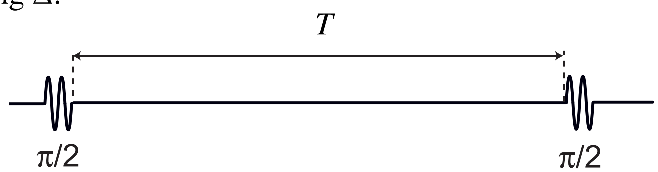
Physics 566: Quantum Optics I
Problem Set 5

Due: Thursday September 30, 2021

Problem 1: Ramsey fringes and the measurement of T2 times (25 Points)

(a) We seek to measure the coherence of between the computational basis states of a qubit $\{|0\rangle,|1\rangle\}$. Consider a two-pulse Ramsey sequence: A "hard" $\pi/2$ pulse around x-axis with detuning Δ, free evolution for a time T, a second hard $\pi/2$ pulse around x-axis at the same detuning Δ.



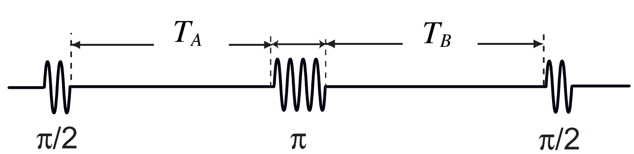
During the free evolution, the coherence ρ_{01} decays exponentially with rate $1/T_2$. Show that, given the qubit initially in $|1\rangle$, the probability to find $|0\rangle$ after the sequence is

$$P_0 = \frac{1}{2}\left[1+\cos(\Delta T)e^{-T/T_2}\right]$$

Explain this using the evolution on the Bloch sphere. Plot this for $\Delta/2\pi = 1$ MHz and T_2=25 μs, for T=0 to 25 μs.

(b) Suppose now that in addition to homogeneous decay, there is inhomogeneous decay T_2^*. Suppose that if the pulses are tuned to frequency ω, the probability the detuning seen by the qubit is Gaussian distributed, $p(\Delta) = e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}/\sqrt{2\pi\delta^2}$, where Δ_0 is the mean detuning and $\delta = 1/T^*_2$ is the spread in detunings. Calculate the probability P_0 in the same two-pulse Ramsey sequence of part (a) for $T_2^* = 5$ μs. Comment on the result.

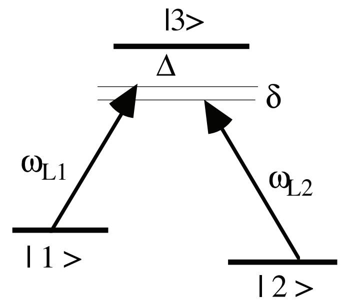
(c) Now consider a three-pulse Hahn spin-echo Ramsey sequence:
A "hard" π/2 pulse around y-axis with detuning Δ, free evolution for a time T_A a "time reverse" hard π pulse around x, free evolution for a time T_B, and then a second hard $\pi/2$ pulse around y-axis at the same detuning Δ.



Show, $P_0 = \frac{1}{2}\left(1-\cos\left[\Delta_0(T_A-T_B)\right]e^{-\delta^2(T_A-T_B)^2/2}e^{-(T_A+T_B)/T_2}\right)$, and plot for $T_A = 10\mu s$, as a function of T_B=0 to 25 μs.

Problem 2: Λ–Transitions and the master equation (25 Points)

Consider a three-level atom in the so-called "lambda" configuration (because it looks like the Greek letter Λ):



Levels $|1\rangle$ and $|2\rangle$ are connected to level $|3\rangle$ on two dipole-allowed transitions driven by lasers at frequencies ω_{L1} and ω_{L2} respectively. Laser-1 is detuned from resonance by Δ. Difference between the detunings of lasers 1 and 2 is $\delta = (\omega_{L1} - \omega_{L2}) - (E_2 - E_1)/\hbar$.

(a) The Hamiltonian for this system (in the RWA) is $H = H_A + H_{AL}$ where

$$H_A = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + E_3|3\rangle\langle 3|\,,$$

$$H_{AL} = \frac{\hbar\Omega_1}{2}\left(e^{-i\omega_{L1}t}|3\rangle\langle 1| + e^{i\omega_{L1}t}|1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(e^{-i\omega_{L2}t}|3\rangle\langle 2| + e^{i\omega_{L2}t}|2\rangle\langle 3|\right).$$

where $\Omega_{1,2}$ are the two Rabi frequencies. Because there are two laser frequencies, the usual unitary transformation to the frame rotating at ω_L does not apply. However, one can perform a unitary transformation that makes *H* time independent. Define a "rotating frame": $|\psi\rangle_{RF} = U^\dagger|\psi\rangle$, $H_{RF} = U^\dagger H U + i\hbar\frac{\partial U^\dagger}{\partial t}U$, where $U = \sum_{j=1}^{3} e^{-i\lambda_j t}|j\rangle\langle j|$.

Show that for appropriate choice of λ_j we can transform *H* to,

$$\hat{H}_{RF} = -\hbar\delta|2\rangle\langle 2| - \hbar\Delta|3\rangle\langle 3| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right).$$

(b) Suppose that level-3 decays to level-1 at a rate Γ_{31} and level-2 with rate Γ_{32}, and the total decay rate from level-3 is $\Gamma = \Gamma_{31} + \Gamma_{32}$. The effective non-Hermitian Hamiltonian is $\hat{H}_{eff} = \hat{H} - i\hbar\frac{\Gamma}{2}|3\rangle\langle 3|$. The (trace preserving) dynamics of the density operator for the system is described by the master equation,

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{eff}\right) + \mathcal{L}_{feed}[\hat{\rho}]\,, \quad \text{where}$$

$$\mathcal{L}_{feed}[\hat{\rho}] = \Gamma_{31}|1\rangle\langle 3|\hat{\rho}|3\rangle\langle 1| + \Gamma_{32}|2\rangle\langle 3|\hat{\rho}|3\rangle\langle 2|\,.$$

Show that the matrix elements evolve according to:

$$\dot{\rho}_{11} = \Gamma_{31}\rho_{33} - i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right),$$

$$\dot{\rho}_{22} = \Gamma_{32}\rho_{33} - i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right),$$

$$\dot{\rho}_{33} = -\Gamma\rho_{33} + i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right) + i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right),$$

$$\dot{\rho}_{23} = -i\left(\Delta - \delta - i\frac{\Gamma}{2}\right)\rho_{23} - i\frac{\Omega_2}{2}\left(\rho_{33} - \rho_{22}\right) + i\frac{\Omega_1}{2}\rho_{21}\,,$$

$$\dot{\rho}_{13} = -i\left(\Delta - i\frac{\Gamma}{2}\right)\rho_{13} - i\frac{\Omega_1}{2}\left(\rho_{33} - \rho_{11}\right) + i\frac{\Omega_2}{2}\rho_{12}\,,$$

$$\dot{\rho}_{12} = -i\delta\,\rho_{12} + i\frac{\Omega_2}{2}\rho_{13} - i\frac{\Omega_1}{2}\rho_{32}$$

These equations describe the full dynamics, including optical pumping (refeeding) and saturation. They're pretty complicated to solve. Often, we can obtain good approximation and physical insight by using solely non-Hermitian Schrödinger evolution, $\frac{\partial}{\partial t}|\psi\rangle = -\frac{i}{\hbar}\hat{H}_{eff}|\psi\rangle$, as we studied in class. When possible, one should do this.