Physics 566: Quantum Optics I
Problem Set 9
Due Thursday, November 11, 2021

Problem 1: The Purcell Effect (25 points)

We have studied two extremes of atom-photon interaction: (i) interaction of an atom with a single quantized mode of the electromagnetic field (Jaynes-Cummings model, JCM); (ii) interaction of an atom with all modes in free space. The JCM is exact only for a *perfect cavity*, i.e., a closed quantum system. We have seen this is a good approximation in the in the "strong coupling regime," where the vacuum Rabi frequency, g, is much larger than the rate of decay of the photon out of the cavity κ, and the spontaneous emission rate out of the cavity Γ'.

The precursor to the JCM is the weak coupling regime, where $\kappa >> g$. In that case the photon leaves the cavity well before the atom coherently reabsorbs it. Nevertheless, the presence of the cavity affects the emission of photons. This is known as the *Purcell effect,* first predicted by Ed Purcell in 1946.

We can get at the Purcell effect using Fermi's Golden Rule. Consider a cavity for electromagnetic radiation. The presence of mirrors *modifies the density of states* of electromagnetic modes. Consider a mode of the cavity $\mathbf{u}(\mathbf{r})$ normalized such $\int d^3r\,|\mathbf{u}(\mathbf{r})|^2 = V$, the volume of the mode. The interaction of a two-level atom with the mode is described by the interaction Hamiltonian, $\hat{H}_{\text{int}} = \hbar g \hat{a}_c \hat{\sigma}_+ + \hbar g^* \hat{a}_c^\dagger \hat{\sigma}_-$, where $\hat{a}_c$ is the annihilation operator of a photon in that mode, and $\hbar g = \sqrt{\frac{2\pi\hbar\omega_c}{V}}\,\mathbf{u}(\mathbf{R})\cdot\mathbf{d}_{eg}$, where $\mathbf{R}$ is the position of the atom and $\mathbf{d}_{eg}$ is the dipole matrix element. The cavity mode has a finite lifetime, and thus decay rate κ. Thus, the cavity resonance is not perfectly sharp. We can model this as a Lorentzian, which implies a density of states near the cavity resonance,

$$D_{cav}(\omega) = \frac{\kappa/2\pi}{(\omega-\omega_c)^2 + \kappa^2/4}.$$

(a) Using Fermi's Golden Rule, show that the rate of spontaneous emission into the cavity mode is

$$\Gamma_{cav} = \Gamma_{free} F_P \frac{\kappa^2/4}{(\omega_{eg}-\omega_c)^2 + \kappa^2/4} \frac{\left|\mathbf{u}(\mathbf{R})\cdot\mathbf{d}_{eg}\right|^2}{\left|\mathbf{d}_{eg}\right|^2}$$

where $F_P = \frac{3}{4\pi^2} Q \frac{\lambda_{eg}^3}{V}$ is known as the *Purcell factor*, with $Q = \frac{\omega_c}{\kappa} \gg 1$, the cavity "quality factor".

(b) Show that exactly on resonance and for $\left|\mathbf{u}(\mathbf{R})\cdot\mathbf{d}_{eg}\right|^2 = \left|\mathbf{d}_{eg}\right|^2$, the spontaneous emission into the cavity mode is *enhanced* by the Purcell factor, $\Gamma_{cav} = \Gamma_{free} F_P$ (on resonance), but when the cavity is far-detuned from the atomic resonance, spontaneous emission into the cavity mode is *suppressed*,
$\Gamma_{cav} = \Gamma_{free} \frac{3}{16\pi^2 Q} \frac{\lambda_{eg}^3}{V} \frac{\omega_c^2}{(\omega_{eg}-\omega_c)^2}$ (far-off resonance $\left|\omega_{eg}-\omega_c\right| \gg \kappa$).

Another way to arrive at the Purcell effect is to use the Wigner-Weisskopf approximation, but now with an *effective nonHermitian Hamiltonian* including decay of the atom and cavity mode.

(c) Let $\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma'}{2}\hat{\sigma}_+\hat{\sigma}_- - i\frac{\hbar\kappa}{2}\hat{a}_c^\dagger\hat{a}_c$, where Γ' is the decay rate of the atom into all modes outside the cavity. We again restrict to the subspace $\{|e,0\rangle, |g,1\rangle\}$, though the dynamics is no longer trace preserving. Show that in the interaction picture, the equations motion are

$$\frac{dc_{e,0}}{dt} = -\frac{\Gamma'}{2}c_{e,0} - igc_{g,1}e^{-i(\omega_c-\omega_{eg})t}$$
$$\frac{dc_{g,1}}{dt} = -\frac{\kappa}{2}c_{g,1} - igc_{e,0}e^{+i(\omega_c-\omega_{eg})t}$$

(d) In the Wigner-Weisskopf (Born-Markov) approximation, equivalent to the "weak coupling regime," $g \ll \kappa, \Gamma'$ show that

$$\dot{c}_{e,0} = -\left[\frac{\Gamma'}{2} + \frac{g^2}{i(\omega_{eg}-\omega_c)+\kappa/2}\right]c_{e,0} = -\left[\frac{\Gamma'}{2} + \frac{\Gamma_{cav}}{2} - i\frac{\delta E_{cav}}{\hbar}\right]c_{e,0}$$

where Γ_{cav} is the spontaneous emission rate of photons into the cavity mode. Interpret δE_{cav}.

