Physics 566: Quantum Optics
Introduction to Quantum Field Theory

The need for a quantum field theory

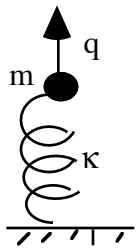
So far we have treated the interaction of matter with light as the interaction of a quantized two-level atom with a classical electromagnetic field. This description allowed us to derive Fermi's Golden rule for the transfer of population from the ground state to the excited state in the short time limit, and coherent Rabi-flopping between these states for longer times. At the end of lecture #5, we added phenomenological damping terms to our optical Bloch equations in order to account for dissipative processes. Foremost among these is the process of spontaneous emission. Given an atom initially in the excited state, in the absence of any light, we know from experience that it will eventually decay to the ground state and "spontaneously" emit light (that is to say, it was not "stimulated" to do so by any light present at the position of the atom). However, according to the Hamiltonian that we've used up to now, in the absence of any light, if the atom starts in the excited state it will remain there forever since the excited state is a *stationary state*. In order to properly account for the phenomenon of spontaneous emission we must go to a more complete description of atom-light interactions.

The missing ingredient in our description so far is that we have treated the electromagnetic field *classically*. As quantum mechanics is supposed to be the fundamental theory of physics, the field must too have a quantum description. In such a theory the uncertainty principle will limit the observables we can know simultaneously about the field, even in principle with ideal measuring devices. This uncertainty can be thought of as fluctuations in the electromagnetic field which effect the dynamics of the atom. Thus, even in the absence of light, i.e. the "vacuum", the atom will evolve; this is the basis of spontaneous emission.

The quantum mechanical description of a field requires a Hamiltonian formulation of the field dynamics. When one does this we will find that the field is equivalent to an infinite collection of harmonic oscillators, each oscillator representing a normal mode of the field with the appropriate boundary conditions. Quantizing the field is then equivalent to quantizing each of the normal mode oscillators. Thus, it is crucial to understand the quantum mechanical description of a simple harmonic oscillator, perhaps the most ubiquitous problem in physics.

The Classical Simple Harmonic Oscillator

The paradigm of a simple harmonic oscillator is a mass *m* on a spring, with "spring constant" κ, and resonant frequency $\omega = \sqrt{\kappa / m}$.



The dynamical coordinate *q*, represents the displacement of the mass from its equilibrium position. The kinetic energy *T* and potential energy *V* of the system are

$$T = \frac{1}{2} m\dot{q}^2, \quad V = \frac{1}{2}\kappa q^2 = \frac{1}{2} m\omega^2 q^2 .$$

The Lagrangian is then,

$$L \equiv T - V = \frac{1}{2} m\dot{q}^2 - \frac{1}{2} m\omega^2 q^2 .$$

The canonical momentum, conjugate to *q* is defined $p \equiv \frac{\partial L}{\partial \dot{q}} = m\dot{q}$, and the Hamiltonian

$$H(q, p) \equiv T(\dot{q} = p / m) + V(q) = \frac{p^2}{2m} + \frac{1}{2} m\omega^2 q^2 .$$

Let us define characteristic scales for the position, momentum, and energy, q_0 , p_0, E_0, respectively, so that $Q \equiv q / q_0$ and $P \equiv p / p_0$ are dimensionless phase space coordinates. If we choose the scales so that $\frac{{p_0}^2}{2m} = \frac{1}{2} m\omega^2 {q_0}^2 = E_0$, where E_0 remains to be chosen, then the Hamiltonian can be expressed as

$$H = E_0\left(Q^2 + P^2\right).$$

Finally, let us define a dimensionless complex amplitude in phase space,

$$\alpha \equiv Q + iP = \frac{q}{q_0} + i\frac{p}{p_0}$$

so that the Hamiltonian can be expressed as, $H = E_0\,\alpha^*\alpha$. Since energy is conserved, $|\alpha|$ is conserved by the dynamics.

The equations of motion can be found from either the Lagrange-Euler equation or the Hamilton equations of motion, $\ddot{q} + \omega^2 q = 0$, with the solution

$$q(t) = q(0)\cos(\omega t) + \frac{p(0)}{m\omega}\sin(\omega t) = q_0\,A\cos(\omega t - \phi)$$
$$p(t) = p(0)\cos(\omega t) - m\omega q(0)\sin(\omega t) = -p_0\,A\sin(\omega t - \phi),$$

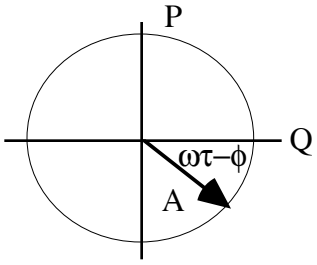
where $A = \sqrt{(Q(0))^2 + (P(0))^2}$, $\phi = \tan^{-1}(P(0)/Q(0))$,

$$Q(0) = q(0)/q_0, \quad P(0) = p(0)/p_0.$$

The complex amplitude thus evolves according to

$$\alpha(t) = (Ae^{i\phi})e^{-i\omega t} = \alpha(0)e^{-i\omega t}.$$

It is helpful to view these dynamics in the (Q,P) plane which can be thought of as phase space, or the complex α plane, with Q the real axis and P the imaginary.



