Physics 566: Quantum Optics I
Problem Set #1 Solutions

Problem 1: Trace operations

(a) Given a basis, $\mathrm{Tr}(\hat{A}) = \sum_{i=1}^{d} \langle e_i | \hat{A} | e_i \rangle$. Consider another orthonormal basis $\{|f_i\rangle\}$ which forms a resolution of the identity $\sum_i |f_i\rangle\langle f_i| = \hat{\mathbb{1}}$

$$\Rightarrow \mathrm{Tr}(\hat{A}) = \sum_{i,j,k} \langle e_i | f_j \rangle \langle f_j | \hat{A} | f_k \rangle \langle f_k | e_i \rangle = \sum_{i,j,k} \langle f_k | \underbrace{e_i \rangle \langle e_i}_{} | f_j \rangle \langle f_j | \hat{A} | f_k \rangle$$

Summing over $i \Rightarrow$ identity

$$\Rightarrow \mathrm{Tr}(\hat{A}) = \sum_{j,k} \underbrace{\langle f_k | f_j \rangle}_{\delta_{k,j}} \langle f_j | \hat{A} | f_k \rangle = \sum_{j=1}^{d} \langle f_j | \hat{A} | f_j \rangle = \text{sum diagonal elements.}$$

(b) Some basic properties

(i) $\mathrm{Tr}(\hat{A}^\dagger) = \sum_i \langle e_i | \hat{A}^\dagger | e_i \rangle = \sum_i (\langle e_i | \hat{A} | e_i \rangle^*) = \left(\sum_i \langle e_i | \hat{A} | e_i \rangle\right)^* = \mathrm{Tr}(\hat{A})^*$

(ii) $\mathrm{Tr}(\hat{A}+\hat{B}) = \sum_i \langle e_i | \hat{A}+\hat{B} | e_i \rangle = \sum_i \langle e_i | \hat{A} | e_i \rangle + \sum_i \langle e_i | \hat{B} | e_i \rangle = \mathrm{Tr}(\hat{A}) + \mathrm{Tr}(\hat{B})$

(iii) $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \sum_i \langle e_i | \hat{A}\hat{B}\hat{C} | e_i \rangle = \sum_{i,j,k} \langle e_i | \hat{A} | e_j \rangle \langle e_j | \hat{B} | e_k \rangle \langle e_k | \hat{C} | e_i \rangle$

$$= \sum_{i,j,k} \langle e_k | \hat{C} | \underbrace{e_i \rangle \langle e_i}_{\text{complete}} | \hat{A} | \underbrace{e_j \rangle \langle e_j}_{\text{complete}} | \hat{B} | e_k \rangle = \sum_k \langle e_k | \hat{C}\hat{A}\hat{B} | e_k \rangle = \mathrm{Tr}(\hat{C}\hat{A}\hat{B})$$

$$= \sum_{i,j,k} \langle e_j | \hat{B} | \underbrace{e_k \rangle \langle e_k}_{\text{Complete}} | \hat{C} | \underbrace{e_i \rangle \langle e_i}_{\text{complete}} | \hat{A} | e_j \rangle = \sum_j \langle e_j | \hat{B}\hat{C}\hat{A} | e_j \rangle = \mathrm{Tr}(\hat{B}\hat{C}\hat{A})$$

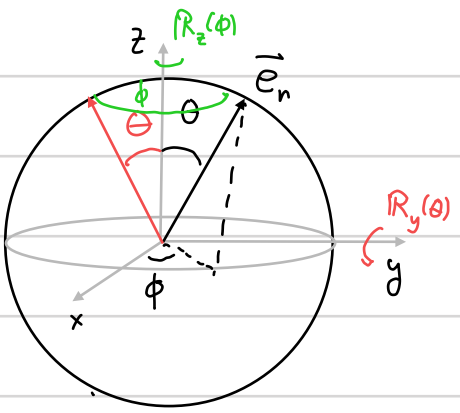
(iv) $\mathrm{Tr}(|\phi\rangle\langle\psi|) = \sum_i \langle e_i | \phi \rangle \langle \psi | e_i \rangle = \sum_i \langle \psi | e_i \rangle \langle e_i | \phi \rangle = \langle \psi | \phi \rangle$