Problem 2: The relative role of vacuum fluctuations and radiation (25 points) – Extra Credit
As discussed in lecture, spontaneous emission and level shifts can be ascribed to the perturbing effect of vacuum fluctuations and/or radiation reaction. These are actually two sides of the same coin – how we apportion the phenomenon to vacuum fluctuations vs. radiation reaction depends on how we analyze the problem. In this problem we will fill in a few details.

(a) Starting with the fundamental Hamiltonian for a two-level atom coupled to the vacuum in the dipole and rotating wave approximation, find the Heisenberg equations of motion show that they can be written as

$$\frac{d}{dt}\hat{a}_{\mathbf{k}\mu} = -i\omega_k\hat{a}_{\mathbf{k}\mu} - ig^*_{\mathbf{k}\mu}\hat{\sigma}_-$$
$$\frac{d}{dt}\hat{\sigma}_+ = i\omega_{eg}\hat{\sigma}_+ - i\sum_{\mathbf{k}\mu} g^*_{\mathbf{k}\mu}\left[s\,\hat{a}^\dagger_{\mathbf{k}\mu}\hat{\sigma}_z + (1-s)\hat{\sigma}_z\,\hat{a}^\dagger_{\mathbf{k}\mu}\right]$$
$$\frac{d}{dt}\hat{\sigma}_z = -2i\sum_{\mathbf{k}\mu}\left(g_{\mathbf{k}\mu}\left[s\hat{\sigma}_+\hat{a}_{\mathbf{k}\mu} + (1-s)\hat{a}_{\mathbf{k}\mu}\hat{\sigma}_+\right]\right) + h.c.$$

Here the *s* is a parameter that we will choose in the range, $0 \le s \le 1$ (don't confuse this with the saturation parameter). When *s*=1, all photon annihilation operators are to the right and all photon creation operators are to the left – the equations are said to be in *"normal order."* When *s*=0, the opposite is true and the equation are said to be in *"anti-normal order."* When *s*=1/2, the equations are said to be in "*symmetric* *order*." The choice of *s* determines the way in which we apportion level shifts and spontaneous decay to vacuum fluctuations vs. radiation reaction.

(b) Show that to first order in the coupling constant (Born approximation),

$$\hat{a}_{\mathbf{k}\mu}(t) = \hat{a}_{\mathbf{k}\mu}^{free}(t) + \delta\hat{a}_{\mathbf{k}\mu}(t)\,, \quad \hat{\sigma}_+(t) = \hat{\sigma}_+^{free}(t) + \delta\hat{\sigma}_+(t)\,, \quad \hat{\sigma}_z(t) = \hat{\sigma}_z^{free}(t) + \delta\hat{\sigma}_z(t)$$

where

$$\hat{a}_{\mathbf{k}\mu}^{free}(t) = \hat{a}_{\mathbf{k}\mu}(0)e^{-i\omega_k t}\,, \quad \delta\hat{a}_{\mathbf{k}\mu}(t) = -ig_{\mathbf{k}\mu}^{*}\int_0^t e^{-i\omega_k(t-t')}\hat{\sigma}_-^{free}(t') = -ig_{\mathbf{k}\mu}^{*}\,\zeta(\omega_k - \omega_{eg})\hat{\sigma}_-^{free}(t)$$

$$\hat{\sigma}_+^{free}(t) = \hat{\sigma}_+(0)e^{i\omega_{eg}t}$$

$$\delta\hat{\sigma}_+(t) = -i\int_0^t dt' e^{i\omega_{eg}(t-t')}\sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}^{*}\hat{a}_{\mathbf{k}\mu}^{\dagger free}(t')\hat{\sigma}_z^{free}(t') = -i\sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}^{*}\,\zeta(\omega_k - \omega_{eg})\hat{a}_{\mathbf{k}\mu}^{\dagger free}(t)\hat{\sigma}_z^{free}(t)$$

$$\hat{\sigma}_z^{free}(t) = \hat{\sigma}_z(0)$$

$$\delta\hat{\sigma}_z(t) = -2i\int_0^t dt'\sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}\hat{\sigma}_+^{free}(t')\hat{a}_{\mathbf{k}\mu}^{free}(t') + h.c. = -2i\sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}\zeta^{*}(\omega_k - \omega_{eg})\hat{\sigma}_+^{free}(t)\hat{a}_{\mathbf{k}\mu}^{free}(t) + h.c.$$

With $\zeta(\omega_{eg} - \omega_k) = \int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')}\,dt' \approx \pi\delta(\omega_{eg} - \omega_k) + iP\left[\frac{1}{\omega_{eg} - \omega_k}\right]$ (the Markoff approx.).

