

Physics 566: Quantum Optics I
Problem Set 7
Due: Thursday, October 28, 2021

Problem 1: Momentum and Angular Momentum in the E&M Field (25 points)

From classical electromagnetic field theory, we know that conservation laws require that the field carry momentum and angular momentum

$$\mathbf{P} = \int d^3x \left(\frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right), \quad \mathbf{J} = \int d^3x \left(\mathbf{x} \times \frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right).$$

(a) Show that when these quantities become field operators, the momentum operator becomes, $\hat{\mathbf{P}} = \sum_{\mathbf{k}, \lambda} \hbar \mathbf{k} \hat{a}_{\mathbf{k}, \lambda}^\dagger \hat{a}_{\mathbf{k}, \lambda}$; interpret.

(b) Show that $\mathbf{J} = \mathbf{J}_{orb} + \mathbf{J}_{spin}$

where $\mathbf{J}_{spin} = \frac{1}{4\pi c} \int d^3x (\mathbf{E}(\mathbf{x}) \times \mathbf{A}(\mathbf{x}))$, $\mathbf{J}_{orb} = \frac{1}{4\pi c} \int d^3x E_i(\mathbf{x}) (\mathbf{x} \times \nabla) A_i(\mathbf{x})$.

(c) Show that

$$\hat{\mathbf{J}}_{spin} = \hbar \sum_{\mathbf{k}} (\hat{a}_{\mathbf{k}, +}^\dagger \hat{a}_{\mathbf{k}, +} - \hat{a}_{\mathbf{k}, -}^\dagger \hat{a}_{\mathbf{k}, -}) \mathbf{e}_{\mathbf{k}}. \quad \text{Interpret this.}$$

Extra credit: Show $\hat{\mathbf{J}}_{orb} = \sum_{\mathbf{k}, \mathbf{k}'} \sum_{\lambda} \hat{a}_{\mathbf{k}', \lambda}^\dagger (i\hbar \nabla_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}') \times \mathbf{k}) \hat{a}_{\mathbf{k}, \lambda}$. Interpret this.

(d) The spin of the photon has magnitude $S=1$, yet there are only two helicity states. Thus, we can map the spin angular momentum onto the Bloch(Poincaré) sphere for $S=1/2$, via

$$\hat{\mathbf{J}}_{spin} = \hat{J}_x \mathbf{e}_x + \hat{J}_y \mathbf{e}_y + \hat{J}_z \mathbf{e}_z,$$

with $J_z = \frac{\hbar}{2} (\hat{a}_{z+}^\dagger \hat{a}_{z+} - \hat{a}_{z-}^\dagger \hat{a}_{z-})$, $J_x = \frac{\hbar}{2} (\hat{a}_{z+}^\dagger \hat{a}_{z-} + \hat{a}_{z-}^\dagger \hat{a}_{z+})$, $J_y = \frac{\hbar}{2i} (\hat{a}_{z+}^\dagger \hat{a}_{z-} - \hat{a}_{z-}^\dagger \hat{a}_{z+})$,

where $(\hat{a}_{z+}, \hat{a}_{z-})$ are the mode operators for positive and negative helicity operators relative to a *space fixed* quantization axis.

(di) Show that these operators satisfy the SU(2) commutation algebra for angular momentum. This relationship is known as the “Schwinger representation” (see Sakurai).

(dii) The mean values of $\hat{J}_x, \hat{J}_y, \hat{J}_z$ are the “Stokes parameters” in classical optics and the Bloch vector components on the Poincaré sphere. Explain the relationship between these operators and the Pauli operators.