Physics 566: Quantum Optics I

Problem Set #2: Solutions

Problem 1:

(a) Consider a statistical mixture

$$\hat{\rho} = P_+ |\Uparrow_z\rangle\langle\Uparrow_z| + P_- |\Downarrow_z\rangle\langle\Downarrow_z|$$

where $P_\pm = \frac{1}{2}\left(1 \pm \frac{1}{\sqrt{2}}\right)$

Density matrix $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$

In basis $|\pm_z\rangle$ $\quad \hat{\rho} \doteq_z \frac{1}{2}\begin{pmatrix} 1+\frac{1}{\sqrt{2}} & 0 \\ 0 & 1-\frac{1}{\sqrt{2}} \end{pmatrix}$

In basis $\left(|\Uparrow_x / \Downarrow_x\rangle = \frac{1}{\sqrt{2}}\left(|\Uparrow_z\rangle \pm |\Downarrow_z\rangle\right) \right)$

$$\hat{\rho} \doteq_x \begin{pmatrix} \langle\Uparrow_x|\hat{\rho}|\Uparrow_x\rangle & \langle\Uparrow_x|\hat{\rho}|\Downarrow_x\rangle \\ \langle\Downarrow_x|\hat{\rho}|\Uparrow_x\rangle & \langle\Downarrow_x|\hat{\rho}|\Downarrow_x\rangle \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

Note: In this basis the density operator has off-diagonal elements. Nonetheless, it is a mixed state:

$$\mathrm{Tr}(\hat{\rho}^2) = \frac{3}{4}$$

The Bloch vector can be seen immediately from the form in the z-basis.

$$\mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_x) = \mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_y) = 0$$

$$\mathrm{Tr}(\hat{\rho}\,\hat{\sigma}_z) = P_{\uparrow_z} - P_{\downarrow_z} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\vec{Q} = \frac{1}{\sqrt{2}}\vec{e}_z} \qquad \text{mixed state } |\vec{Q}| < 1$$

(c) Now we have a state

$$\hat{\rho} = \frac{1}{2}|\uparrow_{n_1}\rangle\langle\uparrow_{n_1}| + \frac{1}{2}|\uparrow_{n_2}\rangle\langle\uparrow_{n_2}|$$

where $|\uparrow_n\rangle\langle\uparrow_n| = \frac{1}{2}(\hat{\mathbb{1}} + \vec{e}_n \cdot \hat{\vec{\sigma}})$ from Prob 1

$$\vec{e}_{n_{1/2}} = \frac{1}{\sqrt{2}}(\vec{e}_z \pm \vec{e}_x)$$

$$\Rightarrow \hat{\rho} = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{4}(\vec{e}_{n_1} + \vec{e}_{n_2}) \cdot \hat{\vec{\sigma}}$$

$$= \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{4}\left(\frac{2}{\sqrt{2}}\vec{e}_z\right) \cdot \hat{\vec{\sigma}}$$

$$= \frac{1}{2}\left(\hat{\mathbb{1}} + \frac{1}{\sqrt{2}}\vec{e}_z\right) \cdot \vec{\sigma} \;\underset{z}{=}\; \begin{bmatrix} \frac{1}{2}(1+\frac{1}{\sqrt{2}}) & 0 \\ 0 & \frac{1}{2}(1-\frac{1}{\sqrt{2}}) \end{bmatrix}$$

Same as $\hat{\rho}$ in part (b)!

<u>Moral of the story</u>: The ensemble decomposition is <u>not</u> <u>unique</u>. In fact, we can take any density matrix for a two-level system, described uniquely in terms of its Bloch vector $\vec{Q}$ and decompose it in terms of an ensemble of any two pure states described by unit vector $\vec{e}_n$ with probability p_n if $\vec{Q} = p_{n_1}\vec{e}_{n_1} + p_{n_2}\vec{e}_{n_2}$.

