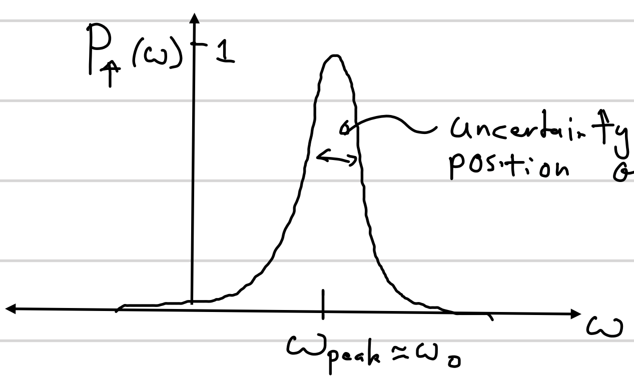
Physics 566: Quantum Optics I
Problem Set 3, Solutions

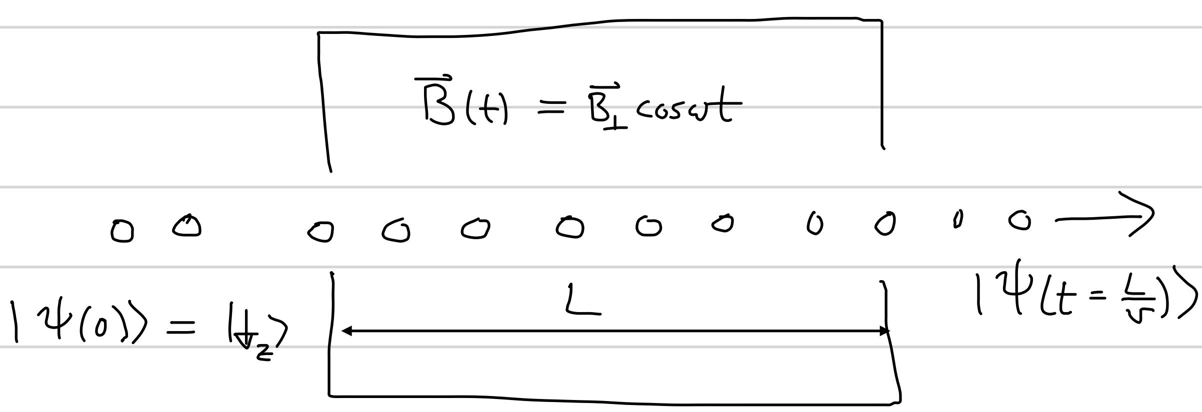
Problem 1: Magnetic Resonance: Rabi vs. Ramsey

A key problem in precision spectroscopy to to tune an oscillator to an absorption two-level resonance. We measure the different between ω + ω_0 by measuring the probability to transfer the (pseudo) spin from $|\downarrow_z\rangle \Rightarrow |\uparrow_z\rangle$ as a function of the detuning. The precision with which we can measure ω_0 depends sharpness of the resulting curve. If the curve $P_\uparrow(\omega)$ is very sharp, then we have small uncertainty the $\omega = \omega_0$ and the peak. If it is broad, then we can be detuned by a substantial amount a still reach high probability

Qualitatively:



(a) In the original Rabi experiment, the spins pass through a cavity we oscillating transverse field $\vec{B}_\perp \cos\omega t$. If the cavity has length L, and the spin moves through the cavity at speed v, the interact for a time $t = L/v$



The evolution of the spin is governed by the spin-resonance Hamiltonian in the RWA

In the rotating frame: $\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x \Rightarrow$ Rabi evolution: $\hat{U} = e^{-\frac{i}{\hbar}\hat{H}t}$

$$\hat{U}(t) = e^{i\vec{\Omega}_{tot}\cdot\frac{\hat{\vec{\sigma}}}{2}} = \cos\frac{\Omega_{tot}t}{2}\hat{\mathbb{1}} + i\frac{\Delta}{\Omega_{tot}}\sin\frac{\Omega_{tot}t}{2}\hat{\sigma}_z - i\frac{\Omega}{\Omega_{tot}}\sin\frac{\Omega_{tot}t}{2}\hat{\sigma}_x$$

The spins start in $|\downarrow_z\rangle$ and evolve according to the Rabi Hamiltonian for a time $T = \frac{L}{v}$.

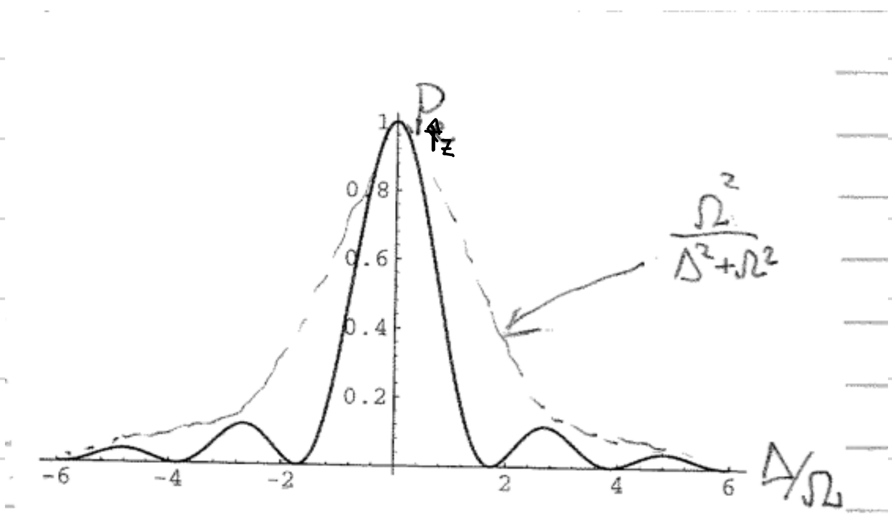
The probability to be in the excited $|\uparrow_z\rangle$ state after emerging from the cavity is thus

$$P_{\uparrow_z} = \left|\langle \uparrow_z | U(t = L/v) | \downarrow_z \rangle\right|^2 = \frac{\Omega^2}{\Omega_{tot}^2} \sin^2\left(\Omega_{tot} L/2v\right)$$

$$= \frac{\Omega^2}{\Omega^2 + \Delta^2} \sin^2\left(\sqrt{\Omega^2 + \Delta^2}\, \frac{L}{2v}\right)$$

