

## Physics 566 Quantum Optics I

### Lecture 13: Introduction to Stochastic Processes

Quantum mechanics is distinguished by two defining properties

- (i) Intrinsic randomness.
- (ii) Interference of indistinguishable processes.

However randomness and interference are features of classical physics and distinguishing those phenomena that have a uniquely quantum origin is a subtle business that has been at the heart of much of the activity in quantum optics since its inception. Certain nonclassical aspects arise from the corpuscular nature of the field, something not present in the classical theory. But this non-classicality can show up in a minimal way, as in Boltzmann's kinetic theory of gases, where the discretization of the fluid into discrete molecules leads to statistical fluctuations. Indeed, Einstein's introduction of "light quanta," i.e. photons, in 1905 arose from his derivation of the Planck law (actually the Wien approximation at short wavelengths) based on the Boltzmann formula for the entropy of a gas of photons; and later for the fluctuation in the energy of blackbody which manifestly has a "wave component" and a "particle component" as we will see when we study the quantum theory of light. Unravelling these various stochastic and wave features and those that have a uniquely quantum description, unavailable in a classical dynamical description, will be one of the important tasks that we will pursue in our studies.

To set the foundation let us begin with a review of some fundamental theory of probability and stochastic processes.

## Probability

A "random variable"  $X$  can take on a value  $x \in S$  for some set  $S$  of possible outcomes.  $S$  can be a discrete set or a continuous set. To each  $x \in S$  we assign a probability  $p(x) \geq 0$ . If  $S$  is discrete (countable), let  $S = \{x_i | i=1, 2, \dots, d\}$  ( $d$  can be  $\infty$ ), then  $p_i \equiv p(x_i)$  is the probability of outcome  $i$ . For the continuous case,  $p(x) dx$  is the probability to have an outcome somewhere in the measurable interval  $x \rightarrow x+dx$ . Then  $p(x)$  is a "probability density" with units  $[\frac{1}{x}]$ . General a probability distribution is normalized:  $\sum_{i=1}^d p_i = 1$ ,  $\int_{x \in S} dx p(x) = 1$ .

### - Probability and logic

Probability is the language of logic. The basic axiom of probability is that if two events  $A$  and  $B$  are mutually exclusive, then  $P(A \text{ or } B) = P(A) + P(B)$ . Quantum mechanics defies logic. Interference means that what appear to be mutually exclusive possibilities, e.g. a photon going one way or other at a beam splitter is not correct. We define nonclassical phenomena to be those that can't be described by probability.

### - Moments and Cumulants

The expected value  $\langle x \rangle = \sum_i x_i p_i$ , and the uncertainty is the root-mean-square deviation  $\Delta x = \sqrt{\Delta x^2}$ ,  $\Delta x^2 = \langle (x - \langle x \rangle)^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2$ : Variance. Generally we define the moments of the distribut  $\langle x^n \rangle = \sum_i (x_i)^n p_i$ . A probability distribution is defined by all of its moments. We define the "moment generating function" or "characterist function"

$$\text{Characteristic fct: } \chi(k) \equiv \langle e^{ikx} \rangle = \sum_i e^{ikx_i} p_i = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle x^n \rangle$$

The characteristic function is the Fourier transform of the probability distribution function and so is well defined since  $\sum_i p_i^2$  is finite (Note, the moment generating function is sometimes defined  $G(k) = \langle e^{kx} \rangle$ , but this not necessarily converge). The moments and the coefficients in the Taylor series around  $k=0$

$$\text{Moments: } \langle x^n \rangle = (-i)^n \left. \frac{d^n \chi(k)}{d k^n} \right|_{k=0}$$

The "cumulants"  $\langle k^n \rangle$  are defined through the generating function

$$\text{Cumulant generating function: } H(k) \equiv \log \chi(k) = \log \langle e^{ikx} \rangle \equiv \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \langle k^n \rangle.$$

Thus  $\langle k^1 \rangle = \langle x \rangle$ ,  $\langle k^2 \rangle = \langle x^2 \rangle - \langle x \rangle^2 = \Delta x^2$ ,  $\langle k^3 \rangle = \langle (x - \langle x \rangle)^3 \rangle$   
 Generally  $\langle k^n \rangle \neq \langle (x - \langle x \rangle)^n \rangle$  for  $n \geq 4$ . The relation to moments is not simple.

### - Multivariate Probability

Given two random variables,  $X, Y$ , we can define the joint probability  $p(x, y) = p(x \cup y)$ .