(c) Two statistical mixtures

$$\hat{\rho}_1 = \sum_n p_n |\hat{+}_n\rangle\langle\hat{+}_n|$$

$$\hat{\rho}_2 = \sum_m q_m |\hat{+}_m\rangle\langle\hat{+}_m|$$

Aside: $|+_n\rangle\langle+_n| = \frac{1}{2}(\hat{\mathbb{1}} + \hat{\sigma}_n)$ $\left(\begin{array}{l}\text{where} \\ \hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}}\end{array}\right)$

Projector

$$\Rightarrow \hat{\rho}_1 = \underbrace{\left(\sum_n p_n\right)}_{=1} \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\underbrace{\left(\sum_n p_n \vec{e}_n\right)}_{\vec{Q}_1} \cdot \hat{\vec{\sigma}}$$

$$\Rightarrow \hat{\rho}_1 = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\vec{Q}_1 \cdot \hat{\vec{\sigma}}$$

Similarly $\hat{\rho}_2 = \frac{1}{2}\hat{\mathbb{1}} + \frac{1}{2}\vec{Q}_2 \cdot \hat{\vec{\sigma}}$

where $\vec{Q}_2 = \sum_m q_m \vec{e}_n$

Thus $\hat{\rho}_1 = \hat{\rho}_2 \iff \vec{Q}_1 = \vec{Q}_2$

Problem 2: Ambiguity of ensemble decomposition

Let $\hat{\rho}_1 = \sum_i p_i |\psi_i\rangle\langle\psi_i|$, $\hat{\rho}_2 = \sum_j q_j |\phi_j\rangle\langle\phi_j|$

Prove $\hat{\rho}_1 = \hat{\rho}_2$ iff $\sqrt{q_j}\,|\phi_j\rangle = \sum_i U_{ji} \sqrt{p_i}\,|\psi_i\rangle$

where U_{ji} are elements of unitary matrix.

Proof:

For convenience, define $|\tilde{\phi}_j\rangle \equiv \sqrt{q_j}\,|\phi_j\rangle$

$|\tilde{\psi}_i\rangle \equiv \sqrt{p_i}\,|\psi_i\rangle$

so $\langle\tilde{\phi}_j|\tilde{\phi}_j\rangle = q_j$ $\quad \langle\tilde{\psi}_i|\tilde{\psi}_i\rangle = p_i$

(i) Assume $|\tilde{\phi}_j\rangle = \sum_i U_{ji} |\tilde{\psi}_i\rangle$ $\quad U_{ji}$ elements of unitary matrix

Consider $\hat{\rho}_2 = \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j| = \sum_{ijk} U^*_{jk} U_{ji} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_k|$

Aside: $(U_{jk})^* = U^\dagger_{kj}$

$$\Rightarrow \hat{\rho}_2 = \sum_{ik} \underbrace{\left(\sum_j U^\dagger_{kj} U_{ji}\right)}_{\delta_{ik}} |\tilde{\psi}_i\rangle\langle\tilde{\psi}_k|$$

$$\Rightarrow \hat{\rho}_2 = \sum_i |\tilde{\psi}_i\rangle\langle\tilde{\psi}_i| = \hat{\rho}_1 \quad ✓$$

(ii) Now assume $\hat{\rho}_1 = \hat{\rho}_2 \equiv \hat{\rho}$

$\hat{\rho}$ being a Hermitian operator can be diagonalized

$$\Rightarrow \quad \hat{\rho} = \sum_\alpha \lambda_\alpha |e_\alpha\rangle\langle e_\alpha|$$

where $\begin{cases} \sum_\alpha \lambda_\alpha = 1 & \text{with } \lambda_\alpha \text{ real}, \quad 0 \le \lambda_\alpha \le 1 \\ \langle e_\alpha | e_\beta\rangle = \delta_{\alpha\beta} \end{cases}$