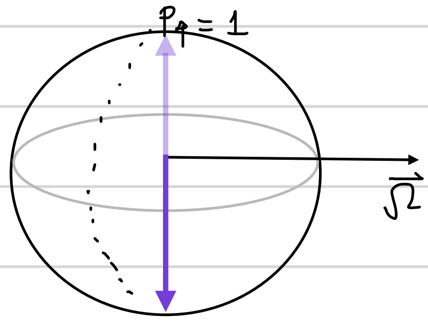
The bare Rabi frequency is chosen so that $\Omega T = \frac{\Omega L}{v} = \pi$ ("π-pulse")

$$\Rightarrow \boxed{P_{\uparrow_z} = \frac{1}{1 + \left(\frac{\Delta}{\Omega}\right)^2} \sin^2\left(\sqrt{1 + \left(\tfrac{\Delta}{\Omega}\right)^2}\, \frac{\pi}{2}\right)}$$

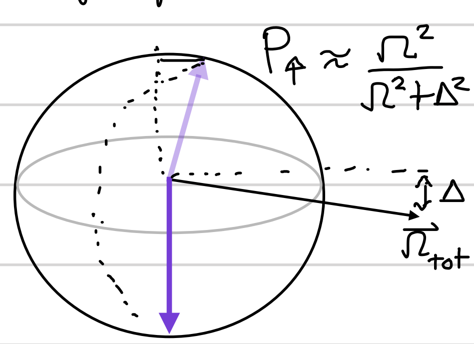


The linewidth of this curve determines the uncertainty of the resonance frequency. For the Rabi method, $\boxed{\delta(\omega - \omega_0) \sim \frac{1}{\Omega} = \frac{T}{\pi}}$. This is just an expression of the Fourier time - frequency uncertainty principle.

Resonance $\Delta = 0$



Slightly off resonance, $|\Delta| << \Omega$



It is clear from these sketches that when Δ is on the order of Ω or larger, the population transfer to the excited state decreases substantially. Since we have $\Omega T = \pi$, we are most sensitive to the detuning by making T longer and Ω smaller, maintaining $\Omega T = \pi$.

(b) Now suppose the spins have a distribution of velocities, characteristic of thermal beams

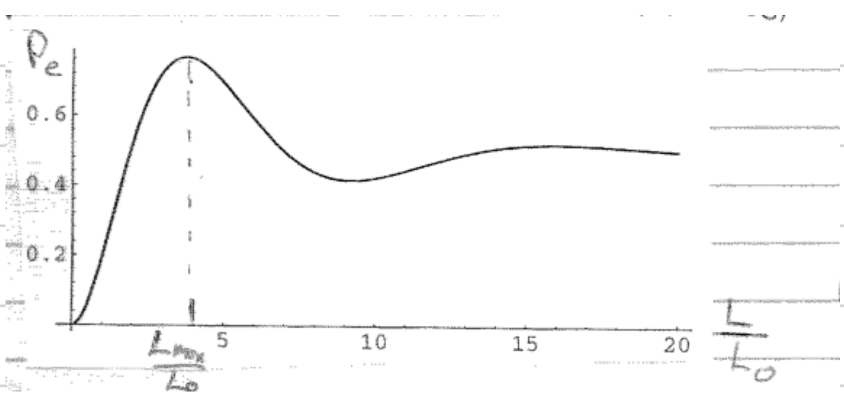
$$f(v) = 2\frac{v^3}{v_0^4} e^{-\frac{v^2}{v_0^2}}, \quad \text{where} \quad v_0 = \sqrt{2k_BT/m}$$

Note: For this distribution:
- Most probable speed $v_p = 1.22 v_0$, • Average speed: $\bar{v} = 1.33 v_0$, • RMS speed $\sqrt{\overline{\Delta v^2}} = 1.42 v_0$.

For a detuning Δ and speed v, we have $P_\uparrow(\Delta, v) = \frac{1}{1+\left(\frac{\Delta}{\Omega}\right)^2} \sin^2\left[\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2} \frac{\Omega L}{2v}\right]$.
At zero detuning $P_\uparrow(0,v) = \sin^2\left(\frac{L\Omega}{2v}\right)$. Now we must average over the velocity distribution:

$$P_\uparrow = \int_0^\infty dv \, f(v) P_\uparrow(0,v) = \int_0^\infty dx \, f(x) \sin^2\left(\frac{\theta_0}{2x}\right) \quad \text{[expressed in dimensionless variables]}$$

where $x = \frac{v}{v_0}$ (dimensionless velocity), $\theta_0 \equiv \frac{\Omega L}{v_0}$ (Angle of precession on the Bloch sphere for spins traveling @ v)

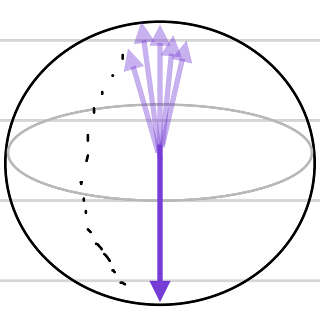


$\left(L_0 \equiv \frac{v_0}{\Omega}\right)$

The maximum $P_\uparrow$ occurs at $L_{max} = 3.77 L_0 \cong 1.20\pi \frac{v_0}{\Omega}$. We can understand this from the fact that the most probable speed is $1.2 v_0$