The motion is that of a phasor rotating clockwise with frequency ω, oscillating every quarter period between pure potential energy (phasor along Q-axis) and pure kinetic energy (phasor along P-axis). The magnitude of this phasor is conserved. Note from the solution for q(t) above, if we have a reference oscillator of the same frequency that oscillates as $\cos(\omega t)$ then q(0) tells us the part of the motion that is in-phase with the reference, and p(0) tells us the part that oscillate 90 degrees out-of-phase ("in-quadrature").

The Quantum Simple Harmonic Oscillator

In the quantum picture of the oscillator, the phase space coordinates become (Hermitian) operators, $q \to \hat{q},\ p \to \hat{p}$, and we impose the fundamental canonical commutator,

$$[\hat{q}, \hat{p}] = i\hbar.$$

If we take the scale length of energy to be $E_0 = \hbar\omega$, then $q_0 = \sqrt{2\hbar / m\omega}$, $p_0 = \sqrt{2m\hbar\omega}$, and the dimensionless phase space operators satisfy $[\hat{Q}, \hat{P}] = \frac{i}{2}$. The complex amplitude is mapped onto a nonHermitian operator

$$\alpha \to \hat{a} = \hat{Q} + i\hat{P} = \sqrt{\frac{m\omega}{2\hbar}}\left(\hat{q} + i\frac{\hat{p}}{m\omega}\right).$$

Using the commutation relations for $\hat{Q}$ and $\hat{P}$ we then have,

$$[\hat{a}, \hat{a}^\dagger] = 1.$$

Substituting the operators $\hat{Q}$ and $\hat{P}$, and the energy scale E_0 into the Hamiltonian yields,

$$\hat{H} = \hbar\omega\left(\hat{Q}^2 + \hat{P}^2\right) = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right).$$

The final form of the Hamiltonian is arrived at by substituting in for $\hat{Q}$ and $\hat{P}$ in terms of $\hat{a}$ and $\hat{a}^\dagger$, and using the commutation relations above.
(Note, in general there is no unique way of going from a classical Hamiltonian $H(q, p)$ to a quantum Hamiltonian $\hat{H}(\hat{q}, \hat{p})$ because of the problem of operator ordering. That is, although q and p commute, $\hat{q}$ and $\hat{p}$ do not, so if the Hamiltonian depends on products of q and p, we must choose some particulart ordering of these variables).

The operator $\hat{N} = \hat{a}^\dagger\hat{a}$ is known as the number operator, satisfying the commutation relations,

$$[\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger, \quad [\hat{N}, \hat{a}] = -\hat{a}\ .$$

Its eigenstates, |n>, are the eigenstates of the Hamiltonian,

$$\hat{N}|n\rangle = n|n\rangle, \quad \hat{H}|n\rangle = \hbar\omega(n+\frac{1}{2})|n\rangle.$$

The ground state satisfies $\hat{a}|0\rangle = 0$. If we act with $\hat{a}$ on the eigenstate $|n\rangle$, the resulting vector is also a eigenstate of $\hat{N}$ (with a different eigenvalue) ,

$$\hat{N}\hat{a}|n\rangle = \hat{a}\hat{N}|n\rangle + [\hat{N},\hat{a}]|n\rangle = \hat{a}\,n|n\rangle - \hat{a}|n\rangle = (n-1)\hat{a}|n\rangle,$$

so we must have

$$\hat{a}|n\rangle = \sqrt{n}\,|n-1\rangle,$$

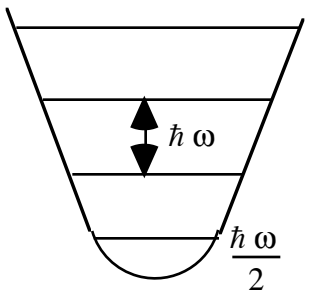
where the $\sqrt{n}$ factor is for normalization. For this reason, $\hat{a}$ is known as the lowering operator, or the annihilation operator. Similarly, one can show,

$$\hat{a}^{\dagger}|n\rangle = \sqrt{n+1}\,|n+1\rangle,$$

so $\hat{a}^{\dagger}$ is known as the raising operator, or the creation operator. The general nth energy eigenstate can be represented by acting n-times on the ground state with $\hat{a}^{\dagger}$,

$$|n\rangle = \frac{(\hat{a}^{\dagger})^n}{\sqrt{n!}}|0\rangle.$$

The energy spectrum is thus an infinite latter of energy levels, all equally spaced by $\hbar\omega$.



