)}(\vec{r}_2 t_2)\, \hat{E}^{(-)}(\vec{r}_1 t_1)\, \hat{E}^{(+)}(\vec{r}_1 t_1) : \rangle$$
$$= \langle \hat{E}^{(-)}(\vec{r}_2 t_2)\, \hat{E}^{(-)}(\vec{r}_1 t_1)\, \hat{E}^{(+)}(\vec{r}_2 t_2)\, \hat{E}^{(+)}(\vec{r}_1 t_1) \rangle$$

= Second order correlation function.

Here, $\hat{I}(\vec{r}, t) = \hat{E}^{(-)}(\vec{r}, t)\, \hat{E}^{(+)}(\vec{r}, t)$ $\qquad : \; : \leftarrow$ Normal ordering

Normal ordering is a short hand for "ignore the commutation relations, and move all of the creation operators to the left, and the annihilation operators to the right.
Thus, e.g., $:\hat{n}^2: \; = \; : \hat{a}^\dagger \hat{a}\, \hat{a}^\dagger \hat{a} : \; = \; \hat{a}^{\dagger 2}\, \hat{a}^2$

The normal ordering is the key difference between quantum and classical photon counting. The two-photon detection implies that if one photon is detected the field is changed (in general), and this effects the detection of a second photon! This is in contrast to the semi-classical theory, for classical fields, which <u>assumed</u> that all photo-counts are uncorrelated. As we will see, non-classical light violates this assumption.

Intensity-Intensity Correlation $\Rightarrow$ Two photon correlations

Semiclassical HBT: $\langle I(\tau) I(0)\rangle = \int d\{\alpha_k\}\, P(\{\alpha_k\})\, \mathcal{E}^*(\tau)\,\mathcal{E}(\tau)\,\mathcal{E}^*(0)\,\mathcal{E}(0)$

Quantum HBT: $\langle :\hat{I}(\tau)\,\hat{I}(0): \rangle = \langle \hat{E}^{(-)}(\tau)\,\hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\,\hat{E}^{(+)}(0)\rangle = \mathrm{Tr}\left(\hat{E}^{(-)}(\tau)\,\hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\,\hat{E}^{(+)}(0)\,\hat{\rho}\right)$

For classical statistical fluctuations $\hat{\rho} = \int d\{\alpha_k\}\, P(\{\alpha_k\})\, |\{\alpha_k\}\rangle\langle\{\alpha_k\}|$

$$\langle :\hat{I}(\tau)\,\hat{I}(0): \rangle = \int d\{\alpha_k\}\, P(\{\alpha_k\})\, |\mathcal{E}(\tau)|^2\, |\mathcal{E}(0)|^2 \leftarrow \text{Exactly the semiclassical result}$$

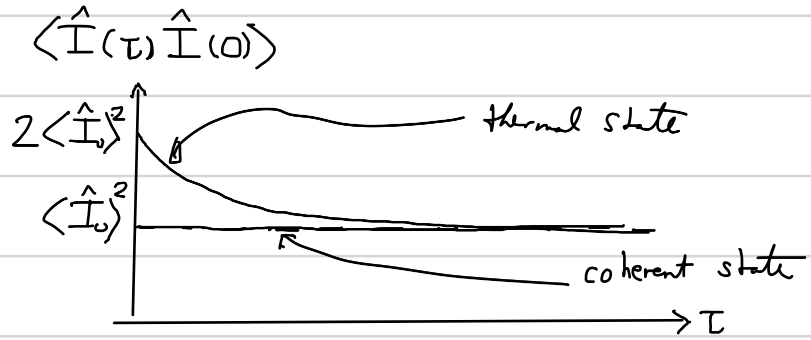
For a "thermal state" the fully quantum theory is exactly the same as the classical prediction

For Gaussian fluctuations in the wave amplitude:

$$\langle \hat{E}^{(-)}(\tau)\,\hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\,\hat{E}^{(+)}(0)\rangle = \underbrace{\langle \hat{E}^{(-)}(\tau)\,\hat{E}^{(+)}(\tau)\rangle}_{\langle \hat{I}_0\rangle}\underbrace{\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(0)\rangle}_{\langle \hat{I}_0\rangle} + \underbrace{\langle \hat{E}^{(-)}(\tau)\,\hat{E}^{(+)}(0)\rangle\langle \hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\rangle}_{\langle I_0\rangle^2 |g^{(1)}(\tau)|^2}$$

