Physics 566: Quantum Optics I
Problem Set 4
Due Thursday, September 23, 2021

Problem 1: Adiabatic rapid passage (15 Points)

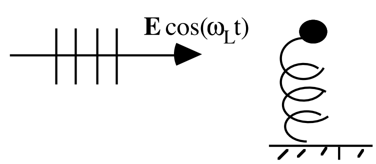
Suppose we have an *inhomogeneously* broadened system, e.g., a system of two-level oscillators with a distribution of resonance energies such as a thermal gas (Doppler broadening) or with a distribution is a solid crystal due to local strain effects. How can we apply a π-pulse to send all of atoms to the excited state with high probability?

(a) Qualitatively suppose we apply a radiation field with a frequency well below resonance ($\Delta < 0$, $|\Delta| >> \Omega$) and sweep the field slowly up through resonance, ending well above resonance ($\Delta >> \Omega$), on a time scale much slower that the Rabi frequency $T >> \Omega^{-1}$, but fast compared to and decay $T << \Gamma^{-1}$. Use the Bloch-sphere magnetic resonance picture to show that population in the ground state will "adiabatically" be transferred to the excited state.

(b) Sketch the eigenvalues of the two-level atom in the RWA as a function of the frequency of the laser. Use the adiabatic theorem of quantum mechanics to explain the transfer of population from the ground to excited state, and the constraints on the time scales. What is the condition that we can invert the entire inhomogeneously broadened sample?

Problem 2: Lorentz oscillator model of scattering (20 points)

Consider the scattering of an electromagnetic wave by a damped Lorentz oscillator



(a) The absorption cross section, σ_{abs}, is defined as the rate at which energy is absorbed by an atom, divided by the incident flux of energy, the intensity $I = \frac{c}{8\pi}|\mathbf{E}_0|^2$ (CGS units). Show that the classical model of absorption gives,

$$\sigma_{abs,class} = \frac{2\pi^2 e^2}{mc} g(\omega_L), \text{ where } g(\omega) = \frac{\Gamma_{rad}/(2\pi)}{(\omega - \omega_0)^2 + \Gamma_{rad}^2/4} \text{ is the line shape.}$$

Assume near resonance so that $\Delta = \omega_L - \omega_0 << \omega_L, \omega_0$.

(b) In the case of radiative damping, all energy absorbed is re-radiated, and is thus *scattered*. Use standard scattering theory to derive the differential scattering cross section for the Lorentz oscillator model, $\frac{d\sigma_{scat}}{d\Omega}$, and after integrating over all solid angles, show that the total scattering cross section equals the absorption cross section found in part (a). Here take $\Gamma_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0^2$.

(c) We can re-derive the expression for the classical natural linewidth Γ_{rad} that we found in class via radiation reaction by looking directly at energy conservation in the scattering process. For the field on resonance, equate the time averaged absorbed power (rate at which field does work on electron, averaged over a period of oscillation) to the Larmor formula for the averaged radiated power to show,

$$\Gamma_{rad} = \frac{2}{3}\frac{e^2}{mc^3}\omega_0{}^2 = \frac{2}{3}\left(k_0 r_c\right)\omega_0,$$ where r_c is the classical electron radius.

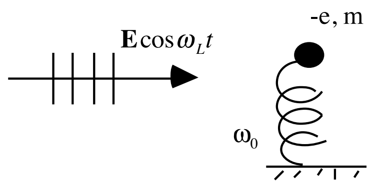
Evaluate this for the case of the sodium "D2 resonance" (the yellow light in stree light), of excitation wavelength is 589 nm. The quantum decay rate is $\Gamma / 2\pi = 9.8$ MHz. What is the oscillator strength of the transition?

(d) Show that the scattering cross section can be rëexpressed as $\sigma_{scat} = \frac{6\pi\lambda\!\!\!^{-}{}_0^2}{1+\left(4\Delta^2/\Gamma_{rad}^2\right)}$, where $\lambda\!\!\!^{-}{}_0 = \lambda / 2\pi$. This expression holds true quantum mechanically as well with $\Gamma_{rad} \to \Gamma$.