(v) $\mathrm{Tr}(|\phi\rangle\langle\psi|\hat{A}) = \sum_i \langle e_i | \phi \rangle \langle \psi | \hat{A} | e_i \rangle = \sum_i \langle \psi | \hat{A} | e_i \rangle \langle e_i | \phi \rangle = \langle \psi | \hat{A} | \phi \rangle$

(vi) $\hat{A} = \sum_a a |a\rangle\langle a|$ (spectral decomposition) $\mathrm{Tr}(\hat{A}) = \sum_a a\, \mathrm{Tr}(|a\rangle\langle a|) = \sum_a a \langle a | a \rangle = \sum_a a$

$= \text{Sum of eigenvalues}$

Problem 2: Spin-½ along arbitrary direction



The unit vector $\vec{e}_n$ specified by the polar angle θ and azimuthal direction ϕ can be obtained by taking the vector along z, $\vec{e}_z$ and rotating it around the y-axis by θ, followed by a rotation around z by ϕ: $\vec{e}_n = \mathbb{R}_z(\phi)\,\mathbb{R}_y(\theta)\,\vec{e}_z$

This implies the state "spin-up along $\vec{e}_n$" is $|\uparrow_n\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)\,|\uparrow_z\rangle$ ← spin up along z

where $\hat{D}_m(\alpha) = e^{-i\alpha\vec{e}_m\cdot\vec{S}/\hbar} = e^{-i\frac{\alpha}{2}\hat{\sigma}_m} = \cos(\frac{\alpha}{2})\hat{\mathbb{1}} - i\sin(\frac{\alpha}{2})\hat{\sigma}_m$ is the rotation operator for spin-½ (SU(2) rotation)

(a) $|\uparrow_n\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)\,|\uparrow_z\rangle = \hat{D}_z(\phi)\left(\cos(\frac{\theta}{2})\hat{\mathbb{1}} - i\sin(\frac{\theta}{2})\hat{\sigma}_y\right)|\uparrow_z\rangle$

Aside $\hat{\sigma}_y = \frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i} \Rightarrow \hat{\sigma}_y|\uparrow_z\rangle = +i|\downarrow_z\rangle$ (since $\hat{\sigma}_+|\uparrow_z\rangle = 0$)

$$\Rightarrow |\uparrow_n\rangle = \hat{D}_z(\phi)\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + \sin\frac{\theta}{2}|\downarrow_z\rangle\right) = e^{-i\frac{\phi}{2}\hat{\sigma}_z}\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + \sin\frac{\theta}{2}|\downarrow_z\rangle\right)$$

$$= e^{-i\frac{\phi}{2}}\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{+i\frac{\phi}{2}}\sin\frac{\theta}{2}|\downarrow_z\rangle \quad \text{since } \hat{\sigma}_z|\uparrow_z\rangle = |\uparrow_z\rangle,\ \hat{\sigma}_z|\downarrow_z\rangle = -|\downarrow_z\rangle$$

$$= e^{-i\frac{\phi}{2}}\left(\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle\right) \equiv \boxed{\cos\frac{\theta}{2}|\uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\downarrow_z\rangle}$$

(all states equivalent up to overall phase)

Similarly $|\downarrow_n\rangle = \hat{D}_z(\phi)\left(\cos(\frac{\theta}{2})\hat{\mathbb{1}} - i\sin(\frac{\theta}{2})\hat{\sigma}_y\right)|\downarrow_z\rangle = e^{-i\frac{\phi}{2}\hat{\sigma}_z}\left(\cos\frac{\theta}{2}|\downarrow_z\rangle - \sin\frac{\theta}{2}|\uparrow_z\rangle\right)$