Note: For a coherent state $P(\{\alpha_k\}) = \delta(\{\alpha_k - \alpha_k^0\})$

$$\langle \hat{E}^{(-)}(\tau)\,\hat{E}^{(-)}(0)\,\hat{E}^{(+)}(\tau)\,\hat{E}^{(+)}(0)\rangle = \mathcal{E}^*(\tau)\,\mathcal{E}^*(0)\,\mathcal{E}(\tau)\,\mathcal{E}(0) = |\mathcal{E}(\tau)|^2\,|\mathcal{E}(0)|^2 = I(\tau)\,I(0) = \langle \hat{I}_0\rangle^2 : \text{Independent of } \tau$$



General Correlation Functions

The joint probability to detect n-photons at n-space/time points $x = (\vec{r}, t)$

$$p^{(n)}(x_1, x_2, \dots, x_n) \propto \langle :\hat{I}(x_1)\,\hat{I}(x_2)\cdots\hat{I}(x_n): \rangle = \langle \hat{E}^{(-)}(x_1)\,\hat{E}^{(-)}(x_2)\cdots\hat{E}^{(-)}(x_N)\,\hat{E}^{(+)}(x_1)\,\hat{E}^{(+)}(x_2)\cdots\hat{E}^{(+)}(x_N)\rangle$$

Define: $G^{(n)}(x_1, x_2, \dots; x_n, x_1', x_2', \dots, x_n') = \langle \hat{E}^{(-)}(x_1)\,\hat{E}^{(-)}(x_2)\cdots\hat{E}^{(-)}(x_n)\,\hat{E}^{(+)}(x_1')\,\hat{E}^{(+)}(x_2')\cdots\hat{E}^{(+)}(x_n')\rangle$

A field, is said to be n^{th}-order coherent if the n^{th}-order correlation function factories

- First-order coherence: Depends only only spectral density: Coherent = narrow band (e.g. laser, single photon state in single mode, filtered thermal light)
- Second order coherence: depends on quantum statistics
- A coherent state $|\{\alpha_k\}\rangle$ exhibits "coherence" to all orders.

Normalized Correlation function:

$$g^{(n)}(x_1,x_2,\cdots,x_n;x_1',x_2',\cdots x_n') = \frac{G^{(n)}(x_1,x_2,\cdots,x_n;x_1',x_2',\cdots,x_n')}{\left[G^{(1)}(x_1,x_1)\,G^{(1)}(x_2,x_2)\cdots G^{(1)}(x_n,x_n)\;G^{(1)}(x_1',x_1')\,G^{(1)}(x_2',x_2')\cdots G^{(1)}(x_n',x_n')\right]^{1/2}}$$

The n-fold coincidence function:

$$g^{(n)}(x_1,x_2,\cdots x_n) = \frac{G^{(n)}(x_1,x_2\cdots x_n;x_1,x_2,x_n)}{G^{(1)}(x_1,x_1)\,G^{(1)}(x_2,x_2)\cdots G^{(1)}(x_n,x_n)}$$

For a coherent state: $g^{(n)}(x_1,x_2,\cdots;x_n) = 1$

First-order coherence: $\langle \hat{I}_{out}\rangle = \frac{1}{4}\left[\langle \hat{E}^{(-)}(t_1)\hat{E}^{(+)}(t_1)\rangle + \langle \hat{E}^{(-)}(t_2)\hat{E}^{(+)}(t_2)\rangle + \langle \hat{E}^{(-)}(t_1)\hat{E}^{(+)}(t_2)\rangle + \langle \hat{E}^{(-)}(t_2)\hat{E}^{(+)}(t_1)\rangle\right]$

$$= \frac{1}{4}\left[G^{(1)}(t_1,t_1) + G^{(1)}(t_2,t_2) + G^{(1)}(t_1,t_2) + G^{(2)}(t_2,t_1)\right]$$

$$= \frac{1}{2}\langle \hat{I}_0\rangle\left[1 + \mathrm{Re}\left[\underbrace{g^{(1)}(t_1,t_2)}_{\equiv g^{(1)}(\tau)}\right]\right] \quad \text{(for stationary statistics)}$$

$$\tau = t_2 - t_1$$

Second-Order coherence: $\langle :\hat{I}(\tau)\hat{I}(0):\rangle = \langle \hat{E}^{(-)}(\tau)\hat{E}^{(-)}(0)\hat{E}^{(+)}(\tau)\hat{E}^{(+)}(0)\rangle \equiv G^{(2)}(\tau)$

$$g^{(2)}(\tau) \equiv \frac{\langle \hat{E}^{(-)}(\tau)\hat{E}^{(-)}(0)\hat{E}^{(+)}(\tau)\hat{E}^{(+)}(0)\rangle}{\langle \hat{E}^{(-)}(\tau)\hat{E}^{(+)}(\tau)\rangle\langle \hat{E}^{(-)}(0)\hat{E}^{(+)}(0)\rangle} = \frac{G^{(2)}(\tau)}{\langle \hat{I}_0\rangle^2}$$