$$\Rightarrow \frac{\Omega L_{max}}{1.2 v_0} = \pi$$

On the Bloch Sphere



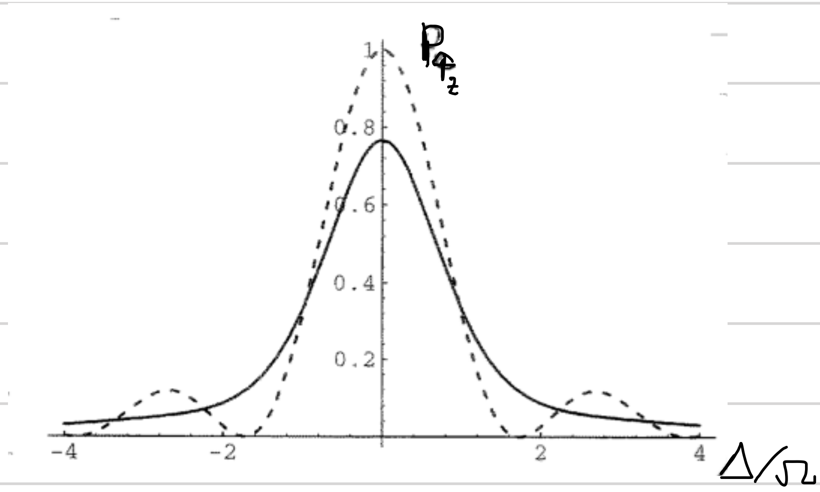
Distribution of speeds → Distribution of interaction times → Distribution of rotation angles on the Bloch sphere.

This is an example of "inhomogeneous broadening." The "damped" Rabi oscillations plotted above are not due to irreversible decay. Rather they are due to the spread in rotation angles due to the spread in velocities.

The total probability to excite the spin as a function of detuning is the the average of $P_{\uparrow}(\Delta, v)$ over the distribution of speeds $f(v)$. We set $L = L_{max}$, so $\Theta_0 = 1.2\pi$

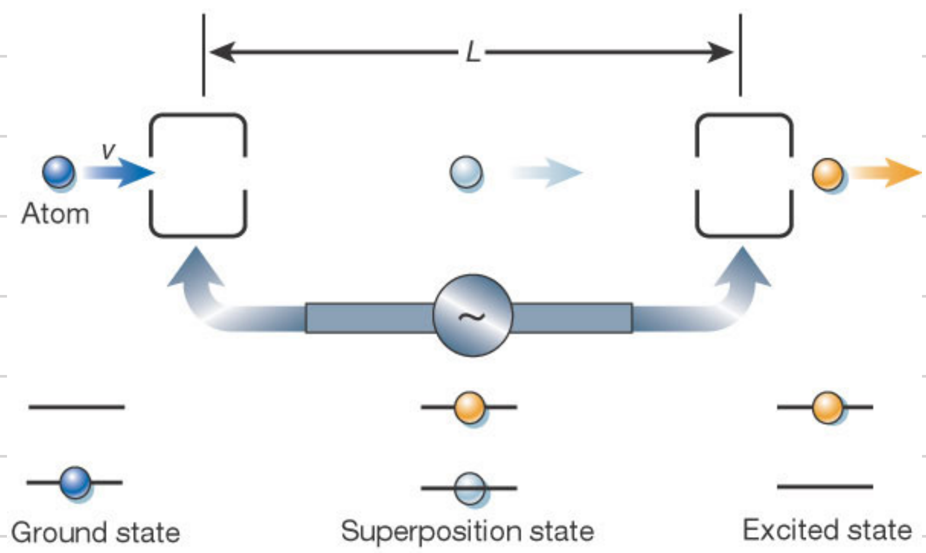
$$P_{\uparrow}(\Delta) = \frac{1}{1+(\frac{\Delta}{\Omega})^2} \int_0^{\infty} \sin^2\left[\left(1+\frac{\Delta^2}{\Omega^2}\right)^{1/2} \frac{\Theta_0}{2} \frac{1}{x}\right] \left(2x^3 e^{-x^2}\right) dx$$

Solve numerically. Here the solution is plotted as a function of Δ/Ω with $L = L_{max}$.



For reference, I have added the curve for mono-energetic spins (dashed curve). We see that the spread in speeds leads to a broadening of the resonance lineshape.

(c) We now consider the Ramsey separated zone method.



The key idea is to separate two short interaction zones that achieve $\pi/2$ rotations by a long interaction-free zone. As we will show, the precission with which we can measure $\omega - \omega_0$ is set by L, which can be made much longer than for the single zone case.

Given the Hamiltonian in the rotating frame $\hat{H} = -\frac{\hbar\Delta}{2}\hat{\sigma}_z + \frac{\hbar\Omega}{2}\hat{\sigma}_x$, with $|\psi(0)\rangle = |\downarrow_z\rangle$.

In the interaction zones we achieve a $\frac{\pi}{2}$-pulse with a very short interaction time and $\Omega\tau = \pi/2$. Because Ω is large (τ is small), $\Omega \gg |\Delta|$ for all detunings of interest $\Rightarrow \Omega_{tot} = \sqrt{\Omega^2+\Delta^2} \approx \Omega$ in the interaction zone.

$\Rightarrow$ After the first interaction zone $|\psi(\tau = \ell/v)\rangle = e^{-i\frac{\pi}{2}\frac{\hat{\sigma}_x}{2}}|\downarrow_z\rangle = \left(\frac{1}{\sqrt{2}}\hat{\mathbb{1}} - i\frac{1}{\sqrt{2}}\hat{\sigma}_x\right)|\downarrow_z\rangle$

$$\boxed{\Rightarrow |\psi(\tau = \tfrac{\ell}{v})\rangle = \frac{1}{\sqrt{2}}\left(|\downarrow_z\rangle - i|\uparrow_z\rangle\right)}$$