Let $|\bar{e}_\alpha\rangle = \sqrt{\lambda_\alpha}\, |e_\alpha\rangle \quad \Rightarrow \quad \hat{\rho} = \sum_\alpha |\bar{e}_\alpha\rangle\langle \bar{e}_\alpha|$

$$\Rightarrow \quad \sum_i |\Psi_i\rangle\langle\Psi_i| = \sum_\alpha |\bar{e}_\alpha\rangle\langle\bar{e}_\alpha| = \sum_j |\phi_j\rangle\langle\phi_j|$$

we seek the relationship between $\{|\Psi_i\rangle\}$ and $\{|\phi_j\rangle\}$

First note $\{|e_\alpha\rangle\}$ form a basis for the Hilbert space (with $\lambda_\alpha = 0$ for those vectors not in $\hat{\rho}$)

$$\Rightarrow \quad |\Psi_i\rangle = \sum_\alpha |e_\alpha\rangle\langle e_\alpha|\Psi_i\rangle = \sum_\alpha |\bar{e}_\alpha\rangle \frac{\langle e_\alpha|\Psi_i\rangle}{\sqrt{\lambda_\alpha}}$$

$$= \sum_\alpha M_{i\alpha}\, |\bar{e}_\alpha\rangle$$

where $M_{i\alpha} = \dfrac{\langle e_\alpha|\Psi_i\rangle}{\sqrt{\lambda_\alpha}}$

Now: $\sum_i M_{i\alpha} M^*_{i\beta} = \sum_i \frac{\langle e_\alpha | \Psi_i \rangle \langle \Psi_i | e_\beta \rangle}{\sqrt{\lambda_\alpha \lambda_\beta}}$

$$= \frac{1}{\sqrt{\lambda_\alpha \lambda_\beta}} \langle e_\alpha | \left(\sum_i |\Psi_i\rangle\langle\Psi_i| \right) | e_\beta \rangle$$

$$= \frac{1}{\sqrt{\lambda_\alpha \lambda_\beta}} \langle e_\alpha | \hat{\rho} | e_\beta \rangle = \frac{\lambda_\alpha \delta_{\alpha\beta}}{\sqrt{\lambda_\alpha \lambda_\beta}} = \delta_{\alpha\beta}$$

$\Rightarrow$ When arranged in a matrix, the columns of $M_{i\alpha}$ are orthonormal

(Subtle point: $M_{i\alpha}$ need not be square here, since # of pure states in the $\{|\Psi_i\rangle\}$ need not be the dimension of Hilbert space. However, we can always append extra columns in the orthogonal space to make $M_{i\alpha}$ unitary.)

Formally the matrix M is a "partial isometry"

Thus since $|\Psi_i\rangle = \sum_\alpha M_{i\alpha} |\bar{e}_\alpha\rangle$

$$|\Phi_j\rangle = \sum_\beta N_{j\beta} |\bar{e}_\beta\rangle$$

$$|\Phi_j\rangle = \sum_i U_{ji} |\Psi_i\rangle$$

where $U = NM^\dagger$ q.e.d.

Problem 3

The two dimensional vector space the specifies the polarization state of a photon defines a qubit. We make the association:

$$\vec{e}_+ = \frac{\vec{e}_H + i\vec{e}_V}{\sqrt{2}} \Rightarrow |\Uparrow_z\rangle$$

$$\vec{e}_- = \frac{\vec{e}_H - i\vec{e}_V}{\sqrt{2}} \Rightarrow |\Downarrow_z\rangle$$

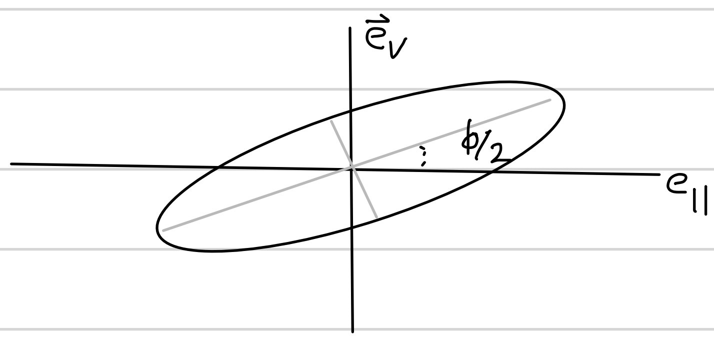
(a) $|\Updownarrow_x\rangle = \dfrac{|\Uparrow_z\rangle \pm |\Downarrow_z\rangle}{\sqrt{2}} \iff \dfrac{\vec{e}_+ \pm \vec{e}_-}{\sqrt{2}} \Rightarrow$ $\boxed{\begin{array}{l} |\Uparrow_x\rangle \Leftrightarrow \vec{e}_H \text{ (linear horizontal)} \\ |\Downarrow_x\rangle \Leftrightarrow i\vec{e}_V \equiv \vec{e}_V \text{ (linear vertical)} \end{array}}$