Thermal light $G^{(2)}(\tau) = \underbrace{\langle \hat{E}^{(-)}(\tau)\hat{E}^{(+)}(\tau)\rangle\langle \hat{E}^{(-)}(0)\hat{E}^{(+)}(0)\rangle}_{\langle \hat{I}_0\rangle^2} + \langle \hat{E}^{(-)}(0)\hat{E}^{(+)}(\tau)\rangle\langle \hat{E}^{(-)}(\tau)\hat{E}^{(+)}(0)\rangle = \langle \hat{I}_0\rangle^2\, g^{(2)}(\tau)$

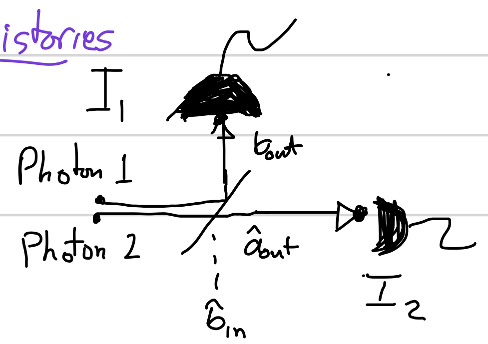
$$g^{(2)}(\tau) = 1 + |g^{(1)}(\tau)|^2$$

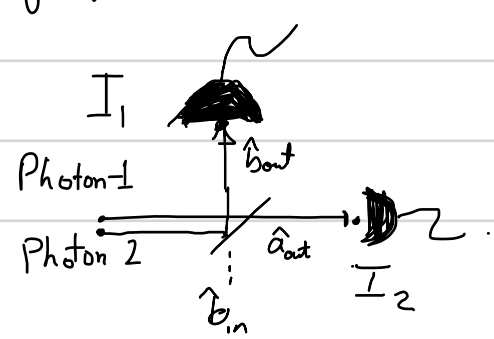
Coherent state $G^{(2)}(\tau) = \langle \hat{I}_0\rangle^2 \Rightarrow g^{(2)}(\tau) = 1$

HBT – The particle interference picture

We now come to the heart of the matter. We have explained the HBT effect in terms correlations of fluctuations of stochastic wave amplitudes. But how do we describe the effect in terms of photons?

Conder a two photon state incident on the beam splitter. The are two histories that can lead to a coincidence count. The probability of the coincidence depends on interference of histories





More quantitatively, let us write the two photon input

$$|\psi\rangle_{in} = \frac{1}{\sqrt{1+|\langle\psi|\phi\rangle|^2}}\,\hat{a}^\dagger[\psi]\,\hat{a}^\dagger[\phi]\,|0\rangle$$ where $\phi(t)$ and $\psi(t)$ are temporal packets

The transformation for 50-50 beam splitter is a scattering: $\hat{S}\hat{a}^\dagger\hat{S}^\dagger = \left(\frac{\hat{a}^\dagger - i\hat{b}^\dagger}{\sqrt{2}}\right)e^{ikl_1}$; $\hat{S}\hat{b}^\dagger\hat{S}^\dagger = \left(\frac{\hat{b}^\dagger - i\hat{a}^\dagger}{\sqrt{2}}\right)e^{ikl_2}$

$$\Rightarrow |\psi\rangle_{out} = \hat{S}|\psi\rangle_{in} = \frac{1}{2\sqrt{1+|\langle\phi|\psi\rangle|^2}}\Big[\hat{a}^\dagger[\phi]\,\hat{a}^\dagger[\psi] - \hat{b}^\dagger[\phi]\,\hat{b}^\dagger[\psi]$$
$$+\, i\,\underbrace{\left(\hat{a}^\dagger[\phi]\,\hat{b}^\dagger[\psi] + \hat{a}^\dagger[\psi]\,\hat{b}^\dagger[\phi]\right)}\,e^{-ik(l_1+l_2)}\Big]|0\rangle$$

