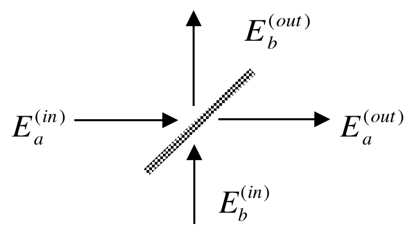
Physics 566, Quantum Optics
Problem Set #10
Due: Thursday Nov. 18, 2021

Problem1: The beam splitter and other linear transformations (25 points)
Consider a symmetric beam splitter



In the first weeks of lecture, we showed that the pair $\left(E_a^{(out)}, E_b^{(out)}\right)$ is related to $\left(E_a^{(in)}, E_b^{(in)}\right)$ through a unitary "scattering matrix"

$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

where $|t|^2 + |r|^2 = 1$, $Arg(t) = Arg(r) \pm \dfrac{\pi}{2}$, so that a possible transformation is,

$$E_a^{(out)} = \sqrt{T}\, E_a^{(in)} + i\sqrt{1-T}\, E_b^{(in)}, \quad E_b^{(out)} = \sqrt{T}\, E_b^{(in)} + i\sqrt{1-T}\, E_a^{(in)}, \text{ where } T = |t|^2.$$

Classically, if we inject a field only into one input port, leaving the other empty, the field in that mode will become attenuated, e.g., $E_a^{(out)} = \sqrt{T}\, E_a^{(in)} < E_a^{(in)}$.

(a) Consider now the quantized theory for these two modes, $E_a \Rightarrow \hat{a}$, $E_b \Rightarrow \hat{b}$. Suppose again that a field is injected only into the "a-port". Show that

$\hat{a}^{(out)} = \sqrt{T}\hat{a}^{(in)}$ is inconsistent with the quantum uncertainty.

(b) In order to preserve the proper commutation relations we cannot ignore *vacuum fluctuations* entering the unused port. Show that if the "in" and "out" creation operators are related by the scattering matrix,

$$\begin{bmatrix} \hat{a}^{(out)\dagger} \\ \hat{b}^{(out)\dagger} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} \hat{a}^{(in)\dagger} \\ \hat{b}^{(in)\dagger} \end{bmatrix},$$ the commutator is preserved.

(c) Suppose a single photon is injected into the a-port, so that the "in-state" is $\left|\psi^{(in)}\right\rangle = |1\rangle_a \otimes |0\rangle_b$. The "out-state" is $\left|\psi^{(out)}\right\rangle = \hat{S}\left|\psi^{(in)}\right\rangle$ where $\hat{S}$ is the "scattering operator", defined so that $\hat{S}\hat{a}^{(in)\dagger}\hat{S}^\dagger = \hat{a}^{(out)\dagger}$ and $\hat{S}\hat{b}^{(in)\dagger}\hat{S}^\dagger = \hat{b}^{(out)\dagger}$.

Show that $\left|\psi^{(out)}\right\rangle = t|1\rangle_a \otimes |0\rangle_b + r|0\rangle_a \otimes |1\rangle_b$.

(d) Suppose a coherent state is injected into the a-port $\left|\psi^{(in)}\right\rangle = |\alpha\rangle_a \otimes |0\rangle_b$. Which is the output, $\left|\psi^{(out)}\right\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b$ or $\left|\psi^{(out)}\right\rangle = t\,|\alpha\rangle_a \otimes |0\rangle_b + r\,|0\rangle_a \otimes |\alpha\rangle_b$? Explain the difference between these.

(e) A general linear optical system consisting, e.g., of beam-splitters, phase shifters, mirrors, etalons, etc. can be described by a unitary transformation on the modes

$$E_k^{(out)} = \sum_{k'} u_{kk'} E_{k'}^{(in)} .$$

In the quantum description the mode operators transform by the scattering transformation $\hat{a}_k^{\dagger(out)} = \hat{S}\hat{a}_k^{\dagger(in)}\hat{S}^\dagger = \sum_{k'} u_{kk'}\hat{a}_{k'}^{\dagger(in)}$, where $u_{kk'}$ is a unitary matrix.

Show that if we start with a multimode coherent state $\left|\psi^{(in)}\right\rangle = \left|\left\{\alpha_k^{(in)}\right\}\right\rangle$, the output state is also a coherent state, $\left|\psi^{(out)}\right\rangle = \left|\left\{\alpha_k^{(out)}\right\}\right\rangle$, with $\alpha_k^{(out)} = \sum_{k'} u_{kk'}\alpha_{k'}^{(in)}$.

