Physics 566 – Lecture 5

Optical Bloch Equations – Phenomelogical Damping

So far we have considered only "coherent evolution" of a two-level system governed by unitary dynamics. The resulting evolution exhibited a simple sinusoidal time dependence for all time – it is reversible. In real situations, this is never the case. The system will eventually "relax" to some steady state where "memory" of the initial condition is lost. This type of irreveribble evolution is a fact of nature. How do we take irreversibility into account in quantum theory whose basic time evolution is given by a unitary map? This is a fundamental problem that we will address in more detail later. For now we will take a phenomenological approach. As an aside, the problem of obtaining effectively irreversible dynamics at the "macroscopic level" when the microscopic laws of physics are time-reverse invariant is a fundamental issue in classical physics as well. How does the second-law of thermodynamics arise from Newton's laws and Maxwell's equations?

Optical Bloch Eqns

Consider a two-level system. There are two kinds of "relaxtions"

- Relaxing of the populations ρ_{ee}, ρ_{gg}
- Relaxation of the coherences ρ_{eg}, ρ_{ge}

We have names for these effects that originate from NMR

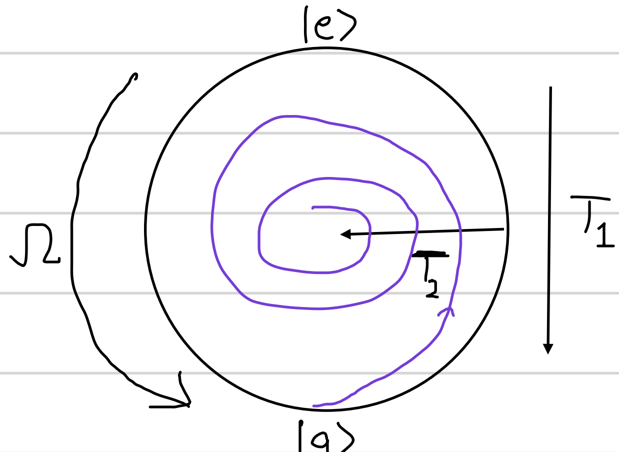
1) $T_1 \equiv$ decay time constant of the population inversion, $w \equiv \rho_{ee} - \rho_{gg}$
   (sometimes known as $T_{\parallel}$: "Longitudinal relaxation")

2) $T_2 \equiv$ decay time constant of the coherences (u, v) : ($T_{\perp}$: "transverse relaxation")

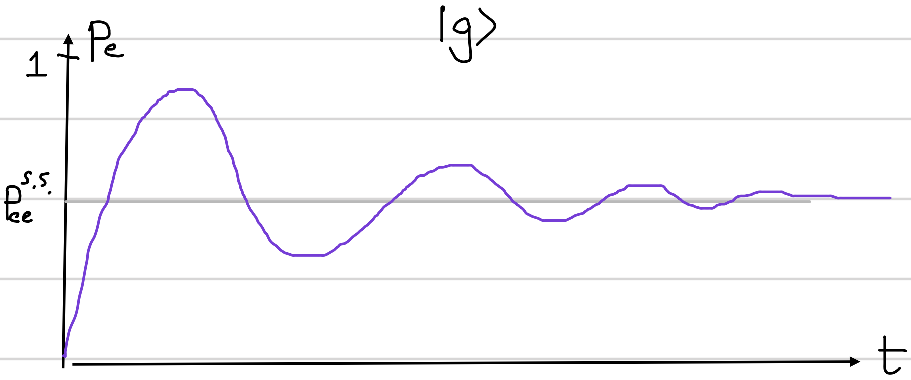
Bloch equations with relaxation

$$\boxed{\begin{aligned} \dot{u} &= +\Delta v - \frac{1}{T_2} u \\ \dot{v} &= -\Delta u - \Omega w - \frac{1}{T_2} v \\ \dot{w} &= -\frac{1}{T_1}(w - w_{s.s.}) + \Omega v \end{aligned}}$$

Trajectory on the Bloch sphere with relaxation



The trajectory moves off the surface of the Bloch sphere and into the interior of the Bloch ball. The state is now longer pure — it is a mixed state.