→ two different histories that lead to detection at a and b

Coincidence rate in a time window Δt

$$C = \int_0^{\Delta t} d\tau\,\langle:\hat{I}_a(0)\,\hat{I}_b(\tau):\rangle = \int_0^{\Delta t} d\tau\,\langle\hat{a}^\dagger(0)\,\hat{b}^\dagger(\tau)\,\hat{a}(0)\,\hat{b}(\tau)\rangle = \int_0^{\Delta t} d\tau\,\|\hat{a}(0)\hat{b}(\tau)|\psi_{out}\rangle\|^2$$

$$= \int_0^{\Delta t} d\tau\,\frac{|\overbrace{\phi(\tau)\psi(0) + \psi(\tau)\phi(0)}^{\text{Two boson wave function (exchange symmetry)}}|^2}{4(1+|\langle\phi|\psi\rangle|^2)} = \int_0^{\Delta t} d\tau\,\frac{|\phi(\tau)|^2|\psi(0)|^2 + |\psi(\tau)|^2|\phi(0)|^2 \overset{\text{Bosons!}}{+} \overbrace{2\,\mathrm{Re}(\phi^*(\tau)\psi(0)\psi^*(\tau)\phi(0))}^{\text{Exchange}}}{4(1+|\langle\phi|\psi\rangle|^2)}$$

This is the first example of two-photon interference, first emphasized by the famous atomic physicist Ugo Fano in 1961.

Consider then coincidence counting in the HBT experiment at $\tau = 0$. We count photons in the same spatial & temporal mode, so I will drop the mode label. In Heisenb. pict.

$$\langle:\hat{I}(0)\,\hat{I}(0):\rangle = G^{(2)}(0) \propto \langle\psi_{in}|\hat{a}^\dagger_{out}\hat{b}^\dagger_{out}\,\hat{a}_{out}\,\hat{b}_{out}|\psi_{in}\rangle \propto \langle\psi_{in}|\,\hat{a}^\dagger_{in}\hat{a}^\dagger_{in}\hat{a}_{in}\hat{a}_{in}|\psi_{in}\rangle$$

$$= \sum_n |\langle n|\hat{a}\hat{a}|\psi\rangle|^2 \approx \sum_{n=2}^{\infty}(n+1)(n+2)P_{n+2}$$ ← probability of n+2 photons in mode

Two paths ('Bose enhancement')

For short counting time, with $\langle\hat{n}\rangle \ll 1$, $G^{(2)}(0) \approx 2p_2$: probability of 2-photons in mode.

<u>Thermal state</u>: $P_n = \frac{\langle\hat{n}\rangle^n}{(1+\langle\hat{n}\rangle)^{n+1}} \Rightarrow P_2 = \frac{1}{(1+\langle\hat{n}\rangle)^3}\langle\hat{n}\rangle^2 \approx \langle\hat{n}\rangle^2$

<u>Coherent State</u>: $P_n = \frac{1}{n!}\langle\hat{n}\rangle^n e^{-\langle\hat{n}\rangle} \Rightarrow P_2 = \frac{1}{2}\langle\hat{n}\rangle^2 e^{-\langle\hat{n}\rangle} \approx \frac{1}{2}\langle\hat{n}\rangle^2$

The probability of finding two photons in a mode of a thermal state is twice that of the coherent state for the same mean photon number $\langle\hat{n}\rangle$. The photons are thus twice as likely to arrive at the same time — the photons in the thermal state are bunched.

This bunching can be interpreted in terms of the differing photon statistics of these states. The deviation of the coincident counts from "accidental" product of uncorrelated counts

$$\langle \vdots \Delta \hat{I}^2 \vdots \rangle = \langle : \hat{I}(0)\hat{I}(0) : \rangle - \langle : \hat{I}(0) : \rangle \langle : \hat{I}(0) : \rangle = \langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2$$

$$= \langle \hat{a}^\dagger \hat{a}\, \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle - \langle \hat{a}^\dagger \hat{a} \rangle^2 = \langle \hat{n}^2 \rangle - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle = \Delta n^2 - \langle \hat{n} \rangle$$

For a thermal state: $\Delta n^2 = \langle \hat{n} \rangle^2 + \langle \hat{n} \rangle$ (Bose-Einstein dist.) $\Rightarrow \langle \Delta \hat{I}^2 \rangle = \langle \hat{n}^2 \rangle$

For a coherent state: $\Delta n^2 = \langle \hat{n} \rangle$ (Poisson) $\Rightarrow \langle \Delta \hat{I}^2 \rangle = 0$

Classical vs. Nonclassical Light

The photon statistics of the field is one way for us to distinguish "classical light" vs. "non classical light. We have seen that a classical deterministic current leads to the creation of a coherent state, the quasiclassical state. A classically noisy current leads to a statical mixture of coherent states with a positive $P(\{\alpha_k\})$ - representation.