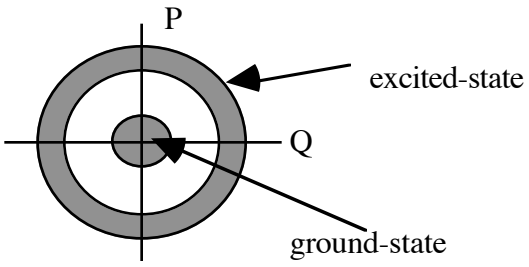
The ground state energy $E_{ground} = \frac{1}{2}\hbar\omega$ is known as the zero point energy. That is, quantum mechanically it is forbidden to have the oscillator exactly at rest at the equilibrium

position, for this would violate the uncertainty principle. The ground state is a minimum uncertainty wave packet state $\Delta q\,\Delta p = \frac{1}{2}\hbar$.

If we want to depict the energy eigenstates in a phase space diagram as we did for the classical oscillator on Page 3, we can no longer represent it as a well defined phasor in the Q,P plane since these variables no longer commute. In fact,

$$\langle n|\hat{Q}|n\rangle = \langle n|\hat{P}|n\rangle = 0,$$

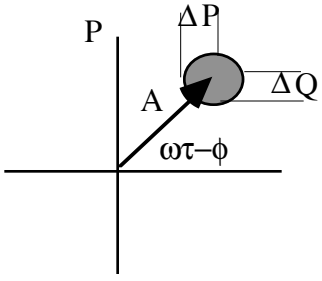
for the stationary states. Instead we must picture these states a "fuzzy" rings in phase space, whose radius is *n* and whose, thickness depends on the uncertainties $\Delta\hat{Q}$ and $\Delta\hat{P}$.
The ground state will be a fuzz-ball center on the origin with the minimum uncertainties $\Delta\hat{Q} = \Delta\hat{P} = \frac{1}{2}$.



If we construct a wave packet as a superposition of different energy eigenstates

$$|\psi\rangle = \sum_n c_n |n\rangle,$$

then we can obtain a state whose expectation values of $\hat{Q}$ and $\hat{P}$ oscillate in time.



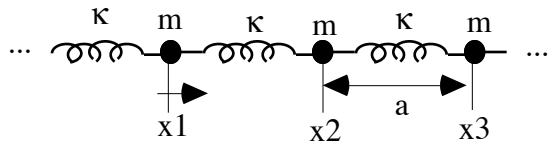
However, the degree to which we can specify the trajectory in phase space is limited by the uncertainty principle. In fact, the more precise we try to specify the phase of the oscillator, the more uncertain the amplitude will become, This can be stated in terms of an (approximate) uncertainty relation between the number of excitations (i.e. the amplitude of the oscillation) and its phase

$$\Delta\hat{N}\Delta\hat{\phi} \sim 1.$$

An eigenstate of the number operator thus has the complete uncertainty in the phase, as is represented by the ring in phase-space. A state with a more well defined phase, will have a large uncertainty in number.

Lagrangian formulation of a one dimensional scalar field theory

Consider now a collection of N identical masses attached to one another in a linear chain by springs of length a.



Linear chain of oscillators

The configuration space for this system is given by the set of positions for the N oscillators $\{x_i\}$. The kinetic energy and potential energy for the system are, respectively,

$$T = \sum_i \frac{1}{2} m\dot{x}_i^2, \quad V = \sum_i \frac{1}{2}\kappa(x_{i+1} - x_i)^2$$

From the Lagrangian L=T–V we can determine the coupled equations of motion of the masses.

We now want to go to a limit so that our chain of oscillators becomes a continuous elastic rod. We do so by taking the limit as $N\to\infty$, $a\to dx$, $m/a\to\mu$ (the linear mass density),

and $\kappa a \to Y$ (the Young's modulus of the rod). In doing so, the discrete set of dynamical variables $\{x_i\}$ are replaced by a continuous field $\eta(x)$ describing the displacement from equilibrium of a differential element of the rod at position x by an amplitude η.

$$\{x_i\} \Rightarrow \eta(x)$$

When we take this limit the kinetic and potential energies of the system become,

$$T = \lim \sum_i a \frac{1}{2}\left(\frac{m}{a}\right)\dot{x}_i^{\,2} = \int dx \;\; \frac{1}{2}\mu\left(\frac{\partial \eta}{\partial t}\right)^2$$
$$V = \lim \sum_i a \frac{1}{2}\kappa a\left(\frac{x_{i+1} - x_i}{a}\right)^2 = \int dx \;\; \frac{1}{2}Y\left(\frac{\partial \eta}{\partial x}\right)^2 .$$

The equation of motion which follows from this Lagrangian is

$$\frac{\partial^2 \eta}{\partial t^2} - \frac{Y}{\mu}\frac{\partial^2 \eta}{\partial x^2} = 0 .$$

This is none other than the wave equation for longitudinal excitations on elastic rod, whose phase velocity is $v = \sqrt{Y/\mu}$.

