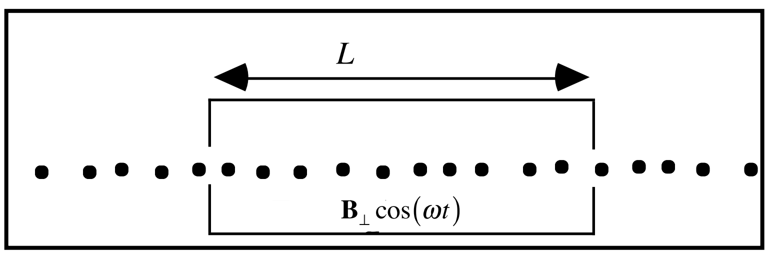
Physics 566: Quantum Optics I
Problem Set 3

Due: Wednesday Sept. 15, 2021

Problem 1: Magnetic Resonance: Rabi vs. Ramsey (25 Points)

The technique of measuring transition frequencies with magnetic resonance was pioneered by I. I. Rabi in the late 30's. It was modified by Ramsey (his student) about 10 years later, and now serves as the basis for atomic clocks and the SI definition of the second. All precision atomic measurements, including modern atom-interferometers and quantum logic gates in atomic systems, have at their heart a Rabi/Ramsey-type geometry.

(i) Rabi resonance geometry. Consider a beam of two-level "spins" with energy splitting $\hbar\omega_0$ passing through an "interaction zone" of length *L*, in which they interact with a monochromatic field oscillating at frequency ω that drives transitions between $|\downarrow\rangle$ and $|\uparrow\rangle$.

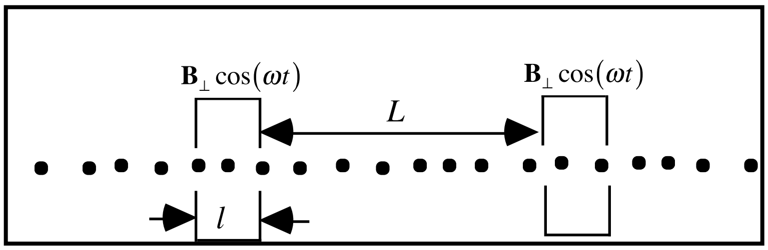


(a) Suppose all the spins start in the state $|\downarrow\rangle$, and have a well-defined velocity *v*, chosen such that $\Omega L / v = \pi$, where Ω is the bare Rabi frequency. Plot the probability to be in the excited state $|\uparrow\rangle$, $P_\uparrow$ as a function of driving frequency ω. What is the characteristic "linewidth" of the curve $P_\uparrow$, i.e., some characteristic frequency at which $P_\uparrow$ falls off substantially? Explain your estimate in terms of the "time energy uncertainty."

(b) Now suppose the spins have a distribution of velocities characteristic of thermal beams: $f(v) = \frac{2}{v_0^4} v^3 \exp(-v^2 / v_0^2)$, where $v_0 = \sqrt{2k_B T / m}$. Plot $P_\uparrow$ vs. L for Δ=0, (you may need to do this numerically). At what *L* is it maximized - explain? Also plot as in (a), $P_\uparrow$ as a function of ω with $L = L_{\max}$. What is the linewidth? Explain in terms of the Bloch-sphere.

(ii) Ramsey separated zone method

As you have seen in parts (a)-(b), assuming one can make the velocity spread sufficiently small, the resonance linewidth is limited by the interaction time *L*/*v*. This is known as "transit-time broadening" and is a statement of the time-energy uncertainty principle. Unfortunately, if we make L larger and larger other inhomogeneities, such as the amplitude of the driving field come into play. Ramsey's insight was that one can in fact "break up" the π-pulse given to the atoms into two π/2-pulses in a time $\tau = l/v$ (i.e. $\Omega\tau = \pi/2$), separated by *no interaction* for a time $T=L/v$. The free interaction time can then made *much* longer.