- If  $X$  and  $Y$  are "statistically independent" then  $p(x, y) = p(x)p(y)$

- Conditional probability  $p(x|y) = p(x)$  Given  $p(y) = 1 = \frac{p(x, y)}{p(y)}$ ; also  $p(y|x) = \frac{p(x, y)}{p(x)}$

$$\Rightarrow \text{Bayes rule } p(y|x) = \frac{p(x|y)p(y)}{p(x)}$$

- "Marginal probability"  $p(x_i) = \sum_j p(x_i, y_j)$ ;  $p(y_j) = \sum_i p(x_i, y_j)$

Generalize to  $N$ -random variables  $\vec{X} = (X_1, X_2, \dots, X_N)$ ;  $p(\vec{x}) = p(x_1 \cup x_2 \cup \dots \cup x_n)$ .

Statistical independence  $\Rightarrow p(\vec{x}) = \prod_{i=1}^N p(x_i)$ .

"Marginalize" over  $n$  variables, e.g.  $p(x_1, x_2) = \sum_{\{x_3, x_4, \dots, x_N\}} p(\vec{x})$

Aside Let  $Y = \sum_{i=1}^N X_i$  (random variable)  $\Rightarrow$

$\Rightarrow \langle y \rangle = \sum_i \langle x_i \rangle$ ,  $\Delta y^2 = \sum_i \Delta x_i^2$ , if  $\{x_i\}$  are statistically indep.

Multivariate characteristic function  $\chi(\vec{k}) \equiv \langle e^{i\vec{k} \cdot \vec{x}} \rangle = \sum_{\{\vec{x}\}} p(\vec{x}) e^{i\vec{k} \cdot \vec{x}}$

$\Rightarrow$  Multivariate moments: E.g.  $\langle x_i^n x_j^m \rangle = (-i)^{n+m} \frac{\partial^{n+m}}{\partial k_i^n \partial k_j^m} \chi(\vec{k}) \Big|_{\vec{k}=0}$

Covariance  $\langle \Delta x_i \Delta x_j \rangle = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle = \langle x_i x_j \rangle - \langle x_i \rangle \langle x_j \rangle$   
 $= \langle \Delta x_i^2 \rangle \delta_{ij}$  if  $x_i$  &  $x_j$  statistically independent.

### - Some fundamental probability distributions

• Bernoulli trial  $S = \{0, 1\}$   $p(x) = \begin{cases} p & x=1 \\ q=1-p & x=0 \end{cases}$

$$\langle x \rangle = p \quad \Delta x = \langle x^2 \rangle - \langle x \rangle^2 = p^2 - p = p q$$

• Binomial distribution  $k = \sum_{i=1}^N x_i$ ,  $X_i = \text{Bernoulli trial}$   $S = \{0, 1, \dots, N\}$

$$p(k) = \binom{N}{k} p^k q^{N-k} = \frac{N!}{k!(N-k)!} p^k (1-p)^{N-k} : \begin{array}{l} N \text{ statistically independent} \\ \text{Bernoulli trial with } k \text{ 1's} \\ \text{N-k 0's} \end{array}$$

$$\langle k \rangle = Np, \quad \Delta k^2 = Np(1-p) \quad (\text{From } N\text{-statistically independent } x_i)$$

• Poisson distribution:

Limit of binomial distribution with  $N \rightarrow \infty$ ,  $p \rightarrow 0$ , such that  $Np \rightarrow \Lambda$

(Infinitely many Bernoulli trials with infinitesimal probability):

Random variable  $k \in \{0, 1, \dots, \infty\}$  (nonnegative integers)

$$p(n) = \lim_{\substack{N \rightarrow \infty, p \rightarrow 0 \\ Np \rightarrow \Lambda}} \frac{N!}{(N-n)! n!} p^n (1-p)^{N-n} = \frac{\Lambda^n e^{-\Lambda}}{n!}$$

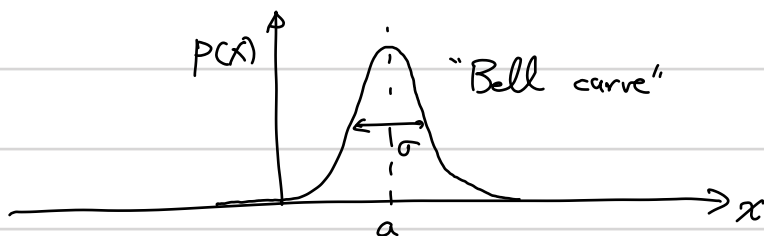
$$\langle n \rangle = \Lambda, \quad \Delta n^2 = \langle n \rangle = \Lambda$$

• Gaussian (normal) distribution

Random variable  $X \in \mathbb{R}$  (reals): Continuous variable

$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-a)^2}{2\sigma^2}} \quad \text{Normalized: } \int_{-\infty}^{\infty} dx p(x) = 1$$

$$\langle x \rangle = a \quad \Delta x = \sigma$$



The normalized form of Gaussian is something every physicist should know by memory.