Normal Modes of the Field

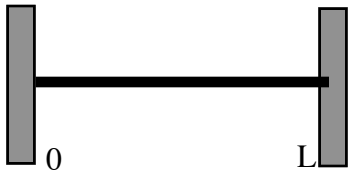
Now consider a finite elastic rod of length L. In order to determine the dynamics of wave on the rod, we decompose it into its normal modes, $\{u_k(x)\}$, satisfying

$$\frac{\partial^2 u_k}{\partial x^2} = -k^2 u_k(x).$$

To solve this, we must specify the boundary conditions.

• "Hard-wall" boundary conditions:

Consider first the case of the elastic rod attached at each end to an infinitely massive wall, so that $\eta(0) = \eta(L) = 0$.



The normal modes are

$$u_{k_n}(x) = \sqrt{\frac{2}{L}} \sin(k_n x), \quad k_n = \frac{n\pi}{L}.$$

These modes are orthonormal,

$$\int dx \;\; u_{k'}(x) \;\; u_k(x) = \delta_{k'k},$$

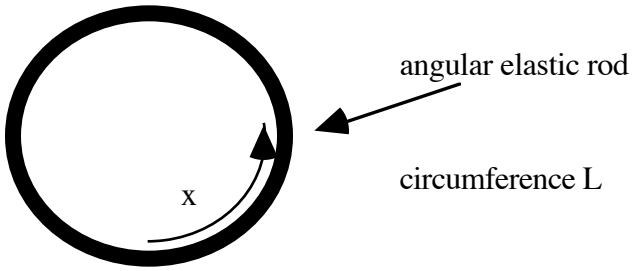
and complete,

$$\int dx \;\; u_k(x') \;\; u_k(x) = \delta(x' - x).$$

Note: I have dropped the subscript n on the mode, though it is to be understood that $k=k_n$.

• "Periodic" boundary conditions

Consider now the case that the rod is wrapped into a ring so that the position along the rod is the azimuthal position.



The field must come back to its after one circumference, $\eta(x+L)=\eta(x)$. The normal modes for these boundary conditions are:

$$u_{kn}(x)=\sqrt{\frac{1}{L}}\,e^{ik_n x},\ k_n=\frac{n2\pi}{L}$$

The orthonormality and completeness relations are,

$$\int dx\ \ u_{k'}{}^{*}(x)\ \ u_k(x)=\delta_{k'k},$$
$$\sum_k u_k{}^{*}(x')\ \ u_k(x)=\delta(x'-x).$$

For either choice of boundary conditions, the completeness relation tells us that we can expand an arbitrary field in terms of the normal modes

$$\eta(x,t)=\sqrt{L}\sum_k q_k(t)\ u_k(x).$$

The factor of $\sqrt{L}$ was added for dimensional purposes so that q_k and η have the same units. This expansion is essentially a Fourier series for the field which satisfies the appropriate boundary conditions. Since η is real, for the hard-wall boundary conditions, the coefficients q_k are real, while for periodic boundary conditions $q_k{}^{*}=q_{-k}$. The difference is that for hard-wall boundary conditions we only retain the "sine" terms in the Fourier series, while for periodic boundary conditions we have both the spatial "cosine" and "sine" terms which represent the real and imaginary parts of q_k respectively. Periodic boundary conditions are more convenient if we want to allow for propagating solutions, rather that the solely standing wave solutions as dictated by hard-wall boundary conditions. If, in the end, we want to take the limit as $L\to\infty$, then the choice of boundary condition becomes unimportant.

Substituting the normal mode expansion into the wave equation, and using the orthonormality relation, we obtain the equation of motion for the normal mode coefficients,

$$\ddot{q}_k+\omega_k^2 q_k=0,\ \ \omega_k=vk=\sqrt{\frac{Y}{\mu}}\,k_n.$$

Thus, each normal mode evolves as a simple harmonic oscillator with frequency ω_k.

Hamiltonian formulation of the field theory

Let us express the kinetic and potential energies in terms of the normal modes. For the discussion to follow I will assume the expansion associated with hard wall boundary conditions since the mode functions and coefficients are real, thereby allowing for simpler calculations. In the end I will generalize this to allow for periodic boundary conditions.