50-50 superposition in the equator of Bloch sphere

In the time between the interaction zones with $\vec{B}_\perp$, the spins freely process around $\vec{B}_{\parallel}$. In the rotating frame (where we are doing all of our calculations), the spins precess as the difference frequency $\Delta = \omega - \omega_0$, governed by the Hamiltonian $\hat{H}_0 = -\frac{\hbar\Delta}{2}\hat{\sigma}_z$

$$\boxed{\Rightarrow |\psi(\tau+T)\rangle = e^{\frac{-i\hat{H}_0 T}{\hbar}}|\psi(\tau)\rangle = e^{i\frac{\Delta T}{2}\hat{\sigma}_z}|\psi(\tau)\rangle = \frac{e^{-i\frac{\Delta T}{2}}|\downarrow_z\rangle - ie^{i\frac{\Delta T}{2}}|\uparrow_z\rangle}{\sqrt{2}}}$$

(Precession around the z-axis by angle ΔT)

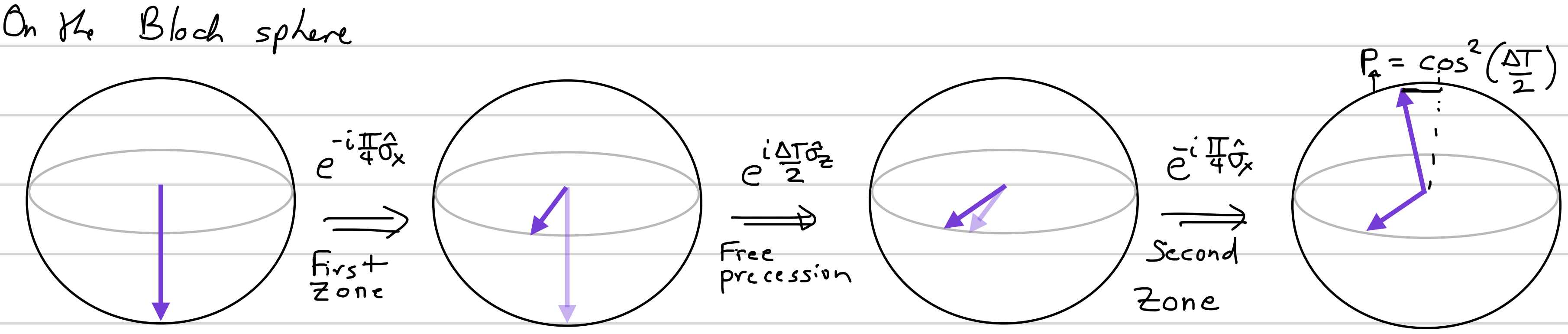
The second interaction zone applies another $\frac{\pi}{2}$ pulse (about the x-axis) to the spins

$$|\psi(2\tau+T)\rangle = e^{-i\frac{\pi}{4}\hat{\sigma}_x}|\psi(\tau+T)\rangle = \frac{1}{\sqrt{2}}\left[e^{-i\frac{\Delta T}{2}}\left(\frac{|\downarrow_z\rangle - i|\uparrow_z\rangle}{\sqrt{2}}\right) - ie^{i\frac{\Delta T}{2}}\left(\frac{|\uparrow_z\rangle - i|\downarrow_z\rangle}{\sqrt{2}}\right)\right]$$

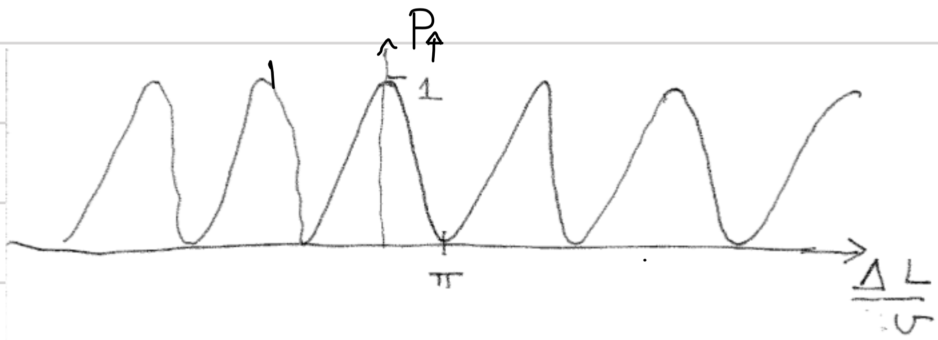
$$\Rightarrow |\psi(2\tau+T)\rangle = \left(\frac{e^{i\frac{\Delta T}{2}} - e^{-i\frac{\Delta T}{2}}}{2}\right)|\downarrow_z\rangle - i\left(\frac{e^{i\frac{\Delta T}{2}} + e^{-i\frac{\Delta T}{2}}}{2}\right)|\downarrow_z\rangle = -i$$

$$\boxed{\Rightarrow |\psi(2\tau+T)\rangle = i\sin\left(\frac{\Delta T}{2}\right)|\downarrow_z\rangle - i\cos\left(\frac{\Delta T}{2}\right)|\uparrow_z\rangle}$$

On the Bloch sphere

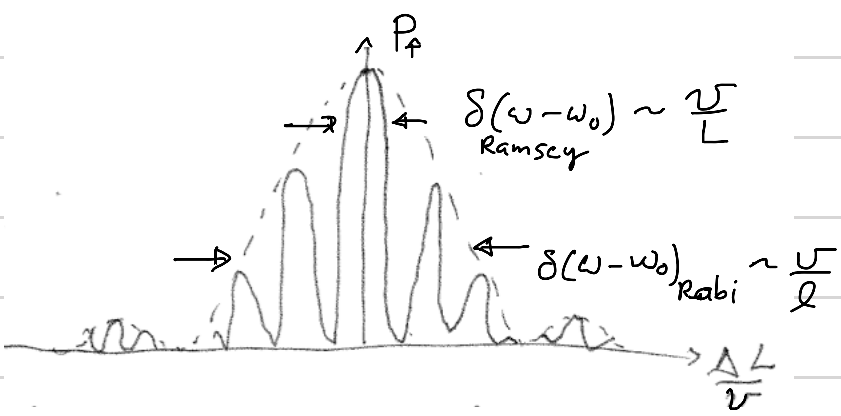


(d) $P_\uparrow(t_{final}) = \cos^2\left(\frac{\Delta T}{2}\right) = \cos^2\left(\frac{\Delta L}{2v}\right)$