(since $\hat{\sigma}_y|\downarrow_z\rangle = \frac{\hat{\sigma}_+}{i}|\downarrow_z\rangle = -i|\uparrow_z\rangle$)

$$\Rightarrow |\downarrow_n\rangle = e^{+i\frac{\phi}{2}}\cos\frac{\theta}{2}|\downarrow_z\rangle - e^{-i\frac{\phi}{2}}\sin\frac{\theta}{2}|\uparrow_z\rangle = -e^{-i\frac{\phi}{2}}\left(\sin\frac{\theta}{2}|\uparrow_z\rangle - e^{\frac{i\phi}{2}}\cos\frac{\theta}{2}|\downarrow_z\rangle\right)$$

$$\equiv \boxed{\sin\frac{\theta}{2}|\uparrow_z\rangle - e^{i\phi}\cos\frac{\theta}{2}|\downarrow_z\rangle}$$

(b) The x-direction $(\theta = \frac{\pi}{2}, \phi = 0) \Rightarrow |\uparrow_x\rangle = \cos(\frac{\pi}{4})|\uparrow_z\rangle + \sin(\frac{\pi}{4})|\downarrow_z\rangle$

$\Rightarrow |\uparrow_x\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle + |\downarrow_z\rangle)$. Similarly $|\downarrow_x\rangle = \cos(\frac{\pi}{4})|\uparrow_z\rangle - \sin(\frac{\pi}{4})|\downarrow_z\rangle = \frac{1}{\sqrt{2}}(|\uparrow_z\rangle - |\downarrow_z\rangle)$

Check: $\hat{\sigma}_x = \hat{\sigma}_+ + \hat{\sigma}_-$, $\hat{\sigma}_x \begin{matrix}|\Uparrow_x\rangle \\ |\Downarrow_x\rangle\end{matrix} = (\hat{\sigma}_+ + \hat{\sigma}_-)\left(\frac{|\Uparrow_z\rangle \pm |\Downarrow_z\rangle}{\sqrt{2}}\right) = \frac{|\Downarrow_z\rangle \pm |\Uparrow_z\rangle}{\sqrt{2}}$

$$= \pm\left(\frac{|\Uparrow_z\rangle \pm |\Downarrow_z\rangle}{\sqrt{2}}\right) = \pm |\Uparrow_x\rangle \checkmark$$

The y-direction: $(\theta = \frac{\pi}{2}, \phi = \frac{\pi}{2}) \Rightarrow |\Uparrow_y\rangle = \cos(\frac{\pi}{4})|\Uparrow_z\rangle + e^{i\frac{\pi}{2}}\sin(\frac{\pi}{4})|\Downarrow_z\rangle$

$\Rightarrow |\Uparrow_y\rangle = \frac{1}{\sqrt{2}}(|\Uparrow_z\rangle + i|\Downarrow_z\rangle)$ and similarly $|\Downarrow_y\rangle = \frac{1}{\sqrt{2}}(|\Uparrow_z\rangle - i|\Downarrow_z\rangle)$

Check: $\hat{\sigma}_y = \frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i} \Rightarrow \hat{\sigma}_y \begin{matrix}|\Uparrow_y\rangle \\ |\Downarrow_y\rangle\end{matrix} = \frac{\hat{\sigma}_+ - \hat{\sigma}_-}{i}\left(\frac{|\Uparrow_z\rangle \pm i|\Downarrow_z\rangle}{\sqrt{2}}\right) = \frac{-|\Downarrow_z\rangle \pm i|\Uparrow_z\rangle}{i\sqrt{2}}$

$$= \frac{\pm|\Uparrow_z\rangle + i|\Downarrow_z\rangle}{\sqrt{2}} = \pm\left(\frac{|\Uparrow_z\rangle \pm i|\Downarrow_z\rangle}{\sqrt{2}}\right) = \pm|\Uparrow_y\rangle \checkmark$$