Substituting the normal mode expansion into the kinetic energy on Page 8,

$$T = \int dx \ \frac{1}{2}\mu\left(\frac{\partial \eta}{\partial t}\right)^2 = \sum_{k,k'} \frac{1}{2}\mu L\, \dot{q}_k \dot{q}_{k'} \underbrace{\int dx\, u_k(x) u_{k'}(x)}_{\delta_{kk'}} = \sum_k \frac{1}{2} M \dot{q}_k{}^2 ,$$

where M=μL. Similarly, for the potential energy

$$\begin{aligned} V &= \int dx \ \frac{1}{2} Y\left(\frac{\partial \eta}{\partial x}\right)^2 = \sum_{k,k'} \frac{1}{2} Y L q_k q_{k'} \int dx\, (\partial_x u_k)(\partial_x u_{k'}) \\ &= \sum_{k,k'} \frac{1}{2} Y L q_k q_{k'} \int dx\, (-\partial^2{}_x u_k) u_{k'}(x) \\ &= \sum_{k,k'} \frac{1}{2} Y L q_k q_{k'}\, k^2 \delta_{k'k} = \sum_k \frac{1}{2} Y \left(\frac{\omega_k}{v}\right)^2 L q_k{}^2 \\ &= \sum_k \frac{1}{2} M \omega_k{}^2 q_k{}^2 . \end{aligned}$$

In the second line we used integration by parts. In the third line we used the normal mode equation on Page 8 and the orthonormality relation. Finally we used the dispersion relation relating k and ω_k, and the relation $v = \sqrt{Y/\mu}$.

Thus, the Lagrangian for the field can be written as

$$L = T - V = \sum_k \left(\frac{1}{2} M \dot{q}_k{}^2 - \frac{1}{2} M \omega_k{}^2 q_k{}^2\right).$$

The canonical momenta are $p_k = \dfrac{\partial L}{\partial \dot{q}_k} = M \dot{q}_k$, and the Hamiltonian is

$$H(\{p_k, q_k\}) = T(q_k = \frac{p_k}{M}) + V(q_k) = \sum_k \left(\frac{p_k{}^2}{2M} + \frac{1}{2} M \omega_k{}^2 q_k{}^2\right).$$

Thus we see explicitly: *the Hamiltonian for the field is equivalent to that for a collection of simple harmonic oscillators , where each oscillator is associated with a normal mode of the field*. Following our analysis of a single oscillator, we define the characteristic units for each mode, $E_{0,k} = \hbar\omega_k$, $q_{0,k} = \sqrt{2\hbar / m\omega_k}$, $p_{0,k} = \sqrt{2m\hbar\omega_k}$, and dimensionless phase space coordinates, $Q_k = q_k / q_{0,k}$, $P_k = p_k / p_{0,k}$. The complex amplitude for each mode is defined analogously, $\alpha_k = Q_k + iP_k$, so

$$
\begin{aligned}
Q_k(t) &= Q_k(0)\cos(\omega_k t) + P_k(0)\sin(\omega_k t), \\
P_k(t) &= P_k(0)\cos(\omega_k t) + Q_k(0)\sin(\omega_k t), \\
\alpha_k(t) &= Q_k(t) + iP_k(t) = \alpha_k(0)e^{-i\omega_k t}.
\end{aligned}
$$

Expressed in terms of the dimensionless coordinates and complex amplitude

$$
\begin{aligned}
H &= \sum_k \hbar\omega_k \left({Q_k}^2 + {P_k}^2 \right) = \sum_k \hbar\omega_k \, {\alpha_k}^* \alpha_k \\
\eta(x,t) &= \sqrt{L}\sum_k q_k(t)\,u_k(x) = \sum_k \sqrt{L} q_{0,k} \frac{\alpha_k(t) + {\alpha_k}^*(t)}{2} u_k(x) \\
&= \sum_k \sqrt{\frac{\hbar}{\mu\omega_k L}} \left(\alpha_k(0)\sin(kx)e^{-i\omega_k t} + {\alpha_k}^*(0)\sin(kx)e^{i\omega_k t} \right),
\end{aligned}
$$

where in the in the final step we substituted for the scale length $q_{0,k}$, and the normal mode function (for hard-wall boundary conditions).

Aside: If we had followed the calculation through with *periodic boundary conditions*, we would have found the following results:

$$
\begin{aligned}
L &= \sum_k \left(\frac{1}{2} M\dot{q}_k \dot{q}_{-k} - \frac{1}{2} M\omega_k{}^2 q_k q_{-k} \right), \quad p_k = \frac{\partial L}{\partial \dot{q}_k} = M\dot{q}_{-k} \\
H(\{p_k, q_k\}) &= \sum_k \left(\frac{p_k p_{-k}}{2M} + \frac{1}{2} M\omega_k{}^2 q_k q_{-k} \right) = \sum_k \hbar\omega_k \left(Q_k Q_{-k} + P_k P_{-k} \right) \\
\alpha_k &= Q_k + iP_{-k}, \qquad H = \sum_k \hbar\omega_k \, {\alpha_k}^* \alpha_k \\
\eta(x,t) &= \sqrt{L}\sum_k q_k(t)\,u_k(x) = \sum_k \sqrt{L} q_{0,k} \frac{\alpha_k(t) + {\alpha_{-k}}^*(t)}{2} u_k(x) \\
&= \sum_k \sqrt{\frac{\hbar}{2\mu\omega_k L}} \left(\alpha_k(0)e^{i(kx-\omega_k t)} + {\alpha_k}^*(0)e^{-i(kx-\omega_k t)} \right).
\end{aligned}
$$