(c) Given a mono-energetic spins with velocity v, internal state $|\psi(0)\rangle = |\downarrow_z\rangle$, and field at a detuning $|\Delta| << \Omega$ so that $\Omega_{tot} = \sqrt{\Omega^2 + \Delta^2} \approx \Omega$ find:

$$|\psi(\tau = l/v)\rangle,\ |\psi(\tau + T = (l+L)/v)\rangle, |\psi(2\tau + T = (2l+L)/v)\rangle$$

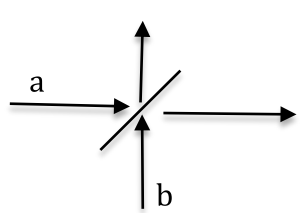
and show that mapping of the state on the Bloch-sphere.

(d) Plot $P_\uparrow(t_{final} = 2\tau + T)$ as a function of ω. Plot also for the case of finite spread in velocity as in part (b). What is the linewidth?

Problem 2: SU(2) Interferometers (25 points)

There is a formal equivalence between a Mach-Zender-type optical interferometer and a so-called Ramsey interferometer for any two-level quantum system, which gets its name from the Ramsey separated zone method of Problem 1. We also call this an SU(2) interferometer.

(a) Consider the following optical transformation: A symmetric beam splitter with transmission amplitude t and reflection amplitude r.



We can encode a qubit in the two orthogonal paths, "a" and "b", of a photon. We then define the standard basis

$$\left|\uparrow_z\right\rangle = \left|1_a, 0_b\right\rangle, \quad \left|\downarrow_z\right\rangle = \left|0_a, 1_b\right\rangle$$

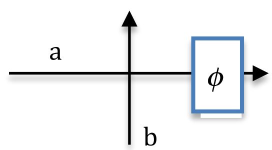
i.e., $\left|\uparrow_z\right\rangle$ is with one photon in path-a an no photons in path-b, and vice versa for $\left|\downarrow_z\right\rangle$. The transformation on the basis states is

$$\left|1_a, 0_b\right\rangle \Rightarrow t\left|1_a, 0_b\right\rangle + r\left|0_a, 1_b\right\rangle, \quad \left|0_a, 1_b\right\rangle \Rightarrow t\left|0_a, 1_b\right\rangle + r\left|1_a, 0_b\right\rangle$$

Show that the conditions for this map to be unitary are: $|t|^2 + |r|^2 = 1$, $tr^* + t^*r = 0$.

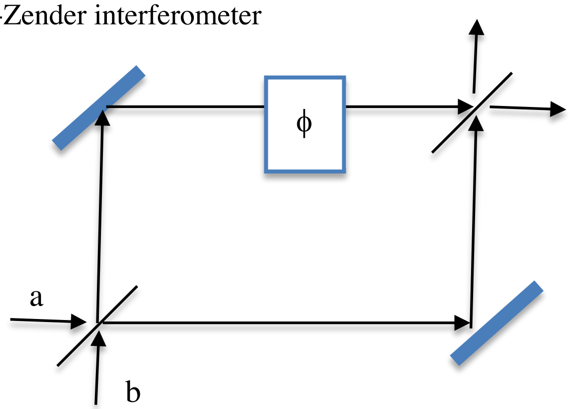
Write this map as an equivalent (up to a negligible phase) SU(2) rotation on the Bloch sphere.

(b) Show that the transformation in which mode-*a* gets a phase shift relative to mode-*b* is an SU(2) rotation. What is the axis and angle of the rotation?



(c) Show how to construct an arbitrary SU(2) rotation with phase shifters and a beam splitter.

Now consider a Mach-Zender interferometer



The 50-50 beam splitters are thin black lines and the mirrors are thick blue lines. We assume here that the optical path lengths of the two arms of the interferometer are equal. A phase shifter ϕ is placed in the upper arm.