Damped Rabi oscillations. The probability to find the atom in the excited level |e⟩ reaches steady state: $P_e^{s.s.}$

"T_1" vs. "T_2" and the density matrix

Consider $\hat{\rho} = \sum_j P_j |\psi_j\rangle\langle\psi_j|$ (ensemble decomposition)

$$\Rightarrow \rho_{\alpha\beta} = \sum_j P_j \, c_\alpha^{(j)} c_\beta^{(j)*} = \sum_j P_j |c_\alpha^{(j)}| \, |c_\beta^{(j)}| \, e^{i(\phi_\alpha - \phi_\beta)}$$

- Diagonal matrix elements $\rho_{\alpha\alpha} = \sum_j P_j |c_\alpha^{(j)}|^2 = \overline{|c_\alpha|^2}$ ensemble average

$\Rightarrow$ T_1 involves the average relaxation of the square-magnitude of probabilities amplitudes.

$$\rho_{\alpha\alpha}(t)\Big|_{relax} = e^{-\Gamma_\alpha t} \rho_{\alpha\alpha}(0)$$

$\Rightarrow$ Γ_α is the rate of decay of $\overline{|c_\alpha|^2}$ $\Rightarrow$ $\overline{|c_\alpha|}(t) = \overline{|c_\alpha|}(0) \, e^{-\Gamma_\alpha t/2}$

- Off-diagonal elements: Consider a case where $|c_\alpha^{(j)}|$ is the same for all j

$$\rho_{\alpha\beta}(t) = |c_\alpha(0)| \, |c_\beta(0)| \, e^{-\frac{(\Gamma_\alpha + \Gamma_\beta)}{2} t} \sum_j P_j \, e^{i(\phi_\alpha^j - \phi_\beta^j)}$$

Consider a process of "dephasing": $\phi_\alpha^j - \phi_\beta^j = (\delta\omega_{\alpha\beta}^j)t$ (Random phase kicks between α & β)

$\sum_j P_j \, e^{i(\delta\omega_{\alpha\beta}^j)t} = e^{-\beta_{\alpha\beta} t}$ $\Rightarrow$ $\boxed{\frac{1}{T_2} = \frac{\Gamma_\alpha + \Gamma_\beta}{2} + \beta_{\alpha\beta}}$: T_2 involves both population decay as well as dephasing

Sources of relaxation – Perturbation by the "environment"

E.g. Collisions of the atoms with "environmental" atoms: Elastic $\Rightarrow$ dephasing
Inelastic $\Rightarrow$ Population change

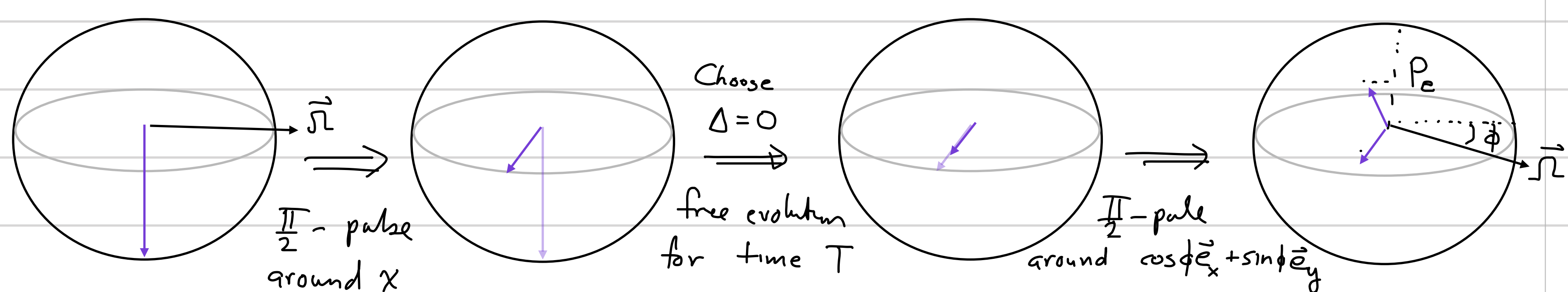
Generally $T_2 \ll T_1$: Small changes in phase occur more easily than pop. change

Note: If we have an ensemble of different unknown environments, then dephasing can occur due to inhomogeneities (see problem set)

Then $\frac{1}{T_2} = \frac{1}{T_2'} + \frac{1}{T_2^*}$, where T_2^* = decay due to inhomogenities
T_2' = decay for homogeneous sample

The decay due to inhomogeneity does not correspond to true irrevisibility – each particle in the ensemble undergoes unitary evolution, but with a slightly different Hamiltonian. The result can be that all the different state vectors get out of phase, but they can be "rephased" with a spin echo (see problem set).