For either choice of boundary condition, we have the general relations,

$$H = \sum_k \hbar\omega_k \, \alpha_k{}^* \alpha_k$$

$$\eta(x,t) = \sum_k \sqrt{L q_{0,k}^2} \left(\alpha_k(0) e^{-i\omega_k t} u_k(x) + \alpha_k{}^*(0) e^{i\omega_k t} u_k{}^*(x) \right)$$

The Quantized Scalar Field

In order to quantize the field we replace each normal mode oscillator by its quantum counter part:

$$q_k \to \hat{q}_k, \;\; p_k \to \hat{p}_k, \;\; \alpha_k \to \hat{a}_k,$$

and we impose the canonical commutation relations on each mode,

$$[\hat{q}_k, \hat{p}_{k'}] = i\hbar\delta_{kk'}, \; [\hat{a}_k, \hat{a}_{k'}^\dagger] = \delta_{kk'}, \;\; [\hat{a}_k, \hat{a}_{k'}] = 0 \, .$$

Note that since each normal mode is an independent degree of freedom, any two operators associated with different modes commute, as is inforced by the Kronecker delta functions in the commutation relations above.

The Hamiltonian for the system is then,

$$\hat{H} = \sum_k \hbar\omega_k \left(\hat{a}_k^\dagger \hat{a}_k + \frac{1}{2} \right),$$

and the quantum field is

$$\hat{\eta}(x) = \sqrt{L} \sum_k \hat{q}_k \, u_k(x) = \sum_k \sqrt{L q_{0,k}^2} \left(\hat{a}_k \, u_k(x) + \hat{a}_k^\dagger u_k^*(x) \right).$$

It is customary to define a "canonical momentum" field, $\hat{\pi}(x)$, conjugate to the field $\hat{\eta}(x)$,

$$\hat{\pi}(x) = \frac{1}{\sqrt{L}} \sum_k \hat{p}_k \, u_k(x) = \; -i \sum_k \sqrt{\frac{p_{0,k}^2}{L}} \left(\hat{a}_k \, u_k(x) - \hat{a}_k^\dagger u_k^*(x) \right).$$

Then, the commutation relation between the field and its canonical conjugate field is

$$[\hat{\eta}(x),\hat{\pi}(x')] = \sum_{k,k'}[\hat{q}_k,\hat{p}_{k'}]u_k(x)u_{k'}(x') = i\hbar\sum_k \delta_{k,k'}\ u_k(x)u_{k'}(x') = i\hbar\,\delta(x-x'),$$

where in the last step we used the completeness of the normal mode functions. This relation is known as the canonical commutation relation for fields. An alternative more sophisticated method for quantizing the field is to start with the Lagrangian for the field in x-space, as on space 8, define the conjugate momentum field $\pi(x)$ as the "functional derivative" of the Lagrangian with respect to the function $\eta(x)$, and then quantize by imposing the field canonical commutator as above. Raising and lowering operators are defined by expanding these fields in terms of their normal modes, and associating the coefficients in this expansion with operators.

The Fock Space of a Quantum Field

We have seen that the quantized field can be interpreted as a collection of quantum harmonic oscillators for each normal mode. The Hilbert space, upon which these operators act, is then the direct product of Hilbert spaces $\{h_k\}$ for each individual mode *k*,

$$\mathcal{H} = h_{k_1}\otimes h_{k_2}\otimes h_{k_3}\otimes\cdots.$$

The Hilbert space $\mathcal{H}$ is known as a **Fock space**.

In the first section of these notes we discussed a complete basis for a single oscillator Hilbert space as the eigenstates of the number operator,

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle,$$

where $|0\rangle$ is the ground state, defined by $\hat{a}|0\rangle = 0$. Thus, a state of the field can be specified by the set of eigenvalues $\{n_{k_1}, n_{k_2}, n_{k_3},\ldots\}$, describing the excitation of each mode k_i with n_{k_i} quanta,

$$\left|\{n_{k1}, n_{k_2}, n_{k_3},\ldots\}\right\rangle = \left|n_{k_1}\right\rangle\otimes\left|n_{k_2}\right\rangle\otimes\left|n_{k_3}\right\rangle\otimes\cdots$$
$$= \frac{(\hat{a}_{k_1}{}^\dagger)^{n_1}}{\sqrt{(n_{k_1})!}}\left|0_{k_1}\right\rangle\otimes\frac{(\hat{a}_{k_2}{}^\dagger)^{n_2}}{\sqrt{(n_{k_2})!}}\left|0_{k_2}\right\rangle\otimes\frac{(\hat{a}_{k_3}{}^\dagger)^{n_3}}{\sqrt{(n_{k_3})!}}\left|0_{k_3}\right\rangle\otimes\cdots$$