Problem 3: The ac-Stark effect (25 points)

Suppose an atom is perturbed by a monochromatic electric field oscillating at frequency ω_L $\mathbf{E}(t) = E_z \cos(\omega_L t)\mathbf{e}_z$ (such as from a linearly polarized laser), rather than the dc-field studied in class. We know that such field can be absorbed and cause transitions between the energy levels; we will systematically study this effect later in the semester. The laser will also cause a *shift* of energy levels of the unperturbed states, known alternatively as the "ac-Stark shift", the "light shift", and sometimes the "Lamp shift" (don't you love physics humor). In this problem, we will look at this phenomenon is the simplest case that the field is near to resonance between the ground state $|g\rangle$ and some excited state $|e\rangle$, $\omega_L \approx \omega_{eg} \equiv \left(E_e - E_g\right)/\hbar$, so that we can ignore all other energy levels in the problem (the "two-level atom" approximation).

(i) <u>The classical picture</u>. Consider first the "Lorentz oscillator" model of the atom – a charge on a spring – with natural resonance ω_0.



The Hamiltonian for the system is $H = \frac{p^2}{2m} + \frac{1}{2}m\omega_0^2 z^2 - \mathbf{d}\cdot\mathbf{E}(t)$, where $d = -ez$ is the dipole.

(a) Ignoring damping of the oscillator, use Newton's Law to show that the induced dipole moment is

$$\mathbf{d}_{induced}(t) = \alpha\mathbf{E}(t) = \alpha E_z \cos(\omega_L t),$$

where $\alpha = \frac{e^2/m}{\omega_0^2 - \omega_L^2} \approx \frac{-e^2}{2m\omega_0\Delta}$ is the polarizability with $\Delta \equiv \omega_L - \omega_0$ the "detuning".

Note: classically, the shift in the energy is

$$\Delta H = -\frac{1}{2}d_{ind}(t)E(t) = -\frac{1}{2}\alpha E^2(t)\text{, or a time average value } \overline{H} = -\frac{1}{4}\alpha E_z^2$$

The factor of ½ arises because the dipole is *induced.*

(ii) Quantum picture. We consider the two-level atom described above. As we will derive later, the Hamiltonian for this system can be written in a time independent form (equivalent to the time-averaging done in the classical case)

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{\text{int}},$$

where $\hat{H}_{atom} = -\hbar\Delta|e\rangle\langle e|$ is the "unperturbed" atomic Hamiltonian, and

$\hat{H}_{\text{int}} = \frac{\hbar\Omega}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$ is the dipole-interaction with $\hbar\Omega \equiv \langle e|\mathbf{d}|g\rangle\cdot\mathbf{E}$ (the Rabi frequency).

(b) Find the *exact* energy eigenvalues and eigenvectors for this simple two-dimensional Hilbert space and plot the levels as a function of Δ. These are known as the atomic "dressed states."

(c) Expand your solution in (b) to lowest nonvanishing order in Ω to find the perturbation to the energy levels. Under what condition is this expansion valid?

(d) Confirm your answer to (c) using perturbation theory. Find also the mean induced dipole moment (to lowest order in perturbation theory), and from this show that the atomic polarizability, defined by $\langle \mathbf{d} \rangle = \alpha \mathbf{E}$ is $\alpha = \frac{-|\langle e|\mathbf{d}|g\rangle|^2}{\hbar\Delta}$, so that the second order perturbation to the ground state is $E_g^{(2)} = -\frac{1}{4}\alpha E_z^2$ as in part (a).

(e) Show that the ratio of the polarizability calculated classical in (a) and the quantum expression in (d) has the form

$$f \equiv \frac{\alpha_{quantum}}{\alpha_{classical}} = \frac{|\langle e|z|g\rangle|^2}{\left(\Delta z^2\right)_{SHO}}, \text{ where } \left(\Delta z^2\right)_{SHO} \text{ the SHO zero point variance.}$$

This ratio is known as the oscillator strength.

Lessons:

- In lowest order perturbation theory an atomic resonance look just like a harmonic oscillator, with a correction factor given by the oscillator strength.
- Harmonic perturbations cause energy level shifts as well as absorption and emission.