(c) Consider an arbitrary pure state of a spin-½ particle $|\psi\rangle = \alpha|\Uparrow_z\rangle + \beta|\Downarrow_z\rangle$, normalized, so $|\alpha|^2 + |\beta|^2 = 1$. Let us write the probability amplitudes as $\alpha = |\alpha|e^{i\varphi_\alpha}$, $\beta = |\beta|e^{i\varphi_\beta}$ $\Rightarrow |\psi\rangle = e^{i\varphi_\alpha}(|\alpha||\Uparrow_z\rangle + |\beta|e^{i(\varphi_\beta - \varphi_\alpha)}|\Downarrow_z\rangle)$

We can thus set $\cos\frac{\theta}{2} = |\alpha|$, $\sin\frac{\theta}{2} = |\beta|$, $\phi = \varphi_\beta - \varphi_\alpha$

$\Rightarrow$ Every pure state of spin-½ particles in spin-up along $\vec{e}_n = (\theta, \phi)$ where $\theta = 2\tan^{-1}\left(\frac{|\beta|}{|\alpha|}\right)$ $0 \le \theta \le \pi$, $\phi = \varphi_\beta - \varphi_\alpha$

Problem 3: Some Algebra with Density Matrices

(a) The state is a statistical mixture of $|\uparrow_z\rangle$ and $|\downarrow_z\rangle$

$$\hat{\rho} = \frac{1}{3}|\uparrow_z\rangle\langle\uparrow_z| + \frac{2}{3}|\downarrow_z\rangle\langle\downarrow_z|$$

(b) $\langle\hat{\sigma}_n\rangle = Tr(\hat{\rho}\hat{\sigma}_n) = \frac{1}{3}\langle\uparrow_z|\hat{\sigma}_n|\uparrow_z\rangle + \frac{2}{3}\langle\downarrow_z|\hat{\sigma}_n|\downarrow_z\rangle$

$$\Rightarrow \langle\hat{\sigma}_z\rangle = \frac{1}{3} - \frac{2}{3} = -\frac{1}{3}, \quad \langle\hat{\sigma}_x\rangle = \langle\hat{\sigma}_y\rangle = 0$$

For a completely mixed state, $\hat{\rho} = \frac{1}{2}\hat{\mathbb{1}}$, $\langle\hat{\sigma}_n\rangle = \frac{1}{2}Tr(\hat{\sigma}_n) = 0$

$\Rightarrow \langle\hat{\sigma}_x\rangle = \langle\hat{\sigma}_y\rangle = \langle\hat{\sigma}_z\rangle = 0$ for completely mixed state

The state prepared is "partially polarized" along z, whereas the completely mixed state is completely unpolarized along any direction

(c) $\langle\hat{\sigma}_n\rangle = \frac{1}{3}\langle\uparrow_z|\hat{\sigma}_n|\uparrow_z\rangle + \frac{2}{3}\langle\downarrow_z|\hat{\sigma}_n|\downarrow_z\rangle$

Now $\hat{\sigma}_n = \vec{e}_n\cdot\hat{\vec{\sigma}}$, $\quad \vec{e}_n = \sin\theta\cos\phi\,\vec{e}_x + \sin\theta\sin\phi\,\vec{e}_y + \cos\theta\,\vec{e}_z$

$$\Rightarrow \langle\hat{\sigma}_n\rangle = \frac{2}{3}\cos\theta - \frac{1}{3}\cos\theta = \frac{1}{3}\cos\theta, \quad \langle\hat{\sigma}_n\rangle_{\text{completely mixed}} = 0$$

Now we consider $\hat{\rho} = \frac{1}{2}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{2}|\uparrow_x\rangle\langle\uparrow_x|$