$|\Updownarrow_y\rangle = \dfrac{|\Uparrow_z\rangle \pm i|\Downarrow_z\rangle}{\sqrt{2}} \iff \dfrac{\vec{e}_+ \pm i\vec{e}_-}{\sqrt{2}} \Rightarrow$ $\boxed{\begin{array}{l} |\Uparrow_y\rangle \Leftrightarrow \frac{1+i}{\sqrt{2}}\left(\frac{\vec{e}_H + \vec{e}_V}{\sqrt{2}}\right) \equiv \frac{\vec{e}_H + \vec{e}_V}{\sqrt{2}} \text{ (liner at } 45^\circ \text{ between } \vec{e}_H \text{ and } \vec{e}_V) \\ |\Downarrow_y\rangle \Leftrightarrow \frac{1-i}{\sqrt{2}}\left(\frac{\vec{e}_H - \vec{e}_V}{\sqrt{2}}\right) \equiv \frac{\vec{e}_H - \vec{e}_V}{\sqrt{2}} \text{ (linear at } -45^\circ \text{ between } \vec{e}_H \text{ and } \vec{e}_V) \end{array}}$

(b) For an arbitrary state of the qubit, $|\Uparrow_n\rangle = \cos\frac{\theta}{2}|\Uparrow_z\rangle + e^{i\phi}\sin\frac{\theta}{2}|\Downarrow_z\rangle$, where (θ, ϕ) is the direction on the Poincaré sphere.

$$\Rightarrow |\Uparrow_n\rangle \equiv \cos\frac{\theta}{2}\,\vec{e}_+ + e^{i\phi}\sin\frac{\theta}{2}\,\vec{e}_- .$$

Recall (e.g. see Jackson 3rd edition Chap 7.2), The polarization is generally eliptical



$$r \equiv \frac{\alpha_+}{\alpha_-} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}e^{i\phi} = \cot\frac{\theta}{2}\,e^{i\phi}$$

Ratio of semimajor/semiminor axis $= \dfrac{1+r}{1-r} = \dfrac{1+\cot\frac{\theta}{2}}{1-\cot\theta/2} = \dfrac{1+\sin\theta}{1-\sin\theta}$

The ellipticity is characterized by $\big||\alpha_+|^2 - |\alpha_-|^2\big| = \left|\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}\right| = |\cos\theta|$

The orientation of the ellipse is shown, making an angle $\phi/2$ w.r.t. $\vec{e}_H$.