A Gaussian is fully specified by its first two moments  $\langle x \rangle$  and  $\langle x^2 \rangle$

Aside: A very useful Gaussian integral in complex plane.

$$\int_{-\infty}^{\infty} dx e^{-\frac{1}{2s^2}(x-z)^2} = \sqrt{2\pi s^2}, \quad s, z \text{ complex, } \text{Re}(s^2) \geq 0$$

$$\text{Characteristic Function } \chi(k) = e^{ika} e^{-\frac{\sigma^2 k^2}{2}} \Rightarrow i \frac{\partial \chi}{\partial k} \Big|_k = \langle x \rangle = 0, \quad -\frac{\partial^2 \chi}{\partial k^2} \Big|_0 = \langle x^2 \rangle = \sigma^2 + a^2 = \Delta x^2 + \langle x \rangle^2$$

$$\text{Cumulant generator } H(k) = \log(\chi(k)) = ika - \frac{\sigma^2}{2} k^2$$

$$\Rightarrow \langle k^1 \rangle = a = \langle x \rangle, \quad \langle k^2 \rangle = \sigma^2 = \Delta x^2, \quad \langle k^n \rangle = 0 \quad n \geq 3$$

⇒ All cumulants of a Gaussian vanish for  $n \geq 3$ . The higher order cumulants are thus a measure of how different the moments are from those of a Gaussian.

When  $\langle x \rangle = 0$ ,  $\chi(k) = e^{-\frac{\sigma^2 k^2}{2}}$ . It then follows  $\langle x^n \rangle = \begin{cases} 0 & n\text{-odd} \\ (n-1)!! \sigma^n & n\text{-even} \end{cases}$

Multivariate Gaussian: Joint probability of  $\vec{x} = (x_1, x_2, \dots, x_N) \in \mathbb{R}^N$

$$p(\vec{x}) = \frac{1}{(2\pi \det \mathbb{C})^{1/2}} \exp\left\{-\frac{1}{2} \sum_{i,j}^N (x_i - a_i) \mathbb{C}_{ij}^{-1} (x_j - a_j)\right\}$$

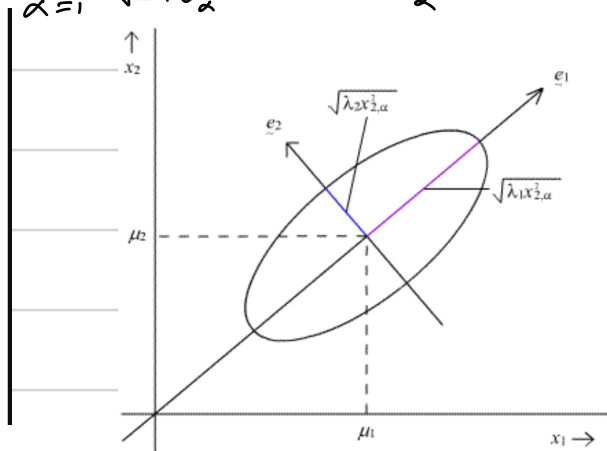
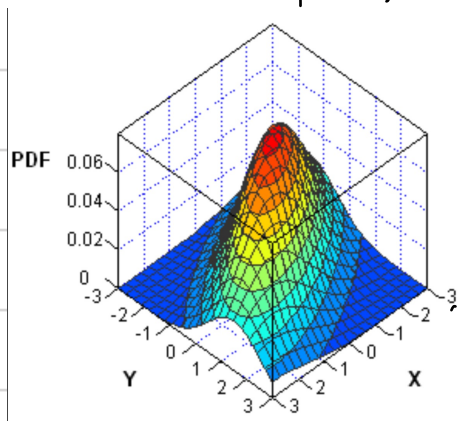
where  $\mathbb{C}$  is a symmetric positive matrix known as the covariance matrix

$$\langle x_i \rangle = a_i \quad \langle \Delta x_i \Delta x_j \rangle = \mathbb{C}_{ij} \quad (\text{covariance})$$

The multivariate Gaussian is fully specified by its mean and covariance matrix. The Gaussian factorizes in terms of the eigenvectors of  $\mathbb{C}$ : Let  $\mathbb{C} \vec{u}^{(\alpha)} = \sigma_\alpha^2 \vec{u}^{(\alpha)}$

$$\text{Let } \vec{x} = \sum \alpha_\alpha \vec{u}^{(\alpha)} \quad \leftarrow \text{coordinates in eigenbasis} \quad \vec{a} = \sum a_\alpha \vec{u}^{(\alpha)}$$

$$\Rightarrow p(\vec{x}) = \prod_{\alpha=1}^N \frac{1}{\sqrt{2\pi\sigma_\alpha^2}} e^{-\frac{(x_\alpha - a_\alpha)^2}{2\sigma_\alpha^2}}$$



Central Limit Theorem:

Let  $Y = \sum_{i=1}^N X_i$  be a random variable, where  $\{X_i\}$  are statistically independent but drawn from the identical probability distribution