The total number operator for the field is defined

$$\hat{N} = \sum_{k_i} \hat{N}_{k_i} = \sum_{k_i} \hat{a}_{k_i}^{\dagger} \hat{a}_{k_i} ,$$

and thus,

$$\hat{N} \left| \{n_{k_1}, n_{k_2}, n_{k_3}, \ldots\} \right\rangle = \sum_{k_i} n_{k_i} \left| \{n_{k_1}, n_{k_2}, n_{k_3}, \ldots\} \right\rangle .$$

A state with a definite number of quantum in each mode is sometimes known as a Fock state, or number state. Spanned over all $\{n_{k_1}, n_{k_2}, n_{k_3}, \ldots\}$, these form a complete basis for the Fock space. The ground state of the field, with zero excitations in all modes, is known as the **vacuum**,

$$|0\rangle \equiv \left|0_{k_1}\right\rangle \otimes \left|0_{k_2}\right\rangle \otimes \left|0_{k_3}\right\rangle \otimes \cdots ,$$

where $\hat{a}_{k_i} |0\rangle = 0,$ for all k_i. Thus, we can write the state with excitations $\{n_{k_1}, n_{k_2}, n_{k_3}, \ldots\}$,

$$\left| \{n_{k_1}, n_{k_2}, n_{k_3}, \ldots\} \right\rangle = \prod_{k_i} \frac{(\hat{a}_{k_i}{}^{\dagger})^{n_i}}{\sqrt{(n_{k_i})!}} |0\rangle ,$$

Then

$$\hat{a}_{k_i} \left| \{n_{k_1}, n_{k_2}, \ldots, n_{k_i}, \ldots\} \right\rangle = \sqrt{n_{k_i}} \left| \{n_{k_1}, n_{k_2}, \ldots, n_{k_i} - 1, \ldots\} \right\rangle ,$$
$$\hat{a}_{k_i}^{\dagger} \left| \{n_{k1}, n_{k_2}, \ldots, n_{k_i}, \ldots\} \right\rangle = \sqrt{n_{k_i} + 1} \left| \{n_{k1}, n_{k_2}, \ldots, n_{k_i} + 1, \ldots\} \right\rangle .$$

In other words, the operator $\hat{a}_{k_i}$ destroys one quantum of excitation in the mode k_i, and is known as the **annihilation operator**. Similarly $\hat{a}_{k_i}^{\dagger}$ creates an excitation in that mode, and is therefore known as the **creation operator**.

Vacuum Fluctuations

We saw in on Page 6 that observes measured in the ground state of a single oscillator have quantum mechanical uncertainties associated with them. That is, although $\langle 0|\hat{q}|0\rangle = \langle 0|\hat{p}|0\rangle = 0$, we can show that $\langle 0|\hat{q}^2|0\rangle \neq 0,$ $\langle 0|\hat{p}^2|0\rangle \neq 0$, due to the noncommutivity of a and $a^{\dagger}$. This fact means that the ground state had a zero point energy of one half the quantum of energy of the oscillator, $E_{ground} = \hbar\omega / 2$.

These facts have important implications for the quantum field, and other systems that interact with the quantum field. In this case, the ground state is the vacuum $|0\rangle$ defined on Page 15. The expectation value of the field vanishes in the vacuum

$$\langle 0|\hat{\eta}(x)|0\rangle = \sum_k \sqrt{Lq_{0,k}^2}\left(\langle 0|\hat{a}_k|0\rangle u_k(x) + \langle 0|\hat{a}_k^\dagger|0\rangle u_k^*(x)\right) = 0\,,$$

where we used $\hat{a}_k|0\rangle = \langle 0|\hat{a}_k^\dagger = 0$. However, the fluctuations of the field do not vanish

$$\begin{aligned}\langle 0|\hat{\eta}(x)^2|0\rangle &= \sum_{k,k'} L\, q_{0,k}q_{0,k'}\langle 0|\hat{a}_k\hat{a}_{k'}^\dagger|0\rangle u_k(x)u_{k'}^*(x)\\ &= \sum_{k,k'} L\, q_{0,k}q_{0,k'}\langle 0|\left(\hat{a}_{k'}^\dagger\hat{a}_k + [\hat{a}_k,\hat{a}_{k'}^\dagger]\right)|0\rangle u_k(x)u_{k'}^*(x)\\ &= \sum_{k,k'} Lq_{0,k}^2\ |u_k(x)|^2 \neq 0\,.\end{aligned}$$

In the first line we includes the only nonvanishing term in the expectation value, and in the final line we used the canonical commutation relation. Thus, even in the vacuum the quantized field will have some finite fluctuations, $\Delta\eta(x)$, because of the uncertainty principle. These fluctuations are known as **vacuum fluctuation**.