(c) Sketch of the Poincaré sphere

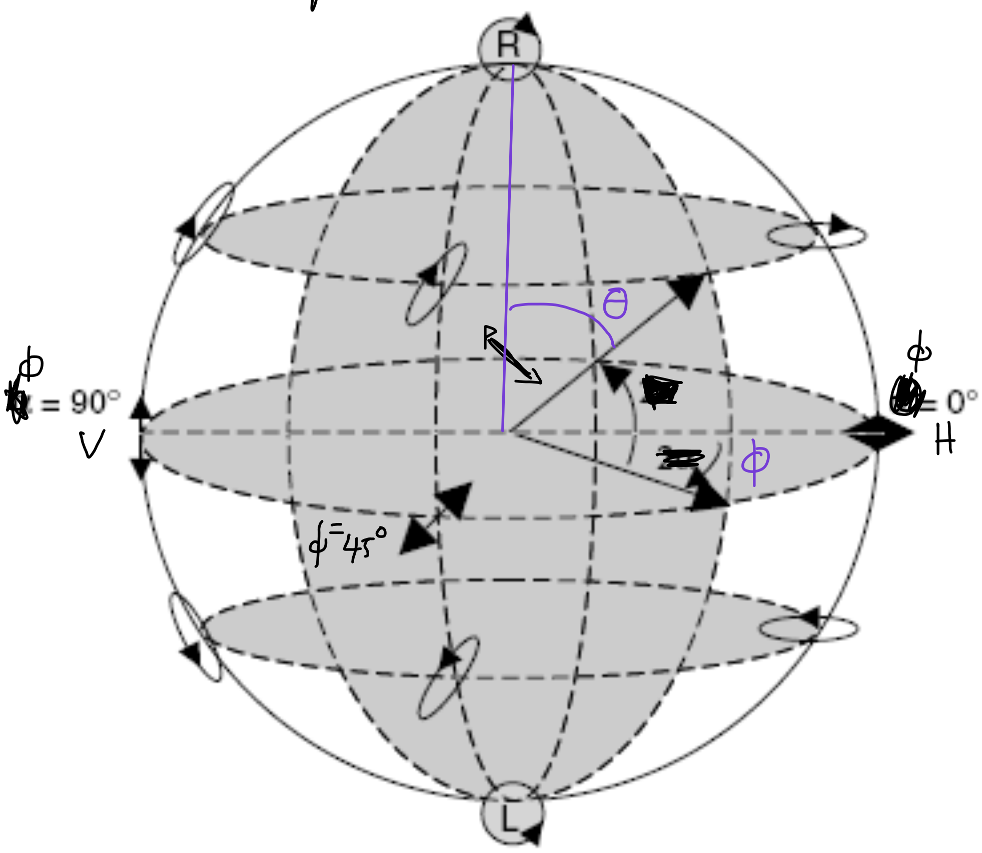


Figure 2.7. Poincaré sphere.

(d) Wave plate induces a phase shift that differs for "ordinary" and "extra ordinary" polarization. The ordinary and extra ordinary directions are the eigenvectors. Thus, defining on the Poincaré sphere the vectors in Hilbert space

$$\vec{e}_o \Rightarrow |\uparrow_n\rangle, \quad \vec{e}_e \Rightarrow |\downarrow_n\rangle,$$

the waveplate performs the transformation

$$\hat{U}^{WP} = e^{i\phi_o} |\uparrow_n\rangle\langle\uparrow_n| + e^{i\phi_e} |\downarrow_n\rangle\langle\downarrow_n|$$

Note: An arbitrary SU(2), $\hat{U} = e^{-i\frac{\phi}{2}} |\uparrow_n\rangle\langle\uparrow_n| + e^{i\frac{\phi}{2}} |\downarrow_n\rangle\langle\downarrow_n|$, for eigenvectors $|\uparrow_n\rangle$ & $|\downarrow_n\rangle$

We can factor out an overall phase that is irrelevant, $\frac{\phi_e + \phi_o}{2}$

$$\Rightarrow \hat{U}^{WP} = \underbrace{\cancel{e^{i\frac{\phi_e+\phi_o}{2}}}}^{\text{neglect}} \left[e^{-i\frac{\Delta\phi}{2}} |\uparrow_n\rangle\langle\uparrow_n| + e^{+i\frac{\Delta\phi}{2}} |\downarrow_n\rangle\langle\downarrow_n| \right]$$

$$= e^{-i\frac{\Delta\phi}{2}\hat{\sigma}_n} \quad \text{where} \quad \Delta\phi = \phi_e - \phi_o$$

Now we want a representation of $\hat{U}_\theta^{WP}(\Delta\phi)$ in the $\{\vec{e}_H, \vec{e}_V\}$ basis. In the language of the pseudospin, that is in the basis $\{|\Uparrow_x\rangle, |\Downarrow_x\rangle\}$

<u>Note:</u> $\vec{e}_o = \cos\theta\,\vec{e}_H + \sin\theta\,\vec{e}_V, \qquad \vec{e}_e = -\sin\theta\,\vec{e}_H + \cos\theta\,\vec{e}_V$