We have already seen  $\langle y \rangle = \sum \langle x_i \rangle = N \langle x \rangle$ ,  $\Delta y^2 = \sum \Delta x_i^2 = N \Delta x^2$

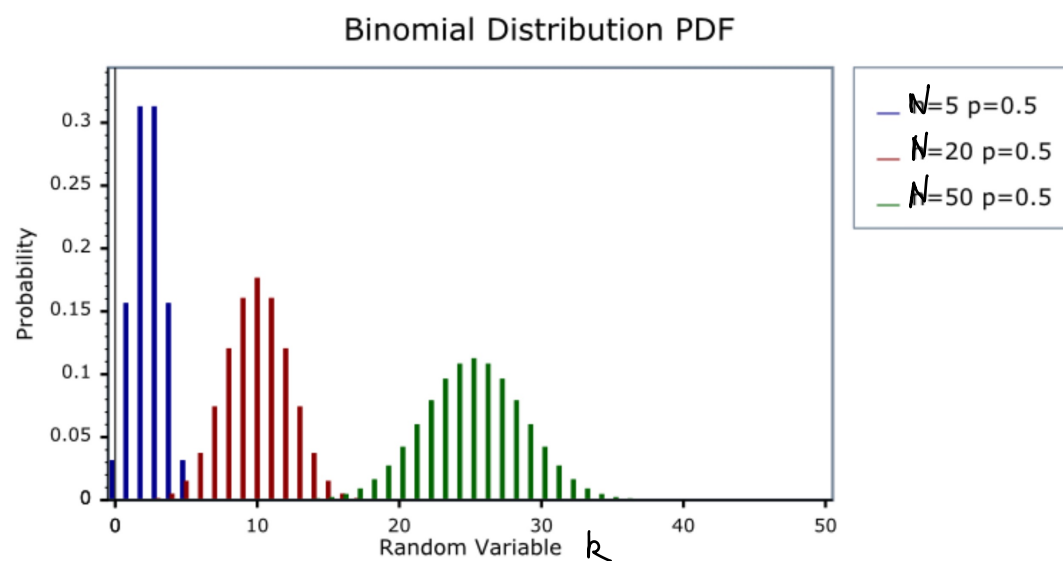
According to the Central Limit theorem, as  $N \rightarrow \infty$

$$p(y) = \frac{1}{\sqrt{2\pi N \Delta x^2}} e^{-\frac{(y - N \langle x \rangle)^2}{2N \Delta x^2}}$$

That is as  $N \rightarrow \infty$   $Y$  is normally distributed, with mean  $N \langle x \rangle$  and variance  $N \Delta x^2$

Example: The binomial distribution is the sum of identical Bernoulli trials of mean  $\langle x \rangle = p$  and variance  $\Delta x^2 = p(1-p)$ . Thus as the number  $N$  of trials  $\rightarrow \infty$ , the binomial distribution converges to a Gaussian

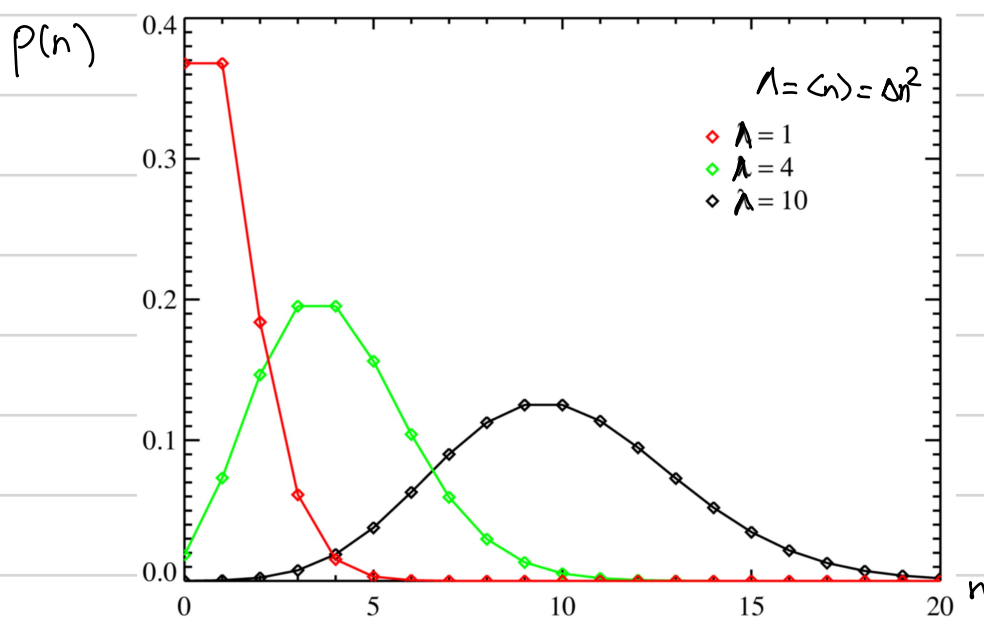
$$\lim_{N \rightarrow \infty} p(k) = \lim_{N \rightarrow \infty} \binom{N}{k} p^k (1-p)^{N-k} \rightarrow \frac{1}{\sqrt{2\pi p(1-p)}} e^{-\frac{(k-Np)^2}{2Np(1-p)}}$$