To measure the "coherence time" one employ a "Ramsey interferometer" (the two-level quantum version of the two-path Mach-Zender inteferometer).
Consider the simplest case without inhomogeneous broadening (we can always add a spin-echo).



Let us suppose that after the free evolution the state is $\hat{\rho}(T)$. We apply a $\frac{\pi}{2}$ pulse with the phase shift ϕ from the first pulse and then measure the fraction of times we find $|e\rangle$ (this should converge to P_e after many trials).

We seek $P_e(\phi, T)$.

$$P_e(\phi, T) = \langle e| e^{-i\frac{\pi}{4}\vec{e}_\phi\cdot\hat{\vec{\sigma}}}\hat{\rho}(T)e^{+i\frac{\pi}{4}\vec{e}_\phi\cdot\hat{\vec{\sigma}}}|e\rangle$$

Aside: $e^{+i\frac{\pi}{4}\vec{e}_\phi\cdot\hat{\vec{\sigma}}}|e\rangle = \left(\cos\frac{\pi}{4}\hat{\mathbb{1}} + i(\cos\phi\,\hat{\sigma}_x + \sin\phi\,\hat{\sigma}_y)\sin\frac{\pi}{4}\right)|e\rangle = \frac{1}{\sqrt{2}}(\hat{\mathbb{1}} + ie^{-i\phi}\hat{\sigma}_+ + ie^{i\phi}\hat{\sigma}_-)|e\rangle$

$$= \frac{1}{\sqrt{2}}(|e\rangle + ie^{i\phi}|g\rangle)$$

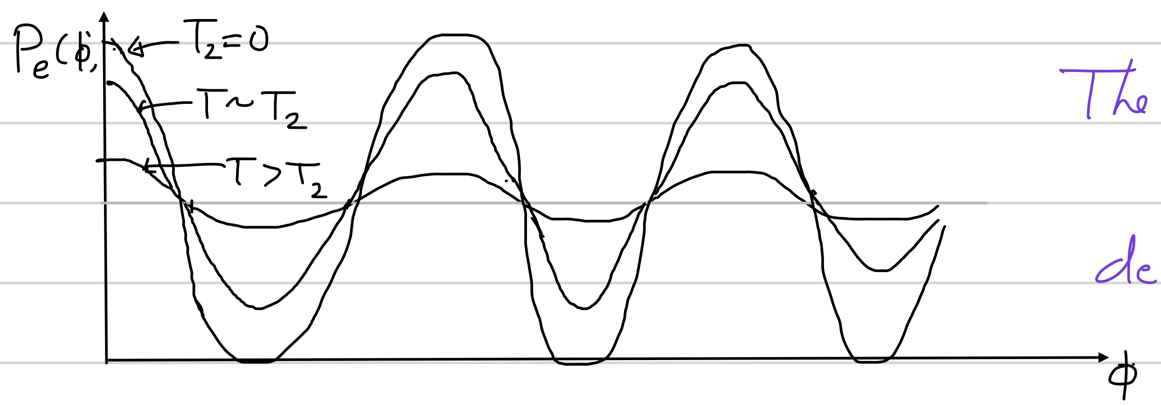
$$\Rightarrow P_e(\phi,T) = \frac{1}{2}(\underbrace{\rho_{ee}(T) + \rho_{gg}(T)}_{=1} + ie^{i\phi}\rho_{eg}(T) + ie^{-i\phi}\rho_{ge}(T))$$

$$\Rightarrow P_e(\phi,T) = \frac{1}{2}(1 - 2\,\mathrm{Im}(\rho_{eg}(T))\cos\phi + 2\,\mathrm{Re}(\rho_{eg}(T))\sin\phi)$$

At $T=0$ $|\psi\rangle = e^{-i\frac{\pi}{4}\hat{\sigma}_x}|g\rangle = \frac{1}{\sqrt{2}}(|g\rangle - i|e\rangle) \Rightarrow \rho_{eg}(0) = \frac{-i}{2}$