(f) The previous part highlights how linear transformations are essentially classical. This was true for input with exactly one photon or for coherent states. However, this is not true for more general inputs. Suppose we send one photon into *both ports*, of a 50-50 beam-splitter T=1/2, $\left|\psi^{(in)}\right\rangle = |1\rangle_a \otimes |1\rangle_b$. Show that the output state is,

$$\left|\psi^{(out)}\right\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_a|0\rangle_b + |0\rangle_a|2\rangle_b\right).$$

This says that the two photons both going to port-a or to port-b, but never one in port-a and one in port-b. This is an effect of Bose-Einstein quantum statistics. Explain in terms of destructive interference between indistinguishable processes.

Problem 2: Collapse a revival in the Jaynes-Cummings model (20 points)

One of the foundational results wich demonstrated the quantum nature of the field was the study of Rabi oscillations of an atom in high-Q cavity (cavity QED).

Suppose at the initial time the atom is in the ground state by the cavity is in a *coherent state*: $|\Psi(0)\rangle_{AF} = |g\rangle \otimes |\alpha\rangle$. The joint atom-field state then evolves according to the Jaynes-Cummings Hamiltonian (we'll neglect here any loss or dissipation).

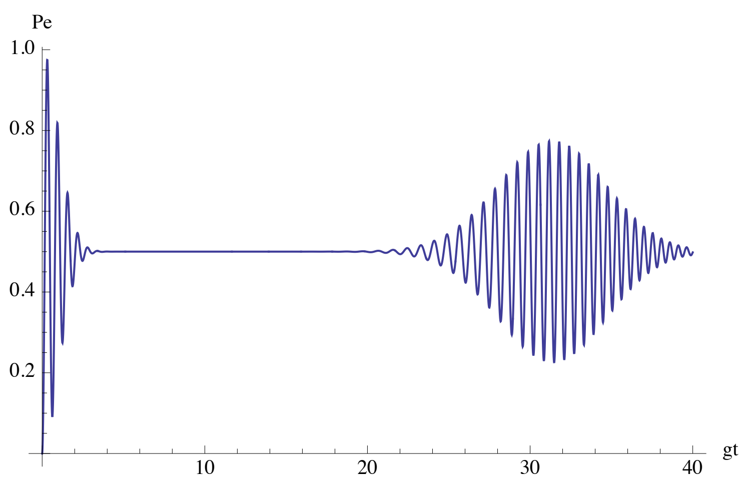
(a) Show that at a later time:

$$|\Psi(t)\rangle_{AF} = c_0 |g\rangle \otimes |0\rangle + \sum_{n=1}^{\infty} c_n e^{-in\omega_0 t} \left(\cos\left(\sqrt{n} gt\right) |g\rangle \otimes |n\rangle + i \sin\left(\sqrt{n} gt\right) |e\rangle \otimes |n-1\rangle \right)$$

where $2g$ is the vacuum Rabi frequency and $c_n = \dfrac{\alpha^n e^{-|\alpha|^2/2}}{\sqrt{n!}}$, and thus show that the probability to be in the excited state at time t, irrespective of photon number, is

$$P_e(t) = \sum_{n=1}^{\infty} \frac{\bar{n}^n e^{-\bar{n}}}{n!} \sin^2\left(\sqrt{n} gt\right), \text{ where } \bar{n} = |\alpha|^2 .$$

(b) Numerically calculate and plot P_e at a function of $0 \le gt \le 40$ for $\bar{n} = 25$. Your result should look as follows:



We see two distinctive features in this plot:

- The Rabi decay (collapse) after a few oscillations.
- After a long time they "revive" and the population starts oscillating again.

(c) The collapse is easily understood because we effectively have "inhomogeneous broadening." That is, we have different Rabi frequencies associated with different numbers of photons, $\Omega_n = \sqrt{n}2g$. We saw this kind of decay of Rabi oscillations early in the semester, when we had a *classical* distribution of intensities. The revival, by contrast, is a purely quantum effect of the field arising from the discrete frequency spectrum (Fourier sum).

Show for large $\bar{n}$, the expected (first) revival time, due to the discreteness of the photons is

$$t_{revive} \approx \frac{2\pi\sqrt{\bar{n}}}{g}$$. Compare with the plot in (b).

Note: The classical limit is of Rabi flopping in free space is intrinsically a multi-mode problem, and will exhibit these collapse and revivals.