If we calculate the energy associated with this vacuum fluctuations by taking the expectation value of the Hamiltonian we find,

$$E_{vac} = \langle 0|\hat{H}|0\rangle = \sum_k \hbar\omega_k\left(\langle 0|\hat{a}_k^\dagger\hat{a}_k|0\rangle + \frac{1}{2}\right) = \sum_k \frac{\hbar\omega_k}{2} = \infty\,.$$

That is, each mode of the field carries a zero point energy of $\hbar\omega_k\ /\ 2$, and for an infinite number of modes, the total vacuum energy would be infinite. A proper treatment of this divergence would require the sophisticated theory of renormalization. For our purpose, we will redefine the zero of energy as that of the vacuum. This is not to say that the vacuum fluctuation energy has no observable effects. In fact, if we could somehow restrict the number of possible normal modes that could be excited in our field then we would change the zero-point fluctuations. We will see that vacuum fluctuations of the electromagnetic field play a central role in the decay of an atom from its excited to ground state.

The Particle Interpretation of a Quantum Field

In the modern description of the quantized field, each quantum of excitation is associated with a **particle**. For the case of our scalar field describing vibrations on an elastic rod, these particles are known as phonons, familiar in the description of vibrations in condensed matter.

quantum of vibrational excitation $\Leftrightarrow$ phonon

These particles have all the properties we usually associate with quantum particles. For example, we can define wave packet states of a single phonon

$$\begin{aligned}\left|\{f_{k_i}\}\right\rangle &= \sum_{k_i} f_{k_i}\left|1_{k_i}\right\rangle \\ &= f_{k_1}\left|1_{k_1},0_{k_2},0_{k_3},\ldots\right\rangle + f_{k_2}\left|0_{k_1},1_{k_2},0_{k_3},\ldots\right\rangle + f_{k_3}\left|0_{k_1},0_{k_2},1_{k_3},\ldots\right\rangle + \cdots.\end{aligned}$$

This state is a one particle state, as can be confirmed by applying the total number operator to this state $\widehat{N}\left|\{f_{k_i}\}\right\rangle = 1\left|\{f_{k_i}\}\right\rangle$. However it is neither a eigenstate of the energy or momentum operators.

We can also consider general two particle states, described by two wave packets $\{f_{k_i}\}$ and $\{g_{k_i}\}$. Of course quantum mechanics imposes *symmetry* considerations on the two particle wave function. That is if the particles are **bosons** then the wave function must be symmetric when the coordinates of the two particles are exchanged, $\psi(x_1,x_2) = \psi(x_2,x_1)$; if there were **fermions** they must satisfy the Pauli exclusion principle, which is enforced by the anti-symmetry of the two-particle wave function $\psi(x_1,x_2) = -\psi(x_2,x_1)$. One of the great achievements of relativistic quantum field theory was to establish the fundamental connection between the statistics (Bose vs. Fermi) of a particle and its spin. The spin of course is the measure of the intrinsic angular momentum carried by a particle. From an abstract point of view the intrinsic angular momentum is defined by the way in which the field transform under rotations, that is its vector nature. The field under consideration here is a scalar, and thus carrier no angular momentum. Therefore, the phonons are zero spin particles, and are thus bosons. An amazing part of the mathematics associate with the Fock space is that the required symmetry of the wave functions is accounted for by the commutation relations of the field operators! In fact we have secretly assumed that our field was associated with bosons by instituting the canonical commutation relations,

$$[\hat{a}_k,\hat{a}^\dagger_{k'}] = \hat{a}_k\,\hat{a}^\dagger_{k'} - \hat{a}^\dagger_{k'}\,\hat{a}_k = \delta_{kk'}: \qquad \textbf{BOSONS}$$

If the field was a vector field whose quanta were fermions, then we would be require to institute *anti*-canonical commutation relations,

$$\{\hat{a}_k, \hat{a}^\dagger_{k'}\} = \hat{a}_k\, \hat{a}^\dagger_{k'} + \hat{a}^\dagger_{k'}\, \hat{a}_k = \delta_{kk'}: \qquad \textbf{FERMIONS}$$

The fact that the Fock space formalism can account for the spin-symmetry automatically without explicit symmetrization of the wave function, is a powerful tool used in problems other than a quantum field theory. In fact, for complicated many-body systems in condensed matter physics, such as electrons in a solid, one often employs a reverse procedure that is the reverse of the one we've introduced here. Where we have associated a collection of particles with a quantum field, in condensed mater theory one often associates a quantum field with the collection of particles!