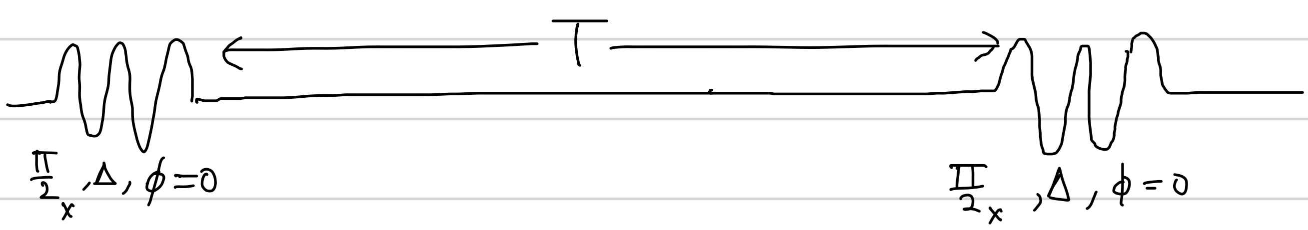
For $T \geq 0$ $\rho_{eg}(T) = \rho_{eg}(0)e^{-T/T_2} = -\frac{i}{2}e^{-T/T_2}$

$$\Rightarrow \boxed{P_e(\phi,T) = \frac{1}{2}(1 + \cos\phi\, e^{-T/T_2})}$$

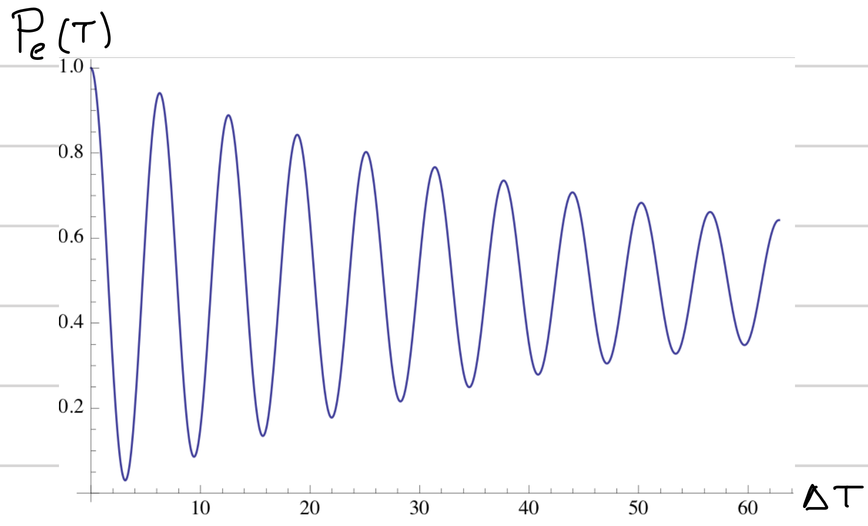


The visibility of the Ramsey fringes determines the coherence time T_2.

In an experimental implementation, typically one does not measure Ramsey fringes at zero detuning because one can be sensitive to small noise fluctations in the detuning. Consider the Ramsey fringes at finite detuning $|\Delta| = |\omega - \omega_0| > 0$. We apply two $\frac{\pi}{2}$ pulses around to "y"-axis with the same phase ϕ, separated by a time T

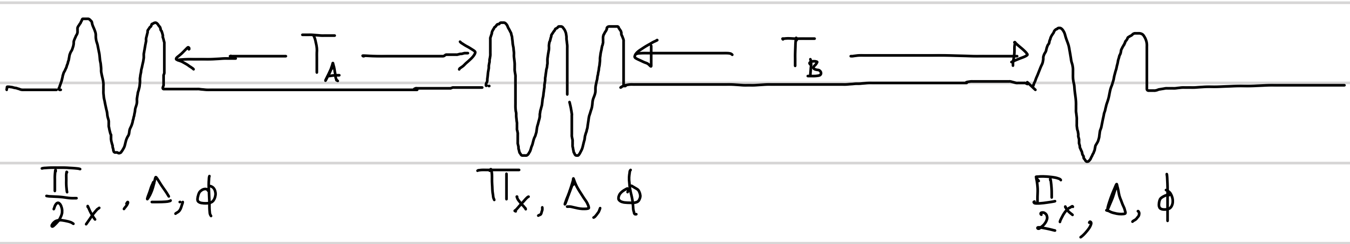


In this case the $P_e(T) = \frac{1}{2}(1 + \cos(\Delta T)\, e^{-T/T_2})$, assuming only homogeneous broadening