Example: The Poisson distribution is limit of a binomial distribution, with  $Np \rightarrow \lambda$ . As  $\lambda \rightarrow \infty$ , the Poisson distribution should then converge to a Gaussian

$$\text{Poisson } p(n) = \frac{\lambda^n e^{-\lambda}}{n!} = \frac{\langle n \rangle^n e^{-\langle n \rangle}}{n!}, \quad \langle n \rangle = \lambda \quad \Delta n^2 = \langle n \rangle = \lambda$$

$$\Rightarrow \text{as } \langle n \rangle \rightarrow \infty \quad p(n) \rightarrow \frac{1}{\sqrt{2\pi \langle n \rangle}} e^{-\frac{(n - \langle n \rangle)^2}{2\pi \langle n \rangle}}$$





## Stochastic Process

A Stochastic Process is a random variable as a function of time  $X(t)$ , i.e.,  $X(t)$  is drawn from some probability distribution

$$p(X(t)=x) = p(x,t)$$

Given a time series, on random variable  $X(t_1), X(t_2), \dots, X(t_n)$ , we define the joint probability distribution  $p(x_1 t_1; \dots, x_n t_n; x, t_1)$  typically ordered  $t_n > \dots > t_2 > t_1$ .

Stationary Process: A stochastic process is said to be "stationary" if the "statistics" of  $X(t)$  do not change with time. Formally, the joint probability distribution

$$p(x_1 t_1; \dots, x_n t_n; x, t_1) = p(x_1 t_1+T; \dots, x_n t_n+T; x, t_1+T)$$

Thus, the joint probabilities depend only on the time difference  $t_i - t_j$ , and not the origin of time.

Thus, the one-time probabilities are independent of time  $p(x,t) = p(x)$

the two time probabilities  $p(x_2 t_2; x_1 t_1) = p(x_2 t_2 - t_1, x_1, 0)$

Ergodic Process: Intuitively, a stochastic process is said to be ergodic if over time one uniformly samples the probability distribution. In that case, an ensemble average over the probability distribution is equal to a time average in the limit that the averaging time goes to infinity. A stochastic process can be ergodic only if it is also stationary, and typically we consider "wide-sense" stationary, meaning stationary for one two-point correlations:

$$\Rightarrow \langle X(t) \rangle = \int dx x p(x,t) = \int dx x p(x) \stackrel{\text{stationary}}{=} \lim_{T \rightarrow \infty} \frac{1}{2T} \int_{-T}^T x(t) dt = \bar{x}$$

$$\langle X(t+\tau) X(t) \rangle = \int dx_1 dx_2 x_1 x_2 p(x_1 t; x_2 t+\tau) = \int dx_1 dx_2 x_1 x_2 p(x_1, 0; x_2 \tau)$$

$$= \lim_{T \rightarrow \infty} \int_{-T/2}^{T/2} dt x(t) x(t+\tau) = \overline{X(t) X(t+\tau)}$$

Markov Process: A Markov Process has "no memory!" Thus the probability of  $X$  taking on a value  $x$  at time  $t$  depends only on the value immediately before, and not the whole history.

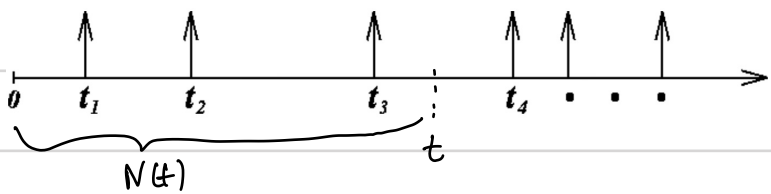
Formally the conditional probability

$$p(x_n t_n | x_{n-1} t_{n-1}; x_{n-2} t_{n-2}; \dots, x_2 t_2; x_1 t_1) = p(x_n t_n | x_{n-1} t_{n-1}) \quad \text{where } t_n > t_{n-1} > t_{n-2} > \dots > t_2 > t_1$$

A "true" Markov process has no memory, even for a differential time interval. There is no such thing physically, but, as we will see, many quantum optical systems are well approximated as Markov processes.

## Example: Poisson Process

A Poisson process describes the counting of random events that arrive at the counter at a constant rate, but event event is random and completely uncorrelated from any other event



(Most randomly spaced events)

This is a stationary, ergodic, Markov process.