> Thus, we can define classical light as those with state $\hat{\rho} = \int d\{\alpha_k\}\, P(\{\alpha_k\})\, |\{\alpha_k\}\rangle\langle\{\alpha_k\}|$ where $P(\{\alpha_k\}) \geq 0$. Otherwise, we call the light "nonclassical"

Poisson Statistics

As we have seen, measuring the intensity fluctuations in a mode, equivalent to measuring the difference between photon correlations and the product of uncorrelated "accidentals" at a given time is:

$$\langle \vdots \Delta \hat{I}^2 \vdots \rangle = \langle : \hat{I}(0)^2 : \rangle - \langle : \hat{I}(0) : \rangle \langle : \hat{I}(0) : \rangle = \Delta n^2 - \langle \hat{n} \rangle \; : \text{(Deviation from Poisson)}$$

For classical light: $\langle : \hat{I}(0)^2 : \rangle = \langle a^\dagger a^\dagger a a \rangle = \mathrm{Tr}\left(\hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} \int d^2\alpha\, P(\alpha)\, |\alpha\rangle\langle\alpha| \right) = \int d^2\alpha\, P(\alpha) \langle \alpha | \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a} | \alpha \rangle$

$$= \int d^2\alpha\, |\alpha|^4\, P(\alpha) = \overline{|\alpha|^4}$$

$$\langle : \hat{I}(0) : \rangle = \langle \hat{a}^\dagger \hat{a} \rangle = \int d^2\alpha\, |\alpha|^2\, P(\alpha) = \overline{|\alpha|^2}$$

$$\langle \vdots \Delta \hat{I}^2 \vdots \rangle = \overline{|\alpha|^4} - \left(\overline{|\alpha|^2}\right)^2 = \left(\Delta |\alpha|^2\right)^2 \geq 0$$

> - For classical light $\Delta n^2 \geq \langle \hat{n} \rangle$ (Super-Poissonian)
> - Sub-Poissonian number statistics $\Delta n^2 < \langle \hat{n} \rangle$ is a signature of non-classical light.

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$$= \int d^2\alpha\, |\alpha|^4\, P(\alpha) = \overline{|\alpha|^4}$$

$$\langle :\hat{I}(0): \rangle = \langle \hat{a}^\dagger\hat{a} \rangle = \int d^2\alpha\, |\alpha|^2\, P(\alpha) = \overline{|\alpha|^2}$$

$$\langle :\Delta\hat{I}^2: \rangle = \overline{|\alpha|^4} - \left(\overline{|\alpha|^2}\right)^2 = \left(\Delta|\alpha|^2\right)^2 \geq 0$$

$$\Rightarrow \Delta n^2 = \underbrace{\langle \hat{n} \rangle}_{\text{Particle uncertainty}} + \underbrace{\left(\Delta|\alpha|^2\right)^2}_{\text{Wave uncertainty}} \geq \langle \hat{n} \rangle \quad : \text{Super-Poissonian Number Fluctuations}$$

- For classical light $\Delta n^2 \geq \langle \hat{n} \rangle$ (Super-Poissonian)
- Sub-Poissonian number statistics $\Delta n^2 < \langle \hat{n} \rangle$ is a signature of non-classical light.

Photon antibunching

Consider the two-time intensity correlation function

$$G^{(2)}(\tau) = \langle :\hat{I}(0)\hat{I}(\tau): \rangle = \int d[\mathcal{E}]\, P[\mathcal{E}]\, |\mathcal{E}(0)|^2 |\mathcal{E}(\tau)|^2 = \left(|\mathcal{E}(0)|^2 \big| |\mathcal{E}(\tau)|^2\right) \quad \text{(inner product of function)}$$

Cauchy-Schwartz inequality: $\left(|\mathcal{E}(0)|^2 \big| |\mathcal{E}(\tau)|^2\right) \leq \sqrt{\left(|\mathcal{E}(0)|^2 \big| |\mathcal{E}(0)|^2\right)} \sqrt{\left(|\mathcal{E}(\tau)|^2 \big| |\mathcal{E}(\tau)|^2\right)} = \sqrt{\langle :\hat{I}^2(0): \rangle} \sqrt{\langle :\hat{I}^2(\tau): \rangle} = \underbrace{\langle :\hat{I}^2(0): \rangle}_{\text{stationary statistics}}$

Photon antibunching

Consider the two-time intensity correlation function

$$G^{(2)}(\tau) = \langle : \hat{I}(0)\hat{I}(\tau) : \rangle = \int d[\mathcal{E}]\; P[\mathcal{E}]\; |\mathcal{E}(0)|^2 \, |\mathcal{E}(\tau)|^2 = \left(|\mathcal{E}(0)|^2 \,\middle|\, |\mathcal{E}(\tau)|^2 \right)$$ (inner product of function)