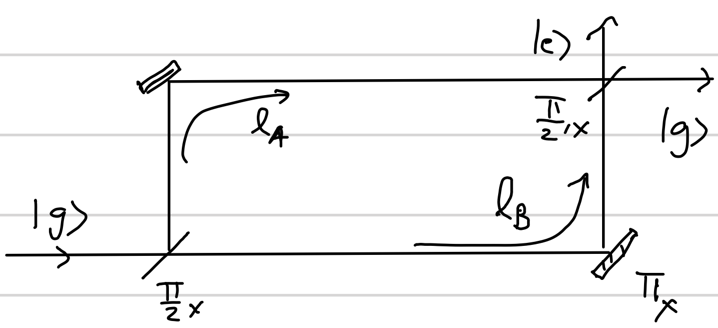


By fitting to the function $\frac{1}{2}\left(1+\cos(\Delta T)\, e^{-T/T_2}\right)$ One can extract the T_2 time

If there are large inhomogenities ($T_2^* = 0$), which in the case of a single qubit could mean that the shot-to-shot variation in ω_0 is large, we can use the Spin-echo technique studied in homework to remove the inhomogeneous decay, and leave only the intrinsic irreversible T_2-decay. Consider a 3-pulse sequence:



The middle π-pulse "time reverses" the two paths, and performs the echo. This is one-to-one correspondence with two path wave interferometer (e.g. a Mach-Zender interferometer)



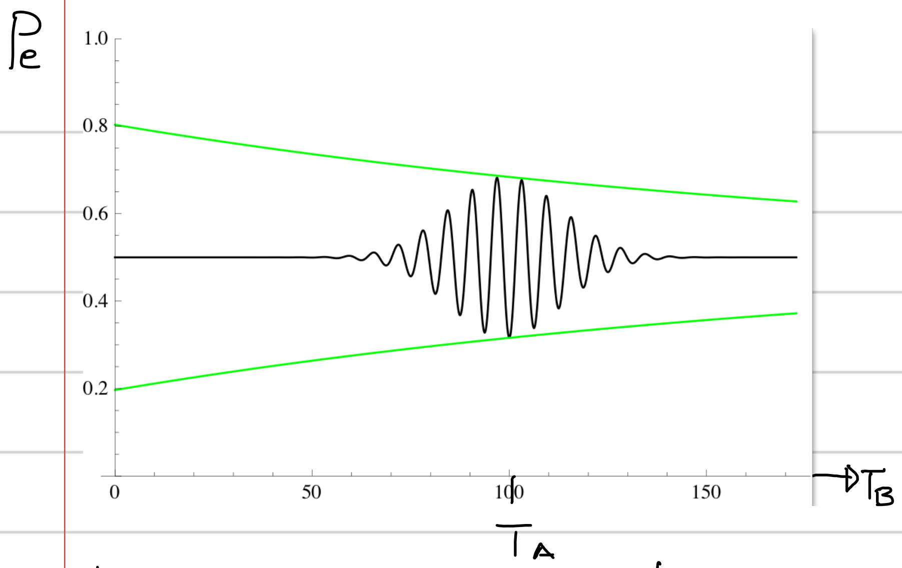
In the absense of any distribution of detunings Δ, the Probabity to find $|e\rangle$ is

$$P_e(T_A, T_B, \Delta) = \frac{1}{2}\left[1 - \cos\{\Delta(T_A - T_B)\}\, e^{-\frac{T_A+T_B}{T_2}}\right]$$

Note if $T_A = T_B$, P_e, independent of Δ.