We define the random variable  $N(t) = \#$  of counts  $[0, t)$  and  $N(t_a, t_b) = \#$  between  $t_a$  and  $t_b = N(t_b) - N(t_a)$ .  $N(t)$  is a process with rate  $\lambda$  in two equivalent definitions:

- (1) The Probability distribution of  $N(t_a, t_b)$  is a Poisson distribution with mean  $\lambda(t_b - t_a)$ , and every nonoverlapping interval is statistically independent.
- (2) In an infinitesimal time interval  $dt$  there can be no more than one count with probability  $P(N(t, t+dt) = 1) = \lambda dt$ , statically independent of the arrivals outside of this interval.



Let us show that definition (2) leads to the Poisson distribution.

Let  $P(n, t) = \text{Probability}(N(t) = n)$ .

$$P(1, dt) = \lambda dt \quad P(0, dt) = 1 - \lambda dt$$

$$\Rightarrow P(0, t+dt) = P(0, t) P(0, dt) = P(0, t)(1 - \lambda dt) \Rightarrow \frac{dP(0, t)}{dt} = -\lambda P(0, t)$$

$$\Rightarrow P(0, t) = C e^{-\lambda t}, \text{ and since } P(0, t=0) = 1, C = 1$$

By similar reasoning,  $P(n, t+dt) = P(n, t)(1 - \lambda dt) + P(n-1, t) \lambda dt$

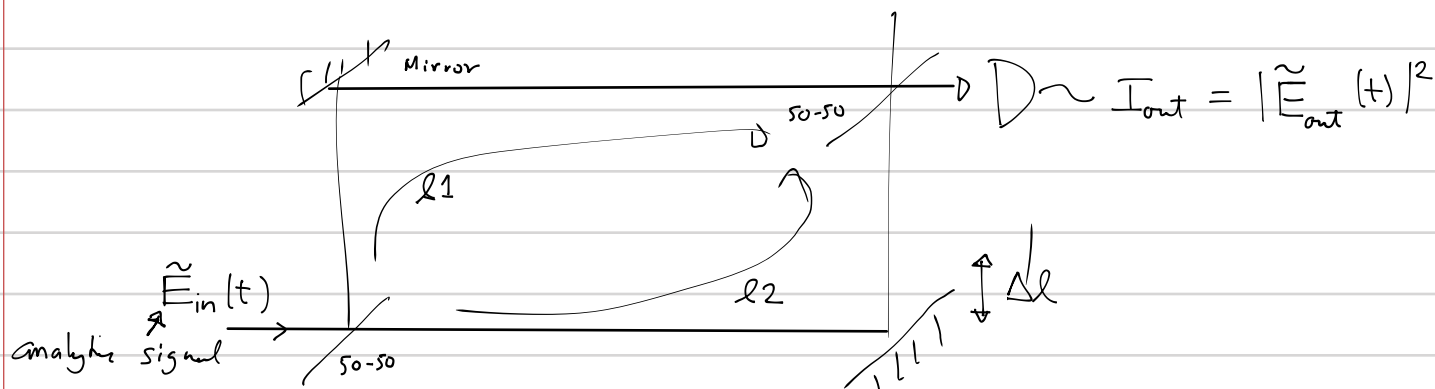
$$\Rightarrow \frac{dP(n, t)}{dt} + \lambda P(n, t) = \lambda P(n-1, t) \Rightarrow \frac{d}{dt} [e^{\lambda t} P(n, t)] = e^{\lambda t} \lambda P(n-1, t)$$

Iterating one finds  $P(n, t) = \frac{(\lambda t)^n}{n!} e^{-\lambda t}$  The Poisson distribution



# Temporal Coherence and Classical Statistical Optics

We have seen that coherence of an electromagnetic field is measured by an <sup>an</sup> interferometer

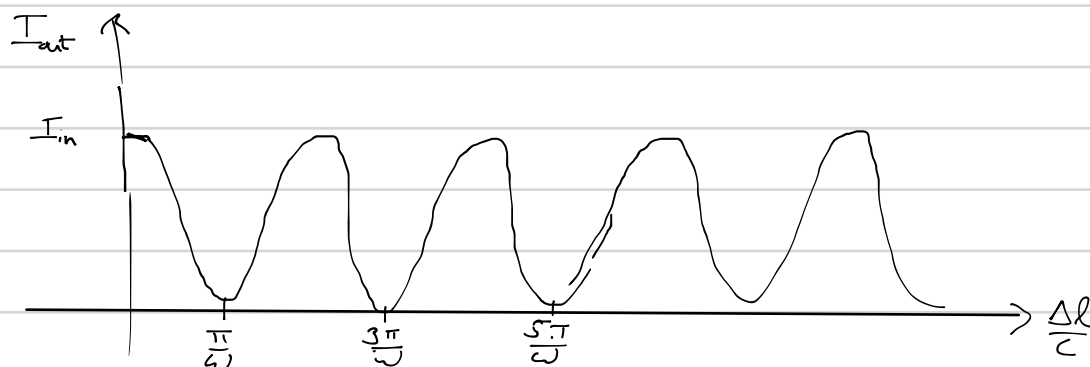


$$\tilde{E}_{out}(t) = -\frac{i}{2} \left( \tilde{E}_{in}(t - \frac{l_1}{c}) + \tilde{E}_{in}(t - \frac{l_2}{c}) \right)$$