Cauchy-Schwartz inequality: $\left(|\mathcal{E}(0)|^2 \,\middle|\, |\mathcal{E}(\tau)|^2 \right) \leq \sqrt{\left(|\mathcal{E}(0)|^2 \,\middle|\, |\mathcal{E}(0)|^2 \right)} \sqrt{\left(|\mathcal{E}(\tau)|^2 \,\middle|\, |\mathcal{E}(\tau)|^2 \right)} = \sqrt{\langle : \hat{I}^2(0) : \rangle} \sqrt{\langle : \hat{I}^2(\tau) : \rangle} = \langle : \hat{I}^2(0) : \rangle$

stationary statistics

$$\Rightarrow \langle : \hat{I}(0)\hat{I}(\tau) : \rangle = G^{(2)}(\tau) \leq \langle : \hat{I}^2(0) : \rangle = G^{(2)}(0)$$

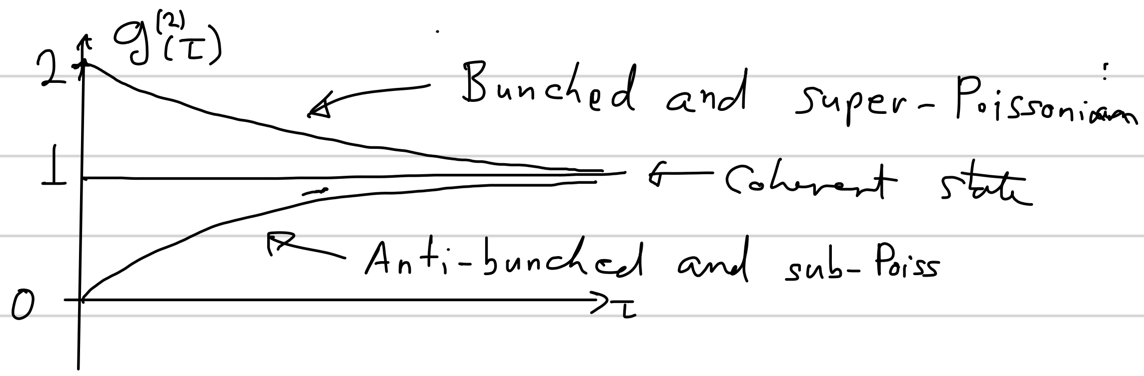
Normalized: $g^{(2)}(0) \geq g^{(2)}(\tau)$: Photon Bunching (classical light)

Nonclassical light $g^{(2)}(0) \leq g^{(2)}(\tau)$:

$\Rightarrow$ Rate of coincidence at zero delay $\leq$ Rate of coincidence at finite delay.

Photons are more likely to arrive separately than together

Nonclassical light $\Rightarrow$ Photon Antibunching



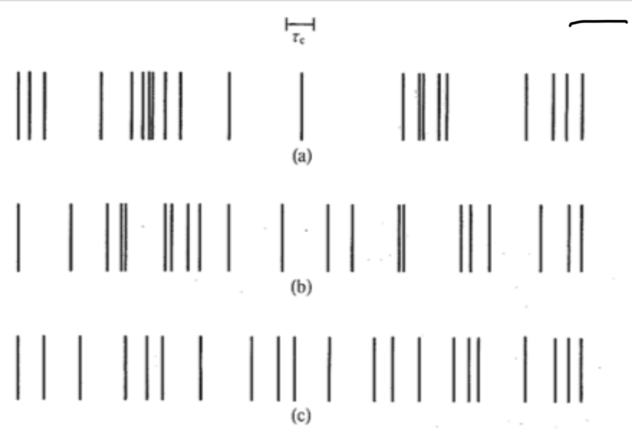


FIG. 6.4. Schematic representations of photon counts as functions of the time for light beams that are (a) bunched with $g^{(2)}(0) > 1$, (b) random with $g^{(2)}(0) = 1$, and (c) antibunched with $g^{(2)}(0) < 1$.

Taken from Loundon "The Quantum theory of light"

The observation of photon -bunching and sub-Poissonian photon statistics was a major milestone in the study of nonclassical light.