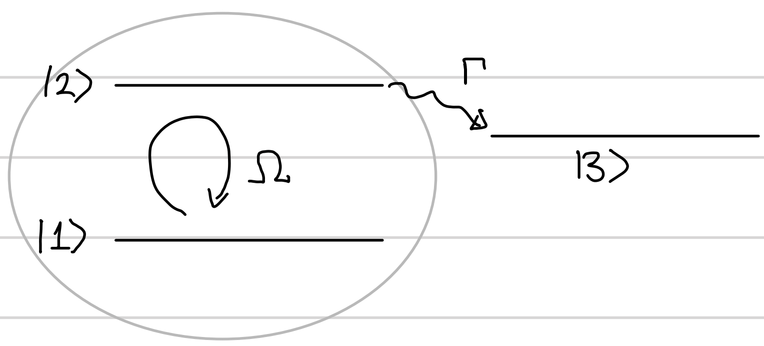
If there is a distribution of Δ, we must integrate over that distribution,

$$P_e(T_A, T_B) = \int d\Delta\; p(\Delta)\, P_e(T_A, T_B, \Delta)$$



The example here shows Ramsey fringes with a spin-echo sequence for an inhomogeneously broadened qubit. The spin is measured as a function of the delay to the second $\frac{\pi}{2}$-pulse T_B, for a fixed T_A. The homogenous T_2-delay is the decay green envelope.

Nonunitary Schrödinger Evolution:



Two-level system with decay to a third

The dynamics in the $\{|1\rangle, |2\rangle\}$ subspace is governed by a non-Hermitian Hamiltonian, to account for the irrevisble behavior.

We add an imaginary part to the excited state energy eigenvalue E_2.

$$\hat{H}_{eff} = E_1 |1\rangle\langle 1| + \left(E_2 - i\frac{\hbar\Gamma}{2}\right)|2\rangle\langle 2|$$

In the absence of any other interaction, if at time $t=0$ $|\psi(0)\rangle = c_1|1\rangle + c_2|2\rangle$ then at a latter time,

$$|\psi(t)\rangle = e^{\frac{-i\hat{H}_{eff}t}{\hbar}}|\psi(0)\rangle = c_1 e^{\frac{-iE_1 t}{\hbar}}|1\rangle + c_2 e^{\frac{-iE_2 t}{\hbar}} e^{-\frac{\Gamma t}{2}}|2\rangle$$

$$\Rightarrow P_2(t) = |\langle 2|\psi(t)\rangle|^2 = |c_2|^2 e^{-\Gamma t}$$

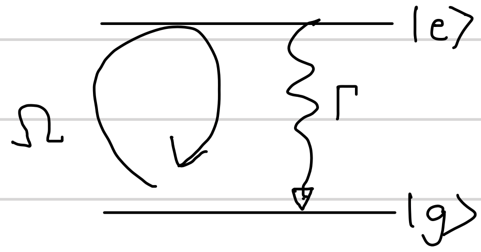
(Note: In this evolution $\|\psi(t)\|$ is not conserved because we are not accounting for the population in $|3\rangle$)

More generally, if $|1\rangle$ & $|2\rangle$ are coupled by an interaction Hamiltonian, then the evolution is described by the Schrödinger eqn. with a non-Hermitian Hamiltonian

$$\frac{\hbar}{-i}\frac{\partial}{\partial t}|\psi\rangle = \hat{H}_{eff}|\psi\rangle \Rightarrow \left.\begin{aligned} \dot{c}_1 &= -i\frac{E_1}{\hbar}c_1 - i\frac{H_{12}}{\hbar}c_2 \\ \dot{c}_2 &= \left(i\frac{E_2}{\hbar} - \frac{\Gamma}{2}\right)c_2 - i\frac{H_{21}}{\hbar}c_1 \end{aligned}\right\} \text{Simple Schrödinger evolution with decay}$$

Decay in a Closed Two-level System

Consider now a two-level "closed transition," with decay from $|e\rangle \to |g\rangle$



Here, the population is fed from $|e\rangle \to |g\rangle$. The total population is constant.

For such a case it is impossible to describe the dynamics by a Schrödinger equation for a wave function. We must turn to density matrix and a Liouville-type evolution. We will return later to explain the deeper reasons why. For now, just note that if we set $c_e(t) = c_e(0)\, e^{-\frac{\Gamma t}{2}}$ and $|c_g(t)|^2 + |c_e(t)|^2 = 1$, there is no $\hat{H}_{eff}$ whose solution is $|c_g(t)|^2 = 1 - |c_e(0)|^2 e^{-\Gamma t}$ independent of $c_g(0)$.