This is known as "Ramsey fringes"

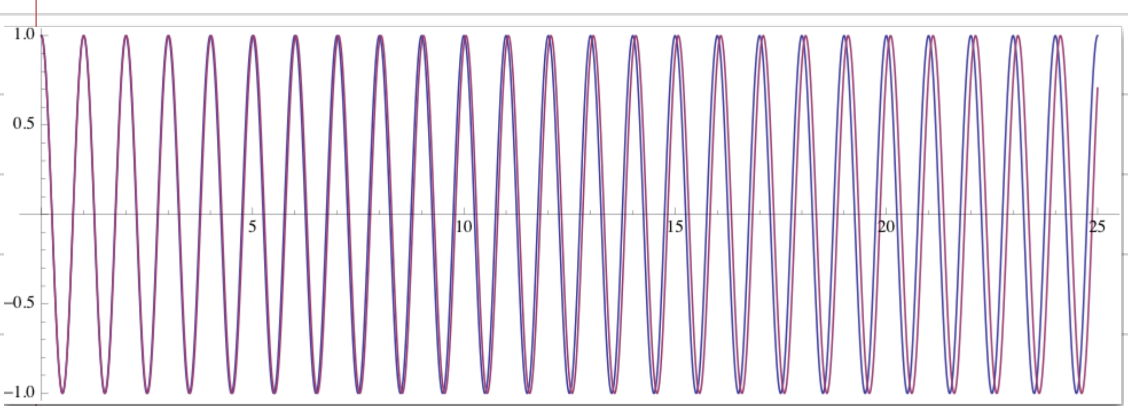
<u>Note:</u> We have assumed $|\Delta| \ll \Omega$, so the $\frac{\pi}{2}$ pulse was perfect. For the more general solution.



The envelope of the fringe pattern is the Rabi pattern of part (a) due to the finite transit-time brodening $\delta(\omega-\omega_0)_{Rabi} \sim \frac{v}{\ell}$

$$\boxed{\text{The linewidth of the Ramsey fringe pattern } \delta(\omega-\omega_0) \sim \frac{v}{L}}$$

A key take-away message from this problem is the nature of the time-energy uncertainty principle. In order measure energy $\hbar\omega_0$, we look for resonant absorption at frequency ω. When the probability of absorption is maximized, we set $\omega = \omega_0$. Our uncertainty is determined by the sensitity of the absorption to small changes in ω. In order to distinguish ω_0 from $\omega_0 + \Delta$, we must interact with the system for at least a time $\sim \frac{1}{\Delta}$. Only then do two oscillators become distinguishable (see below).



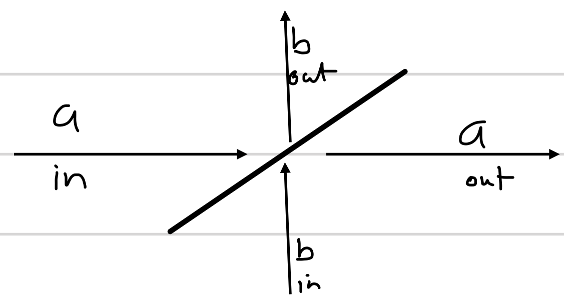
In the Ramsey method we set the spin into free precession at freq ω_0. All the while the cavity oscillates at ω. The longer we can maintain these two oscillators, the more precisely we can distinguish them, and thus measure ω_0.

Problem 2   SU(2) Inteferometers

We define "dual-rail" qubits by two orthogonal paths a photon can take

$$|\uparrow_z\rangle = |1_a, 0_b\rangle; \quad |\downarrow_z\rangle = |0_a, 1_b\rangle$$

(a) A symmetric beam splitter



$$\overset{\text{in}}{|1_a, 0_b\rangle} \Rightarrow \overset{\text{out}}{t|1_a, 0_b\rangle + r|0_a, 1_b\rangle}$$

$$|0_a, 1_b\rangle \Rightarrow t|0_a, 1_b\rangle + r|1_a, 0_b\rangle$$

This transformation of the basis states defines a 2×2 matrix $U = \begin{bmatrix} t & r \\ r & t \end{bmatrix}$.

In order to ensure this basis transformation is unitary, we must have:

$$U^\dagger U = \hat{\mathbb{1}} \Rightarrow \begin{bmatrix} |t|^2 + |r|^2 & tr^* + t^* r \\ t^* r + t^* r & |t| + |r|^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \boxed{|t|^2 + |r|^2 = 1, \quad t^* r + t^* r = 0}$$

Let $t = \sqrt{T} e^{i\varphi_T} = \cos\frac{\theta}{2} e^{i\varphi_T}$ $r = \sqrt{R}\, e^{i\phi_R} = i\sqrt{1-T}\, e^{i\phi_T} = -i\sin\frac{\theta}{2} e^{i\varphi_T}$

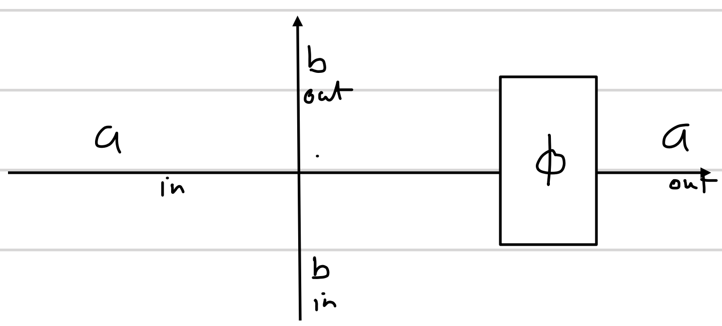
This satisfies the conditions for unitarity

$$\Rightarrow U = \underbrace{e^{i\varphi_T}}_{\substack{\text{negligible} \\ \text{overall phase}}} \begin{bmatrix} \cos\frac{\theta}{2} & -i\sin\frac{\theta}{2} \\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{bmatrix} = \cos\frac{\theta}{2}\hat{\mathbb{1}} - i\sin\frac{\theta}{2}\hat{\sigma}_x$$

$$\Rightarrow \boxed{U = e^{-i\theta\frac{\hat{\sigma}_x}{2}} : \text{A symmetric beam splitter is a rotation of the dual-rail qubit about the "x-axis" by } \theta = 2\cos^{-1}(\sqrt{T})}$$