$|\Uparrow_n\rangle = \cos\theta\,|\Uparrow_x\rangle + \sin\theta\,|\Downarrow_x\rangle, \qquad |\Downarrow_n\rangle = -\sin\theta\,|\Uparrow_x\rangle + \cos\theta\,|\Downarrow_x\rangle$

$$\langle\Uparrow_x|\hat{U}_\theta^{WP}|\Uparrow_x\rangle = e^{-i\frac{\Delta\phi}{2}}|\langle\Uparrow_x|\Uparrow_n\rangle|^2 + e^{+i\frac{\Delta\phi}{2}}|\langle\Downarrow_x|\Uparrow_n\rangle|^2 = e^{-i\frac{\Delta\phi}{2}}\cos^2\theta + e^{i\frac{\Delta\phi}{2}}\sin^2\theta$$

$$= \cos\left(\frac{\Delta\phi}{2}\right)(\cos^2\theta + \sin^2\theta) - i\sin\left(\frac{\Delta\phi}{2}\right)(\cos^2\theta - \sin^2\theta) = \cos\left(\frac{\Delta\phi}{2}\right) - i\cos 2\theta\,\sin\left(\frac{\Delta\phi}{2}\right)$$

$$\langle\Downarrow_x|\hat{U}_\theta^{WP}|\Downarrow_x\rangle = e^{-i\frac{\Delta\phi}{2}}|\langle\Uparrow_x|\Downarrow_n\rangle|^2 + e^{+i\frac{\Delta\phi}{2}}|\langle\Downarrow_x|\Downarrow_n\rangle|^2 = e^{-i\frac{\Delta\phi}{2}}\sin^2\theta + e^{i\frac{\Delta\phi}{2}}\cos^2\theta$$

$$= \cos\left(\frac{\Delta\phi}{2}\right)(\sin^2\theta + \cos^2\theta) - i\sin\left(\frac{\Delta\phi}{2}\right)(\sin^2\theta - \cos^2\theta) = \cos\left(\frac{\Delta\phi}{2}\right) + i\cos 2\theta\,\sin\left(\frac{\Delta\phi}{2}\right)$$

$$\langle\Uparrow_x|\hat{U}_\theta^{WP}|\Downarrow_x\rangle = e^{-i\frac{\Delta\phi}{2}}\langle\Uparrow_x|\Uparrow_n\rangle\langle\Uparrow_n|\Downarrow_x\rangle + e^{+i\frac{\Delta\phi}{2}}\langle\Uparrow_x|\Downarrow_n\rangle\langle\Downarrow_n|\Downarrow_x\rangle = e^{-i\frac{\Delta\phi}{2}}\cos\theta\sin\theta + e^{i\frac{\Delta\phi}{2}}(-\sin\theta)(\cos\theta)$$

$$= -i\sin 2\theta\,\sin\left(\frac{\Delta\phi}{2}\right)$$

$$= \langle\Downarrow_x|\hat{U}_\theta^{WP}|\Uparrow_x\rangle^*$$

$$\Rightarrow \boxed{\hat{U}_\theta^{WP}(\Delta\phi) \doteq \begin{bmatrix} \cos\left(\frac{\Delta\phi}{2}\right) - i\cos 2\theta\sin\left(\frac{\Delta\phi}{2}\right) & -i\sin\left(\frac{\Delta\phi}{2}\right)\sin 2\theta \\ -i\sin\left(\frac{\Delta\phi}{2}\right)\sin 2\theta & \cos\left(\frac{\Delta\phi}{2}\right) + i\cos 2\theta\sin\left(\frac{\Delta\phi}{2}\right) \end{bmatrix}}$$

This is a familiar transformation from optics, which specifies the input-output relation $\vec{E}_{in} = \alpha_{in}\vec{e}_H + \beta_{in}\vec{e}_V \Rightarrow \vec{E}_{out} = \alpha_{out}\vec{e}_H + \beta_{out}\vec{e}_V$

$$\begin{bmatrix} \alpha_{out} \\ \beta_{out} \end{bmatrix} = U_\theta^{WP}(\Delta\phi)\begin{bmatrix} \alpha_{in} \\ \beta_{in} \end{bmatrix}$$

(c) A quarter-wave plate, $L = \frac{\lambda}{4(n_e - n_o)} \Rightarrow \Delta\phi = \frac{\pi}{2}$

$$\Rightarrow U_{QWP} = \begin{bmatrix} \frac{1+i\cos 2\theta}{\sqrt{2}} & \frac{-i}{\sqrt{2}}\sin 2\theta \\ -\frac{i}{\sqrt{2}}\sin 2\theta & \frac{1-i\cos 2\theta}{\sqrt{2}} \end{bmatrix}$$