(d) Purity $= Tr(\hat{\rho}^2) = Tr\left(\frac{1}{4}|\uparrow_z\rangle\langle\uparrow_z| + \frac{1}{4}|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{4}|\uparrow_z\rangle\langle\uparrow_z|\uparrow_x\rangle\langle\uparrow_x| + \frac{1}{4}|\uparrow_x\rangle\langle\uparrow_x|\uparrow_z\rangle\langle\uparrow_z|\right)$

$$= \frac{1}{4} + \frac{1}{4} + \frac{1}{4}\underbrace{|\langle\uparrow_z|\uparrow_x\rangle|^2}_{\frac{1}{2}} + \frac{1}{4}\underbrace{|\langle\downarrow_x|\uparrow_z\rangle|^2}_{\frac{1}{2}} = \frac{3}{4}$$

This is in contrast to the completely mixed state, which has purity $= \frac{1}{d} = \frac{1}{2}$ (for $d=2$, spin-$\frac{1}{2}$)

The mixture here is "purer"

(c) We seek the eigenvectors and eigenvalues of $\hat{\rho}$. We can diagonalize the matrix in $|\pm_z\rangle$ basis.

Characteristic polynomial $= \det(\hat{\rho} - \lambda \hat{1}) = \det \begin{bmatrix} 3/4 - \lambda & \frac{1}{4} \\ 1/4 & \frac{1}{4} - \lambda \end{bmatrix} = \left(\frac{3}{4} - \lambda\right)\left(\frac{1}{4} - \lambda\right) - \frac{1}{16} = 0$

$$\Rightarrow \lambda^2 - \lambda + \frac{1}{8} = 0 \Rightarrow \text{Eigenvalues } \lambda_\pm = \frac{1}{2} \pm \frac{\sqrt{1 - 1/2}}{2} = \frac{1}{2} \pm \frac{1}{2\sqrt{2}} = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)$$

Eigenvectors: $\begin{bmatrix} \frac{3}{4} - \lambda_\pm & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} - \lambda_\pm \end{bmatrix} \begin{bmatrix} \alpha_\pm \\ \beta_\pm \end{bmatrix} = 0 \Rightarrow \left(\frac{3}{4} - \lambda_\pm\right)\alpha_\pm + \frac{\beta_\pm}{4} = 0 \Rightarrow \frac{\beta_\pm}{\alpha_\pm} = -3 + 4\lambda_\pm = -1 \pm \sqrt{2}$

Normalization $|\alpha_\pm|^2 + |\beta_\pm|^2 = 1 = |\alpha_\pm|^2\left(1 + \left|\frac{\beta_\pm}{\alpha_\pm}\right|^2\right) = |\alpha_\pm|^2 (4 \mp 2\sqrt{2}) = 1$

$$\Rightarrow |\alpha_\pm| = \frac{1}{\sqrt{4 \mp 2\sqrt{2}}} = \frac{\sqrt{2 \pm \sqrt{2}}}{2} = \begin{pmatrix} \cos \pi/8 \\ \sin \pi/8 \end{pmatrix}$$

$$|\beta_\pm| = \frac{\sqrt{2 \pm \sqrt{2}}}{2}(-1 + \sqrt{2}) = \frac{\sqrt{2 \mp \sqrt{2}}}{2}$$

$$\boxed{\Rightarrow |\lambda_\pm\rangle = \frac{\sqrt{2 \pm \sqrt{2}}}{2} |+_z\rangle \pm \frac{\sqrt{2 \mp \sqrt{2}}}{2} |-_z\rangle \qquad \Rightarrow \hat{\rho} = \lambda_+ |\lambda_+\rangle\langle\lambda_+| + \lambda_- |\lambda_-\rangle\langle\lambda_-|}$$

Note: We will see later a much simpler solution $\hat{\rho} = \frac{1}{2}(\hat{1} + \vec{Q} \cdot \hat{\vec{\sigma}})$

$\vec{Q} = \frac{1}{2}\vec{e}_z + \frac{1}{2}\vec{e}_x$ = "Bloch vector"