Master Equation for a Two-Level Atom

Consider a closed two-level system, with decay from $|e\rangle \to |g\rangle$ due to spontaneous emission (and no other source of dephasing or decay/relaxation).

$$\boxed{\begin{aligned} &\Gamma = \frac{1}{\tau} = \text{"Einstein A-coefficient"} = \frac{4}{3\hbar}|\langle e|\vec{d}|g\rangle|^2 \left(\frac{\omega_{eg}}{c}\right)^3 \\ &\tau = \text{Natural Lifetime} \end{aligned}}$$

$$\dot{\rho}_{ee}\Big|_{relax} = \frac{d}{dt}\overline{|c_e|^2}\Big|_{relax} = -\Gamma \rho_{ee}\,, \qquad \dot{\rho}_{eg}\Big|_{relax} = \frac{d}{dt}\overline{c_e c_g^*}\Big|_{relax} = -\frac{\Gamma}{2}\rho_{eg}$$

Since $\rho_{ee} + \rho_{gg} = 1 \Rightarrow \dot{\rho}_{gg}\big|_{relax} = -\dot{\rho}_{ee}\big|_{relax} = +\Gamma \rho_{ee}$ $\qquad \left(T_1 = \frac{1}{\Gamma},\ T_2 = \frac{2}{\Gamma}\right)$

Coherent evolution in rotating frame

$$\dot{c}_g = -i\frac{\Delta}{2}c_g - i\frac{\Omega}{2}c_e$$
$$\dot{c}_e = i\frac{\Delta}{2}c_e - i\frac{\Omega}{2}c_g$$

$$\Rightarrow \frac{d}{dt}\rho_{ee}\Big|_{coherent} = \dot{c}_e c_e^* + c_e \dot{c}_e^* = -i\frac{\Omega}{2}(\rho_{ge} - \rho_{eg})$$

$$\frac{d}{dt}\rho_{ge}\Big|_{coherent} = \dot{c}_g c_e^* + c_g \dot{c}_e^* = -i\Delta\rho_{ge} - i\frac{\Omega}{2}(\rho_{ee} - \rho_{gg})$$

⇒ Master equation for two-level atom with spontaneous emission

$$\boxed{\begin{aligned}&\frac{d}{dt}\rho_{ee} = -\Gamma\rho_{ee} - \frac{i}{2}\Omega(\rho_{ge}-\rho_{eg}) = -\frac{d\rho_{gg}}{dt} \quad (\text{since } \rho_{ee}+\rho_{gg}=1 \ \forall t)\\ &\frac{d}{dt}\rho_{ge} = \left(-i\Delta-\frac{\Gamma}{2}\right)\rho_{ge} - \frac{i}{2}\Omega(\rho_{ee}-\rho_{gg}) = \frac{d}{dt}\rho_{eg}^{*}\end{aligned}}$$

These equations are the elements of the following operation equation:

$$\frac{d}{dt}\hat{\rho} = \underbrace{-\frac{i}{\hbar}[\hat{H},\rho]}_{\text{Hamiltonian evolution}} + \underbrace{\mathcal{L}_{diss}[\hat{\rho}]}_{\text{dissipative evolution}}$$

"Master equation" (Liouville equation)

$$\mathcal{L}_{diss}[\hat{\rho}] = -\frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\hat{\rho} + \hat{\rho}\hat{\sigma}_+\hat{\sigma}_-) + \Gamma\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+$$

: Lindblad form

$$= -\frac{i}{\hbar}\left\{-\frac{i\hbar\Gamma}{2}|e\rangle\langle e|, \hat{\rho}\right\}_+ + \Gamma\rho_{ee}|g\rangle\langle g|$$

anti-commutator

The Lindblad form decomposes as

$$\frac{d\rho}{dt} = \underbrace{-\frac{i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{eff}\right)}_{\text{decay}} + \underbrace{\Gamma\hat{\sigma}_-\hat{\rho}\hat{\sigma}_+}_{\text{re-feeding}}$$

$$\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma}{2}|e\rangle\langle e|$$: Non-Hermitian Hamiltonian

We will return to this decomposition next semester when we study open quantum systems in much more detail. This is the foundation of the so-called "quantum trajectory" picture, involving deterministic (but non-unitary) evolution, punctuated by random jumps. The jumps here involve spontaneous jumps from excited to ground states.