Notes: $\hat{a}_{\mathbf{k}\mu}^{free}(t)$ is the "vacuum field" and $\delta\hat{a}_{\mathbf{k}\mu}(t)$ the "source field" leading to radiation reaction.
Because the free field commute at equal times we need not worry about operator ordering here.

(c) Take the Heisenberg state of the joint atom field system to be $|\Psi\rangle_{AF} = |\psi\rangle_A \otimes |0\rangle_F$, i.e. an arbitrary state of the atom and the field in the vacuum. Using the perturbation expansion, show that the expected values of the observables evolve according to

$$\frac{d}{dt}\langle\hat{\sigma}_+\rangle = i\omega_{eg}\langle\hat{\sigma}_+\rangle - i\sum_{\mathbf{k}\mu} g_{\mathbf{k}\mu}^{*}\left[s\left\langle\delta\hat{a}_{\mathbf{k}\mu}^{\dagger}\hat{\sigma}_z^{free}\right\rangle + (1-s)\left(\left\langle\delta\hat{\sigma}_z\,\hat{a}_{\mathbf{k}\mu}^{free\dagger}\right\rangle + \left\langle\hat{\sigma}_z^{free}\delta\hat{a}_{\mathbf{k}\mu}^{\dagger}\right\rangle\right)\right]$$

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = -2i\sum_{\mathbf{k}\mu}\left(g_{\mathbf{k}\mu}\left[s\left\langle\hat{\sigma}_+^{free}\delta\hat{a}_{\mathbf{k}\mu}\right\rangle + (1-s)\left(\left\langle\hat{a}_{\mathbf{k}\mu}^{free}\delta\hat{\sigma}_+\right\rangle + \left\langle\delta\hat{a}_{\mathbf{k}\mu}\hat{\sigma}_+^{free}\right\rangle\right)\right]\right) + h.c.$$

Note the relative contributions of the vacuum field and the source field depending on the operator order we had initially chosen.

(d) Put this all together to show

$$\left(\frac{d}{dt} - i\omega_{eg}\right)\langle\hat{\sigma}_+\rangle = s\underbrace{\left(-\frac{\Gamma}{2} - i\delta\right)}_{radiation-reaction}\langle\hat{\sigma}_+\rangle + (1-s)\left(\underbrace{-2\left(\frac{\Gamma}{2} + i\delta\right)}_{vacuum\,contribution} + \underbrace{\left(\frac{\Gamma}{2} + i\delta\right)}_{radiation-reaction}\right)\langle\hat{\sigma}_+\rangle = -\left(\frac{\Gamma}{2} + i\delta\right)\langle\hat{\sigma}_+\rangle$$

$$\frac{d}{dt}\langle\hat{\sigma}_z\rangle = s\underbrace{\left(-\Gamma\langle\hat{\sigma}_z\rangle - \Gamma\right)}_{radiation-reaction} + (1-s)\left(\underbrace{-2\Gamma\langle\hat{\sigma}_z\rangle}_{vacuum\,contribution} + \underbrace{\left(\Gamma\langle\hat{\sigma}_z\rangle - \Gamma\right)}_{radiation-reaction}\right) = -\Gamma\langle\hat{\sigma}_z\rangle - \Gamma$$

$$\text{where}\ \ \Gamma = 2\pi\sum_{\mathbf{k}\mu}\left|g_{\mathbf{k}\mu}\right|^2\delta(\omega_k - \omega_{eg}), \quad \hbar\delta = \sum_{\mathbf{k}\mu}P\left[\frac{\left|g_{\mathbf{k}\mu}\right|^2}{E_g + \hbar\omega_k - E_e}\right]$$

<u>We learn for this the following lessons:</u>

(i) The decay of populations and coherences can be calculated in the Heisenberg picture.

(ii) The way we apportion the relative contributions to levels shifts arising from vacuum fluctuations and radiation reaction depends on operator ordering, but the total result is independent of operator ordering, as expected.

(iii) In normal ordering we attribute the whole of the level shift and decay rate to radiation reaction.

(iv) In antinormal ordering both vacuum fluctuations and radiation reaction contribute to both level shifts and decay rates.

(v) In symmetric ordering, the entire level shift is due to vacuum fluctuations, but decay has contributions from vacuum fluctuation and radiation reaction.

(vi) In no ordering is the decay attributable solely to vacuum fluctuations.