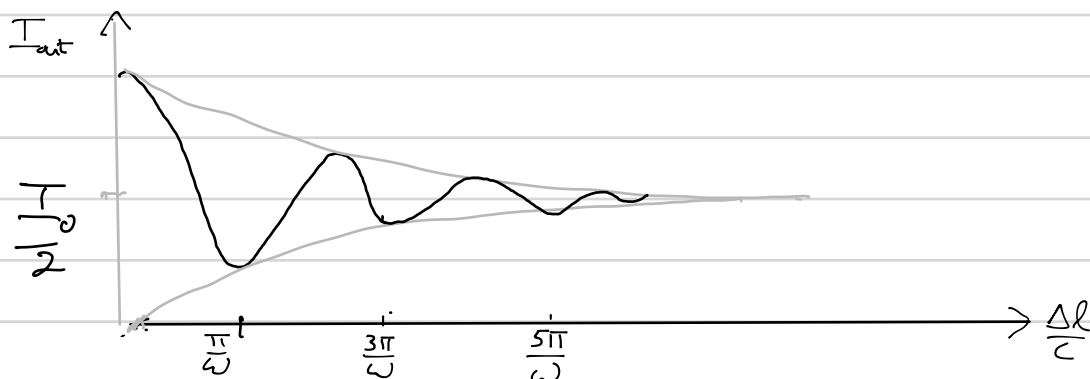
$$\tilde{I}_{out}(t) = \frac{1}{4} I_{in}(t - \frac{l_1}{c}) + \frac{1}{4} I_{in}(t - \frac{l_2}{c}) + \frac{1}{2} \text{Re} \left\langle \tilde{E}_{in}(t - \frac{l_1}{c}), \tilde{E}_{in}^*(t - \frac{l_2}{c}) \right\rangle$$

For a monochromatic signal  $\tilde{E}_{in}(t) = E_0 e^{-i\omega t}$ ,  $I_{in}(t) = |E_0|^2$  (constant)

$$I_{out} = \frac{1}{2} I_{in} + \frac{1}{2} I_{in} \text{Re} \left( e^{i\omega \frac{\Delta l}{c}} \right) = I_{in} \left( \frac{1 + \cos(\frac{\omega \Delta l}{c})}{2} \right)$$



Such a field is said to be perfectly coherent in time. Such a field does not exist in the universe. A more realistic interferometry signal



Why doesn't the interference go on for ever as a function of  $\Delta l$ ?

$\Rightarrow$  Field has only finite coherence, because it is fluctuating

Classical noise. "Natural light": e.g. Starlight or light from a bulb is not arising from a single perfectly oscillating source. It is radiated from an ensemble of "random sources", oscillating with random phases with respect to one another. Moreover, this phase can fluctuate due to other random events

E.g. "Collision-broadened" light source: Consider a gas of radiators and a given pressure, density, and temperature. A molecule in the gas radiates at some given frequency. However, due to collisions between the molecules, the phase is interrupted. We thus have a sequence of random wave trains



From the kinetic theory of gases, the mean-time between collisions

$$(\tau_0)^{-1} = \sigma_0 n \sqrt{\langle v^2 \rangle}, \quad \sigma_0 = \text{collision cross section}, \quad n = \text{density}$$

And the probability the the molecule has uninterrupted oscillation for time  $\tau$  without a collision is  $p(\tau) d\tau = \frac{1}{\tau_0} e^{-\tau/\tau_0} d\tau$

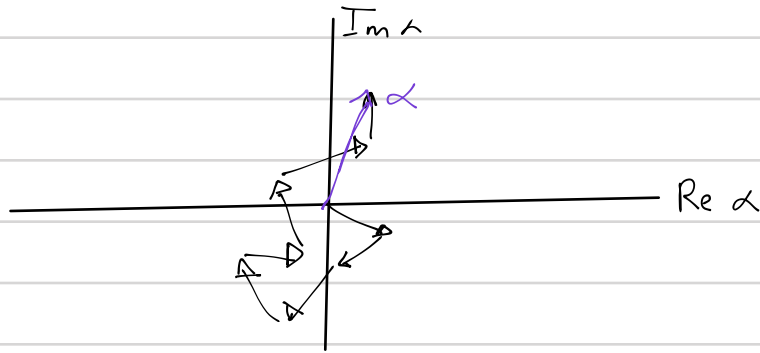
The electric field is thus a random variable, because we don't have full information about the trajectory of the molecules. For a single radiator

$$\tilde{E}_1(t) = E_0 e^{-i\omega t + i\phi_1(t)}, \quad \phi_1(t) \text{ is random}$$