To tranform linear and $\vec{e}_H$ to $\vec{e}_+$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ i \end{bmatrix}$

$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -i \end{bmatrix} = \frac{-i}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix} \equiv \frac{1}{\sqrt{2}}\begin{bmatrix} i \\ 1 \end{bmatrix}$

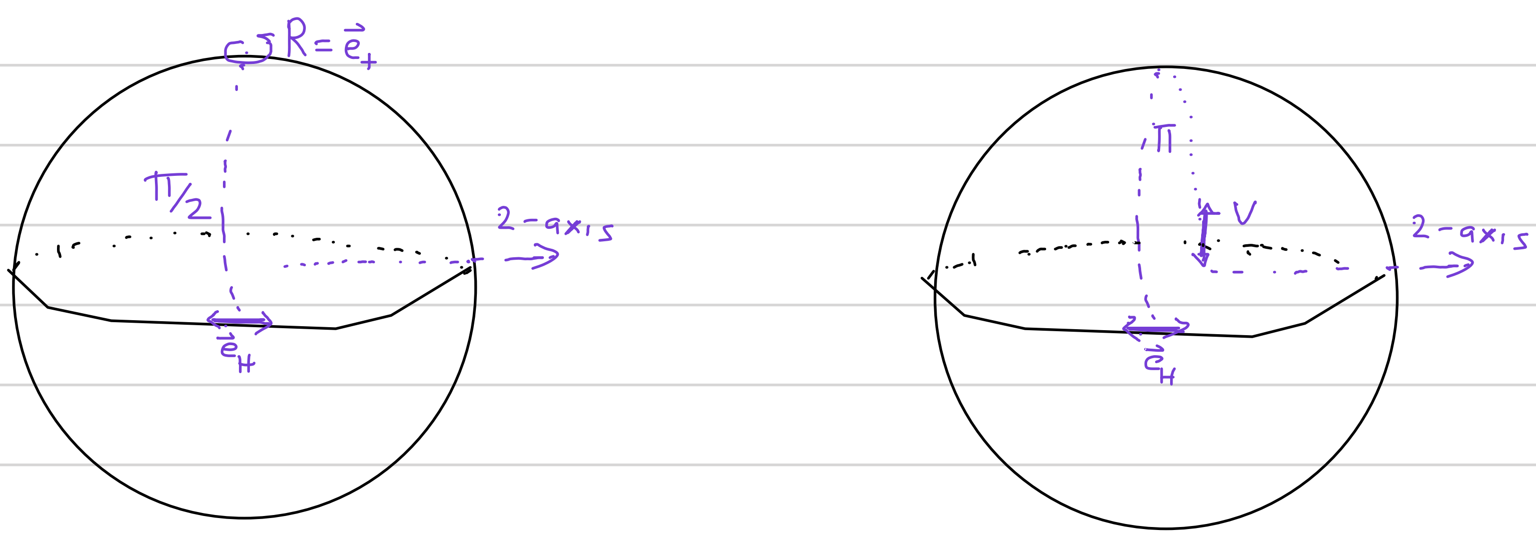
$\Rightarrow$ Choose $\theta = \frac{-\pi}{4}$ (45° between fast & slow axes) $\Rightarrow$

$U_{QWP}(\theta = \frac{\pi}{4}) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$ in $\vec{e}_H$ $\vec{e}_V$ basis $\Rightarrow$ Rotation by $-\frac{\pi}{2}$ around 2-axis of Poincaré

A half-wave plate, $L = \frac{\lambda}{2(n_e - n_o)} \Rightarrow \Delta\phi = \pi$

$$\Rightarrow U_{QWP} = -i\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

To transform $\vec{e}_H \to \vec{e}_V$: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ We achieve this by choosing $\theta = \frac{\pi}{4}$. The transformation is a rotation about 2-axis by π.



Quarter-wave plate　　Half-wave plate

From this geometric construction we see immediately how to orient the wave plate. The axis of rotation (the eigenvector of the waveplate) should be half way between the $\vec{e}_H$ + $\vec{e}_V$ directions. The quarter wave plate then is a $\frac{\pi}{2}$ rotation on the Poincaré sphere, corresponding to $\vec{e}_H \Rightarrow \vec{e}_+$, $\vec{e}_V \Rightarrow \vec{e}_-$. The half-wave plate maps $\vec{e}_H \Rightarrow \vec{e}_V$, $\vec{e}_V \Rightarrow \vec{e}_H$ (overall phase irrelevant).