(b) A phase shifter placed in one of the paths, so $|1_a, 0\rangle \rightarrow e^{i\phi}|1_a, 0\rangle$

$|0_a, 1_b\rangle \rightarrow |0_a, 1_b\rangle$



The transform matrix $U_\phi = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{bmatrix} = \cancel{e^{\frac{i\phi}{2}}} \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix}$

$\Rightarrow$ $\boxed{U_\phi = e^{i\phi\frac{\hat{\sigma}_z}{2}}}$: A relative phase shift of ϕ between modes $a+b$ corresponds to a rotation of $-\phi$ around the "z-axis"

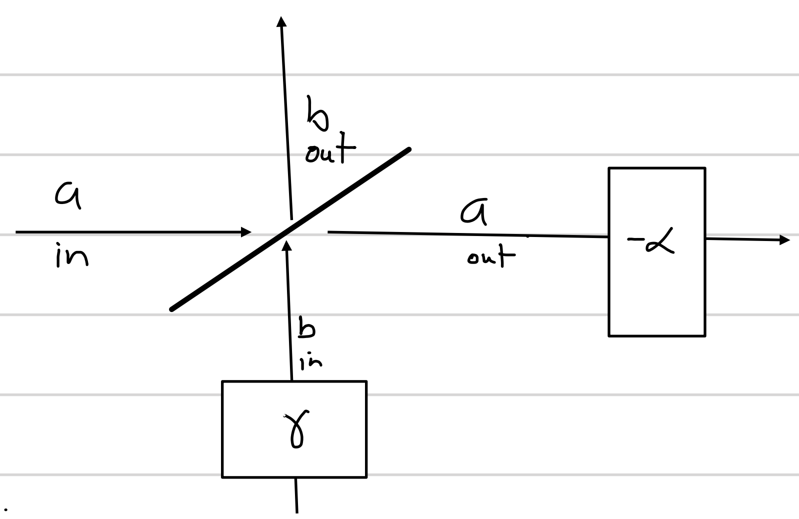
(c) We can use the Euler angle construction to show that an arbitrary SU(2) transformation can be implemented with a beam splitter and phase shifters

$\forall U \in SU(2)$, $\exists\, \alpha, \beta, \gamma$ (Euler angles) such that

$$U(\alpha, \beta, \gamma) = e^{-i\alpha\frac{\hat{\sigma}_z}{2}} e^{-i\beta\frac{\hat{\sigma}_x}{2}} e^{-i\gamma\frac{\hat{\sigma}_y}{2}}$$

(Note: In the usual Euler angle construction, the middle rotation is around the y-axis but the theorem holds for x, as above)

$\Rightarrow$ The following optical elements implement $U(\alpha, \beta, \gamma)$



Phase shifters γ & $-\alpha$ on input and output. The rotation about $-x$ by angle β is achieved by the beam splitter where $\beta = 2\cos^{-1}(\sqrt{T})$.

(d) A 50-50 balanced beam splitter corresponds to the unitary $U_{BS} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \underbrace{e^{-i(\frac{\pi}{2})\frac{\hat{\sigma}_x}{2}}}_{\frac{\pi}{2}\text{-rotation about } x.}$

The mirrors flip the modes: $a_{in} \to b_{in}$ $\quad b_{in} \to a_{in}$ $\Rightarrow U_{mirrors} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \equiv \cancel{-i} \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = e^{i\pi\frac{\hat{\sigma}_x}{2}}$

The phase shift: $a_{in} \to e^{i\phi} a_{in}$, $b_{in} \to b_{in} \Rightarrow U_{phase} = \begin{bmatrix} e^{i\phi} & 0 \\ 0 & 1 \end{bmatrix} = e^{i\phi/2} \begin{bmatrix} e^{i\phi/2} & 0 \\ 0 & e^{-i\phi/2} \end{bmatrix} = \cancel{e^{i\phi/2}} e^{i\phi\frac{\hat{\sigma}_z}{2}}$

Thus the Mach-Zender interferometer corresponds to the sequence

$\frac{\pi}{2}$ x-rotation $\to$ π x-rotation $\to$ ϕ z-rotation $\to$ $\frac{\pi}{2}$ x-rotation

$$\boxed{\text{The Ramsey separated zone sequence is equivalent to the Mach-Zender interferometer without the intermediate } \pi\text{-rotation, and phase rotation } \phi = \Delta \cdot T = \frac{\Delta L}{v}.}$$

(e) The Mach-Zender interferometer corresponds to the SU(2) rotation:

$$U_{MZ} = e^{i\frac{\pi}{2}\hat{\sigma}_x} e^{i\frac{\phi}{2}\hat{\sigma}_z} e^{i\pi\sigma_x} e^{i\frac{\pi}{2}\hat{\sigma}_z} = e^{i\frac{\pi}{2}\hat{\sigma}_x} e^{i\frac{\phi}{2}\hat{\sigma}_z} e^{-i\frac{\pi}{2}\hat{\sigma}_z} \quad \left(\text{using } \tfrac{3\pi}{2} \equiv -\tfrac{\pi}{2} \text{ on sphere}\right)$$

$$= \exp\Big\{ i\frac{\phi}{2} \underbrace{e^{i\frac{\pi}{2}\hat{\sigma}_x} \hat{\sigma}_z e^{-i\frac{\pi}{2}\hat{\sigma}_x}}_{=\hat{\sigma}_y} \Big\} = \boxed{e^{+i\frac{\phi}{2}\hat{\sigma}_y} : \text{ The Mach-Zender interferometer corresponds to a rotation by } \phi \text{ about } -y}$$