Further thoughts about photons statistics

We have defined "classical light" as field generated by "classical currents," i.e. c-number current $\vec{J}(\vec{x},t)$ with no quantum fluctuations, or back action from the quantum field. If we generalize to all $\vec{J}(\vec{x},t)$ to a classical random variable, then this includes classical statistical optics, where $\tilde{\vec{E}}(\vec{x},t)$ is a random field. In the quantum decription, the quantum state is a statistical mixture of coherent states,

$$\hat{\rho} = \int d^2\{\alpha\} \; P(\{\alpha\}) \; |\{\alpha\}\rangle\langle\{\alpha\}|$$

Where $P(\{\alpha\})$ is a classical probability distribution (positive Glauber P-representation).

Consider a single mode, $\hat{\rho} = \int d^2\alpha \; P(\alpha) \; |\alpha\rangle\langle\alpha|$.

- For a thermal state, $P(\alpha) = \frac{1}{\pi\langle\hat{n}\rangle} e^{\frac{-|\alpha|^2}{\langle\hat{n}\rangle}}$ where $\langle\hat{n}\rangle = \frac{1}{e^{\frac{\hbar\omega}{kT}}-1}$ is the average number of photon at temperature T. Note as $\langle\hat{n}\rangle \to 0$ $P(\alpha) \to \delta^{(2)}(\alpha)$, the vacum state, which is a delta-function at the origin
- For a general coherent state $\hat{\rho} = |\alpha_0\rangle\langle\alpha_0|$, $P(\alpha) = \delta^{(2)}(\alpha-\alpha_0)$, Gaussian at α_0 of zero width (delta function)

These Gaussian fluctuations in the field amplitude, plus the fundamental fluctuations of the "vacuum noise" translate in different photon statistics.

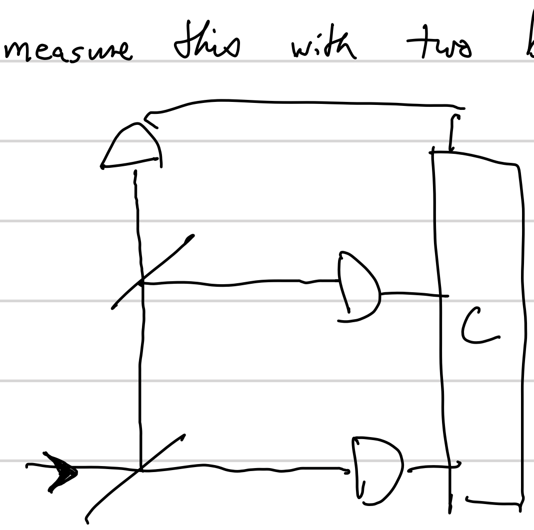
Consider the n-photon coincidence correlation function

$$G^{(n)}(0) = \langle :\hat{n}^n: \rangle = \langle \hat{a}^{\dagger n} \hat{a}^n \rangle$$