The total field from the gas is the is the sum of the field radiated from all of the atoms, each with its own random phase

$$\begin{aligned} \tilde{E}_c(t) &= \tilde{E}_1(t) + \tilde{E}_2(t) + \dots + \tilde{E}_N(t) \quad (\text{total complex amplitude}) \\ &= E_0 e^{-i\omega t} \underbrace{\sum_{i=1}^N e^{i\phi_i(t)}}_{\text{Random complex number } \propto(t)} \end{aligned}$$

The random phases add up like a random walk in phase space



The probability distribution is thus Gaussian:

$$P(\alpha) = \frac{1}{\pi N} e^{-\frac{|\alpha|^2}{N}} \quad : \quad N\text{-step random walk}$$

The intensity is not constant, it fluctuates.

The interferometer thus measures the ensemble average over all the oscillators

$$I_{int} = \langle |E(t - \frac{\Delta l}{c})|^2 \rangle + \langle |E(t + \frac{\Delta l}{c})|^2 \rangle + 2 \operatorname{Re} \langle E^*(t) E(t + \frac{\Delta l}{c}) \rangle$$

Note: I have assume here that the statistics are "stationary", i.e. the nature of the fluctuations is not changing in time, so the correlations depend only on the time-differences, and not the difference in times.

$$\text{Consider } \langle E^*(t) E(t+\tau) \rangle = E_0^2 e^{-i\omega_0 \tau} \langle \{e^{-i\phi_1(t)} + \dots + e^{-i\phi_N(t)}\} \{e^{i\phi_1(t+\tau)} + \dots + e^{i\phi_N(t+\tau)}\} \rangle$$

The term  $\langle e^{-i\phi_i(t)} e^{i\phi_j(t+\tau)} \rangle_{i \neq j} = 0$ , no phase relationship between different atomic radiators

$$\Rightarrow \langle E^*(t) E(t+\tau) \rangle = N \langle E_i^*(t) E_i(t+\tau) \rangle \text{ for any given molecules}$$

$$\Rightarrow \langle |E(t)|^2 \rangle = N E_0^2 = I_{in} \quad (N \text{ times intensity of given molecule})$$

$$\begin{aligned} \langle E_i^*(t) E_i(t+\tau) \rangle &= E_0^2 e^{-i\omega_0 \tau} N \underbrace{\langle e^{i(\phi(t+\tau) - \phi(t))} \rangle}_{\text{probability that no collision happened for a time of least } \tau} \\ &= \int_{-\infty}^{+\infty} p(\tau) d\tau = e^{-\tau/\tau_0} \end{aligned}$$

$$\Rightarrow \langle E^*(t) E(t+\tau) \rangle = N E_0^2 e^{-i\omega_0 \tau} e^{-\tau/\tau_0} = I_{in} e^{(-i\omega_0 - \frac{1}{\tau_0})\tau}$$

$$\text{The output intensity: } I_{out} = I_{in} \left( \frac{1 + e^{-\tau/\tau_0} \cos(\omega_0 \tau)}{2} \right)$$

$$\text{where } \tau = \frac{\Delta l}{c}$$

We thus see for  $\Delta l/c \gg \tau_0$ , there is no interference.  $\tau_0$  determines the coherence time the field maintains coherence with itself only for time  $\tau$  short compared to  $\tau_0$ .

The key point is that the phase relation between  $E(t)$  and  $E(t+\tau)$  was noisy. This made us lose coherence.

$$\Gamma(\tau) = \langle E^*(t) E(t+\tau) \rangle = E_0 \left( \langle \alpha(t) | \alpha(t+\tau) \rangle \underbrace{e^{i\phi(t+\tau)} e^{-i\phi(t)}}_{\text{randomized phase}} \right)$$

The field is "temporally coherent" when there is phase memory. The effect of the collisions is to erase the phase of oscillators. The time scale over which the oscillators maintain a well determined phase (here the mean time between collisions) sets the coherence time of the field.

The level of coherence in the field is exhibited by the degree of interference can be quantified by the visibility of the interference fringes, defined as

$$V = \frac{I_{max} - I_{min}}{I_{max} + I_{min}} \quad 0 \leq V \leq 1$$

$$\text{For our model of "natural light": } I_{max} = \frac{I_{in}}{2} (1 + e^{-\frac{\Delta l}{c\tau_0}}); \quad I_{min} = \frac{I_{in}}{2} (1 - e^{-\frac{\Delta l}{c\tau_0}})$$

$$\Rightarrow \text{The fringe visibility } V = e^{-\frac{\Delta l}{c\tau_0}} \begin{cases} \rightarrow 1 & \text{for } \Delta l \ll c\tau_0 \text{ (coherence length)} \\ \rightarrow 0 & \text{for } \Delta l \gg c\tau_0 \end{cases}$$