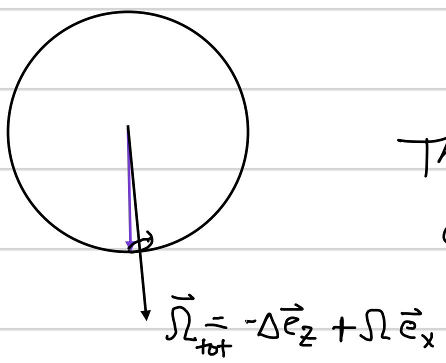
With $|\uparrow_z\rangle = |1_a, 0_b\rangle$, the probability to find $|1, 0_a\rangle$ in the output port:

$$\boxed{P_{\uparrow_z} = |\langle \uparrow_z | U_{MZ} | \uparrow_z \rangle|^2 = |\langle \uparrow_z | e^{i\frac{\phi}{2}\hat{\sigma}_y} | \uparrow_z \rangle|^2 = \cos^2\frac{\phi}{2} = \frac{1-\cos\phi}{2}}$$

Problem 3 : Adiabatic Rapid Passage

Using adiabatic evolution, we can robustly transfer population from ground to excited state without perfect knowledge of the parameters of the Hamiltonian

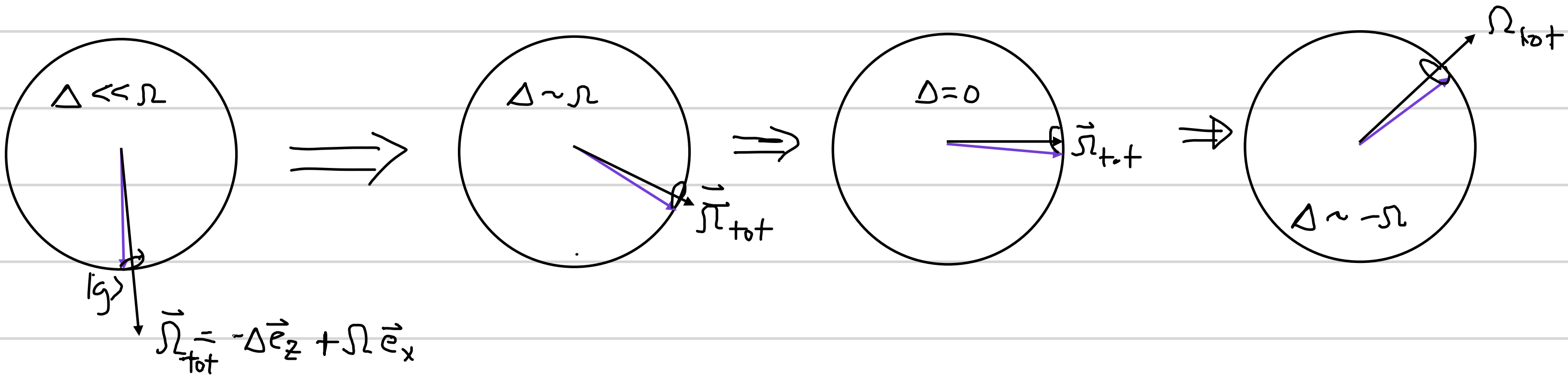
(a) Consider atoms initially in the ground state.
We apply our laser field that couples $|g\rangle$ and $|e\rangle$ at a detuning well below resonance

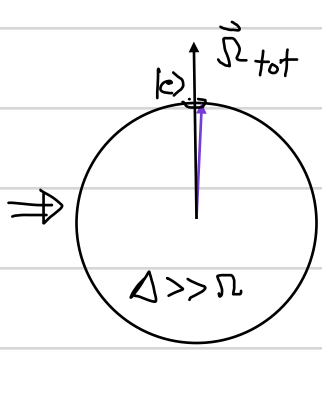


$|\Delta| >> \Omega$.

The Bloch vector will precess around the torque vector $\vec{\Omega}_{eff}$ close to the south pole.

Now suppose we sweep Δ through resonance slowly compared to $\sqrt{\Omega^2+\Delta^2}$. The spin will then "adiabatically follow" the direction of the torque vector regardless of the true value of Δ.
The slowest rate is for $\Delta = 0 \Rightarrow$ we require the time scale over which we change $\vec{\Omega}$ to be slow compared to Ω. However, we must accomplish the transfer $|g\rangle \rightarrow |e\rangle$ much faster than the rate of spontaneous emission Γ. This kind of transfer is known as "adiabatic rapid passage" We must be adiabatic with respect to the coherent dynamics characterized by + frequency Ω, but rapid with respect to the decay process that occurs at rate Γ.





As long as the Bloch vector rapidly rotates around the torque vector, we can adiabatically rotate $|g\rangle \rightarrow |e\rangle$.

(b) Quantitatively, we can turn to the adiabatic theorem of quantum mechanics.

Our time-dependent Hamiltonian in the rotating frame is

$$\hat{H}_{RF}(t) = \hbar \vec{\Omega}_{tot}(t) \cdot \frac{\vec{\sigma}}{2} \quad , \quad \vec{\Omega}_{tot}(t) = -\Delta(t)\vec{e}_z + \Omega \vec{e}_x$$