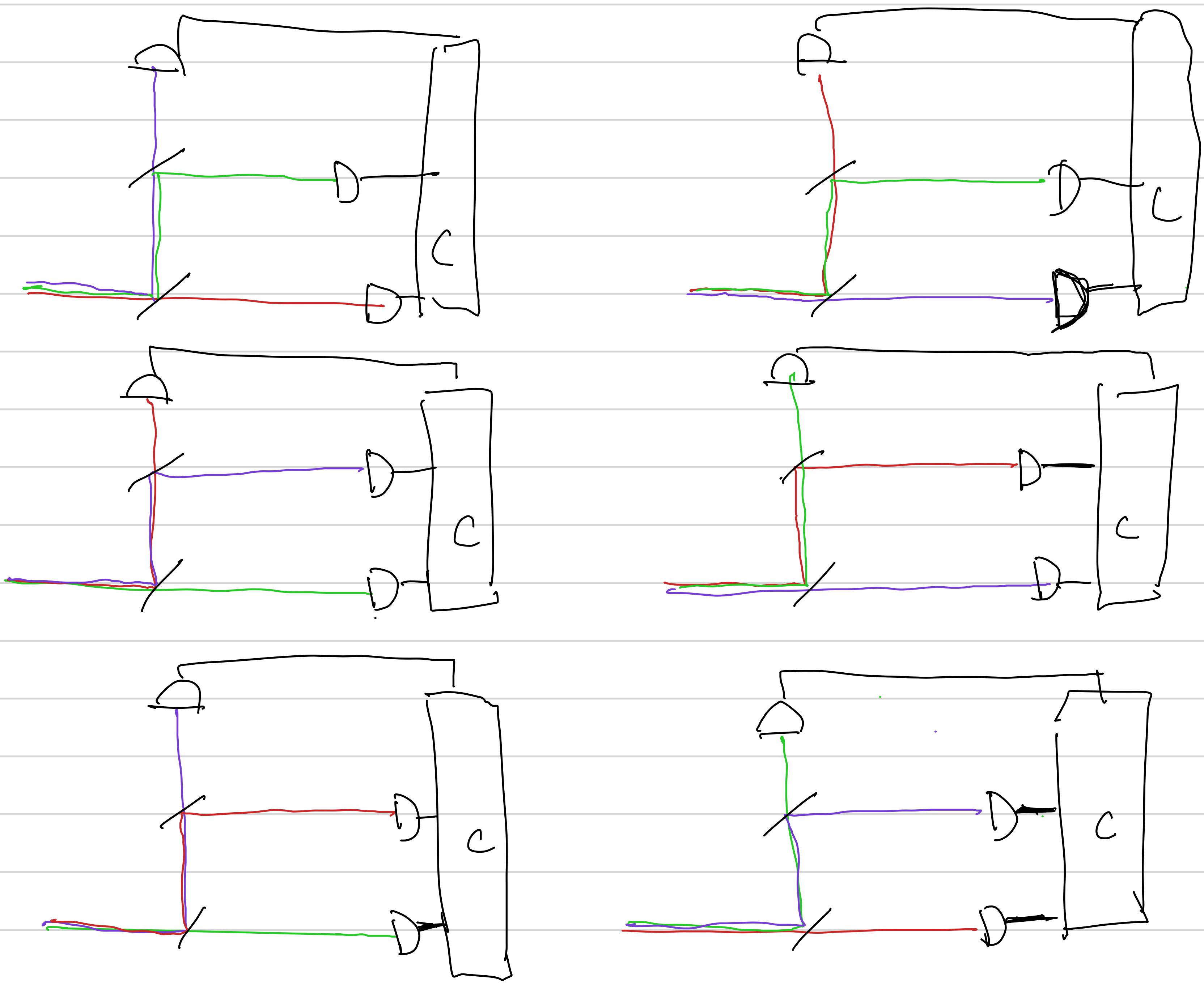
For an exactly n-photon Fock State $|n\rangle$, $G^{(n)}(0) = \langle n|\hat{a}^{\dagger n}\hat{a}^n|n\rangle = n!$. This represents the n indistinguishable permutations of the n bosons. This can be seen in the n! different "histories" that lead to coincidence counts. For example, consider $G^{(3)}(0) = \langle \hat{a}^{\dagger 3} \hat{a}^3 \rangle$

(Next Page)

We might measure this with two beam splitters



There are $3! = 6$ different indistinguishable paths leading to coincidence



The different colors are use to indicate different potential paths, but here are photons are the same "color;" they are indistinguishable.

Now, for a general state $G^{(n)}(0) = \langle \hat{a}^{\dagger n} \hat{a}^n \rangle = \sum P_m \langle m | \hat{a}^{\dagger n} \hat{a}^n | m \rangle$

If $\langle n \rangle << 1$, the dominant term is $G^{(n)}(0) = n! \, P_n$ (lowest non vanishing)

For the thermal state, $P_n = \dfrac{\langle \hat{n} \rangle^n}{(\langle \hat{n} \rangle + 1)^{n+1}} \simeq \langle \hat{n} \rangle^n$

$\Rightarrow G^{(n)}(0) \simeq n! \langle \hat{n} \rangle^n$ (this would be true in general for arbitrary $\langle \hat{n} \rangle$)

That is, for single mode $\langle : \hat{I}^n : \rangle = \langle : \hat{n}^n : \rangle = \langle \hat{a}^{\dagger n} \hat{a}^n \rangle = n! \langle \hat{n} \rangle^n$

This is exactly what we expected from the classical statistical theory of random waves

$$\langle I^n \rangle = n! \langle I \rangle^n \quad \text{for "chaotic light"}$$

In contrast, for a coherent state $P_n = e^{-\langle n \rangle} \dfrac{\langle n \rangle^n}{n!} = \dfrac{\langle n \rangle^n}{n!}$. The denominator $n!$ is essential. Then $G^{(n)}(0) = \langle : \hat{I}^n : \rangle = \langle : \hat{n}^n : \rangle = \langle \hat{n} \rangle^n$

The thermal state is "bunched". The excess fluctuations in intensity can be understood as the Bose thermal state's tendency to have more bosons in the same mode that for distinguishable statistics. The coherent state, with its Poisson statistics counters the Bose-Einstein statistics – it is not an equilibrium state of Bosons. The perfect coherence of a single amplitude $|\alpha\rangle$ corresponds to Poisson photon statistics, with no correlations, and no bunching.