(f) From part (d), the SU(2) rotation corresponding to a wave plate, with the crystal axes oriented at angle θ w.r.t $\vec{e}_H, \vec{e}_V$ direction is $U^{WP}_\theta(\Delta\phi) = e^{-i\frac{\Delta\phi}{2}\vec{e}(\theta)\cdot\hat{\vec{\sigma}}} = \cos\frac{\Delta\phi}{2}\hat{\mathbb{1}} - i(\cos 2\theta\,\hat{\sigma}_1 + \sin 2\theta\,\hat{\sigma}_2)\sin\frac{\Delta\phi}{2}$. (Note: A subtle point — in part (d) we wrote the matrices in the $\vec{e}_H, \vec{e}_V$ basis this defines $\hat{\sigma}_1$ in the usual Poincaré sphere) Thus, for a quarter waveplate and half waveplate respectively:

$$\text{QWP: } U^{WP}_\theta\left(\tfrac{\pi}{2}\right); \qquad \text{HWP: } U^{WP}_\theta(\pi)$$

We seek to show that we can construct an arbitrary SU(2) rotation on the Poincaré sphere with two QWP and one HWP. To do this I will employ an Euler angle parameterization. Recall

$$U \in SU(2) \Rightarrow \exists\, \alpha, \beta, \gamma \text{ (Euler angles) s.t. } U = \hat{D}_3(\alpha)\,\hat{D}_2(\beta)\,\hat{D}_3(\gamma)$$

(rotation about 3-axis, then 2-axis, then 3-axis)

(Note this is one Euler decomposition)

Thus a sequence of QWP. HWP. QWP

$$\Rightarrow U^{WP}_{\theta_a}\left(\tfrac{\pi}{2}\right) U^{WP}_{\theta_b}(\pi)\, U^{WP}_{\theta_c}\left(\tfrac{\pi}{2}\right)$$

$$= \left[\hat{D}_3(2\theta_a)\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3^\dagger(2\theta_a)\right]\left[\hat{D}_3(2\theta_b)\,\hat{D}_1(\pi)\,\hat{D}_3^\dagger(2\theta_b)\right]\left[\hat{D}_3(2\theta_c)\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3^\dagger(2\theta_c)\right]$$

$$= \hat{D}_3(2\theta_a)\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3(2\theta_b - 2\theta_a)\,\underbrace{\hat{D}_1(\pi)\,\hat{D}_3(2\theta_c - 2\theta_b)}\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3^\dagger(2\theta_c)$$

$$= \hat{D}_3(2\theta_a)\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3(2\theta_b - 2\theta_a)\,\underbrace{\hat{D}_3(-2\theta_c + 2\theta_b)\,\hat{D}_1(-\pi)}\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3^\dagger(2\theta_c)$$

(flip sign with π-rotation)

$$= \hat{D}_3(2\theta_a)\,\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3(2\theta_b - 2\theta_a)\,\hat{D}_3(-2\theta_c + 2\theta_b)\,\hat{D}_1^\dagger\left(\tfrac{\pi}{2}\right)\hat{D}_3^\dagger(2\theta_c)$$

$$= \hat{D}_3(2\theta_a)\,\underbrace{\hat{D}_1\left(\tfrac{\pi}{2}\right)\hat{D}_3(4\theta_b - 2\theta_a - 2\theta_c)\,\hat{D}_1^\dagger\left(\tfrac{\pi}{2}\right)}\,\hat{D}_3^\dagger(2\theta_c) : \quad \text{rotatate } 2 \to 1$$

$$= \hat{D}_3(\underbrace{2\theta_a}_{\alpha})\,\hat{D}_2(\underbrace{4\theta_b - 2\theta_a - 2\theta_c}_{\beta})\,\hat{D}_3(\underbrace{-2\theta_c}_{\gamma})$$

Q.E.D. Phew!