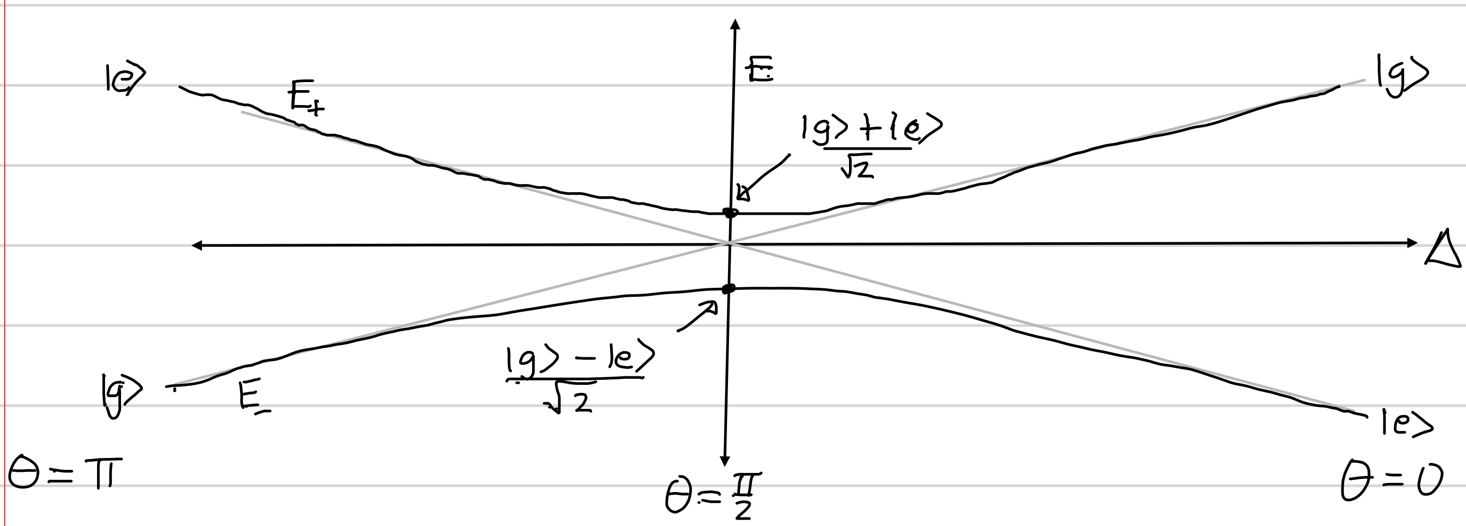
$$\Rightarrow \hat{H}_{RF}(t) = \hbar \Omega_{tot} \vec{e}_n(t) \cdot \hat{\vec{\sigma}} \quad , \quad \Omega_{tot} = |\vec{\Omega}_{tot}(t)|, \quad \vec{e}_n(t) = \frac{\vec{\Omega}_{tot}(t)}{|\vec{\Omega}_{tot}(t)|} = \cos\theta(t)\vec{e}_z + \sin\theta(t)\vec{e}_x$$

$$\tan\theta(t) = \frac{\Omega_{tot}(t)}{-\Delta(t)} ; \quad \cos\theta(t) = \frac{-\Delta(t)}{\Omega_{tot}(t)}, \quad \sin\theta(t) = \frac{\Omega}{\Omega_{tot}(t)}$$

We can immediately write the eigenvectors and eigenvalues of $\hat{H}_{RF}(t)$

$$E_{\pm}(t) = \pm \frac{\hbar \Omega_{tot}(t)}{2} = \pm \frac{\hbar}{2}\sqrt{\Delta^2(t) + \Omega^2}$$

$$|\pm(t)\rangle = |\updownarrow_{\vec{e}_n(t)}\rangle = \cos\left[\frac{\theta(t)}{2}\right]|g\rangle \pm \sin\left[\frac{\theta(t)}{2}\right]|e\rangle$$



According to the adiabatic theorem of quantum mechanics, for a time-dependent Hamiltonian, if at time $t=0$ $|\psi(0)\rangle = |u_n(0)\rangle$ where $\hat{H}(0)|u_n(0)\rangle = E_n(0)|u_n(0)\rangle$, then at a later time, $|\psi(t)\rangle \approx |u_n(t)\rangle$, where $\hat{H}(t)|u_n(t)\rangle = E_n(t)|u_n(t)\rangle$, if the evolution is "adiabatic". In other words, if we start in an eigenstate of the Hamiltonian, then we stay in the time-local eigenstate of the Hamiltonian if the Hamiltonian changes slowly enough. For the case at hand, the adiabatic eigenstate are $|\pm(t)\rangle$. These are sometimes known as the dressed states because the atomic levels are "dressed" by the laser field field. In contrast, the states $|g\rangle, |e\rangle$ are known as the "bare states."

The adiabatic theorem of quantum mechanics state that the evolution will remain adiabatic in the sense described above if

$$\frac{\left|\langle +(t)| \frac{d\hat{H}}{dt} |-(t)\rangle\right|}{(E_+(t) - E_-(t))^2/\hbar} << 1$$

The denominator is the "gap" between the energies of the dressed energy levels.

$$\hat{H}(t) = \hbar |\vec{\Omega}_{tot}(t)| \, \vec{e}_n(t) \cdot \hat{\vec{\sigma}} \qquad \vec{e}_n(t) = \cos\theta(t)\, \vec{e}_z + \sin\theta(t)\, \vec{e}_x$$

$$\Rightarrow \frac{d\hat{H}}{dt} = \hbar \dot{\Omega}_{tot} \, \vec{e}_n(t) \cdot \hat{\vec{\sigma}} + \hbar \Omega_{tot} \dot{\theta} \left[-\cos\theta\, \hat{\sigma}_z + \sin\theta\, \hat{\sigma}_x\right]$$

$$\Rightarrow \langle +(t)| \hbar \frac{d\hat{H}}{dt} |-(t)\rangle = \hbar^2 \Omega_{tot} \dot{\theta}$$

$$\Rightarrow \text{Adiabatic if } \frac{|\hbar^2 \Omega_{tot} \dot{\theta}|}{\Omega_{tot}^2} << 1 \quad \Rightarrow |\dot{\theta}| << \Omega_{tot} \geq \Omega \text{ (the minimum gap)}$$

$$\Rightarrow \boxed{\text{Require } |\dot{\theta}| << \Omega \text{ to be adiabatic}}$$

This is exactly the same condition we saw in the geometric picture on the Bloch sphere. In order to evolve adiabatically, the rate of precession of the Bloch vector, $\Omega_{tot} \geq \Omega$, must, at all times, remain fast compared to the rotation rate of the torque vector, $\dot{\theta}$. Simultaneously we must have $\dot{\theta} >> \Gamma$ to avoid decay.