(d) Show that up to a negligible overall phase, the sequence

beam-splitter → mirrors → phase shift →beam-splitter

is equivalent to the sequence of SU(2) transformations

$\pi/2$ -x-rotation → π -x-rotation → ϕ -z-rotation → $\pi/2$ -x-rotation

(e) Given the initial state $|\downarrow_z\rangle$, sketch the evolution on the Bloch sphere corresponding to this sequence.

(f) Show that up to an overall phase (which is negligible), the transformation from the input is equivalent to the rotation $e^{-i\frac{\phi}{2}\sigma_y}$. Using this, calculate the probability to find the state $|1_a, 0_b\rangle$ at the output port given that the state was in $|1_a, 0_b\rangle$ at the input port.

Problem 3: Inhomogeneous broadening (25 Points)

Many important phenomena occur in a time short compared to the "relaxation" time for coherent phenomena (e.g. spontaneous emission lifetime). Historically, these are known as coherent transients. Here are some examples.

(a) Free induction decay by inhomogeneous broadening: Consider a macroscopic ensemble of spins in a static magnetic field with a spatially inhomogeneous magnitude. The cloud is extended so that spins see an inhomogeneous distribution of B-fields in z-direction, with probability

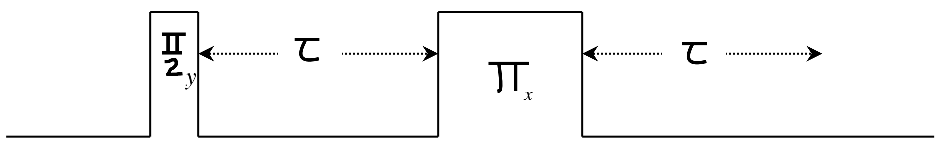
$$P(B) = \frac{1}{\sqrt{2\pi(\delta B)^2}} e^{-\frac{(B-B_0)^2}{2(\delta B)^2}}$$

If the spins start in a coherent superposition of $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$, the sample will radiate magnetic dipole radiation at a mean frequency $\Omega_0 = \gamma B_0$ (γ being the gyromagnetic ratio), but the signal will decay much, much faster than the radiative decay rate. The decay arises because of the inhomogeneity -- different local spins will oscillate at different local frequencies. The resulting radiation from the different components eventually getting out of phase and destructively interfering.

(i) Calculate the characteristic decay time, known as T_2^*, due to inhomogeneity. Take $\gamma B_0 / 2\pi = 1$ MHz, $\gamma(\delta B)/2\pi = 10$ kHz. The characteristic width $1/T_2^*$ is known as the inhomogeneous linewidth.

(ii) If the spins all start in $|\uparrow_z\rangle$, qualitatively describe how to achieve an approximate π/2-pulse to rotate all spins into the *x*-*y* plane (i.e. into a superposition state).

(b) Spin echo: Though inhomogeneous broadening will cause a decay of the ensemble averaged coherence, it is not a truly irreversible process. A procedure for recovering the coherence is known as a "spin echo". Consider the following pulse sequence.



The $\pi/2$-pulse about the *y*-axis acts according to (aiii) to bring all spins onto the *x*-axis of the Bloch sphere. For a time τ, the spins dephase. The π-pulse about the *x*-axis acts to time reverse the process. An "echo" signal will be seen at a time τ later.

Explain this process using this Bloch sphere. **Sketch** the signal one would detect of the radiated fields in rf coils.

(c) A spin-echo sequence is often used in a two-level Ramsey interferometer to make it more robust to inhomogeneities. Consider the following sequence

$\pi / 2$ *y*-rotation →Free evolution for time T/2 → π *x*-rotation → Free evolution for time T/2 → $\pi / 2$ *y*-rotation

Explain the trajectory of the Bloch sphere. To what kind of inhomogeneities is this sequence this robust? This is a *direct* analogy to a Mach-Zender interferometer studied in problem 2. Given what you said about the Ramsey interferometer, to what kinds of inhomogeneity is the Mach-Zender interferometer robust?