Physics 566: Quantum Optics I
Problem Set 4
Due: Thursday Sept. 28, 2023

Problem 1: Inhomogeneous broadening (25 Points)

Rabi oscillations will decay because the two-level system is an open quantum system; coherence decays on a time scale we denote T_2. The oscillations can also decay because of uncertainty in the parameters of Hamiltonian. In modern parlance, we might call these "coherent errors." In particular, when we are measuring an expectation value, we must average over a large ensemble of measurement outcomes. If the value of the Rabi frequency and/or detuning are different for different members of ensemble, this is known "inhomogeneous broadening." This can be true because we have an extended ensemble in space and the fields that define the Hamiltonian are spatially inhomogeneous across the ensemble, or for a single system reprepared and measured, the field varies from shot-to-shot. We study in this problem the dephasing of oscillations on a time scale known as T_2^*, and explore the difference from true decay due to decoherence.

(a) Consider an ensemble of spins undergoing Rabi oscillations (spin magnetic resonance) in a bias static magnetic field B, giving a resonance frequency $\omega_0 = \gamma B$ and driven by an rf-magnetic field transversely oscillating at frequency ω. The cloud is extended so that spins see an inhomogeneous distribution of bias B-fields, which we will take to be Gaussian distributed. Because of this, the detuning from resonance, $\Delta = \omega - \omega_0$, will be inhomogeneous according to a probability distribution

$$P(\Delta) = \frac{1}{\sqrt{2\pi\delta^2}} e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}$$

where Δ_0 is the average detuning and δ^2 is the variance. The measured Rabi oscillation is the average across the ensemble is

$$P(\uparrow_z, t) = \int_{-\infty}^{\infty} d\Delta \; P(\Delta) P(\uparrow_z, t|\Delta, \Omega)$$

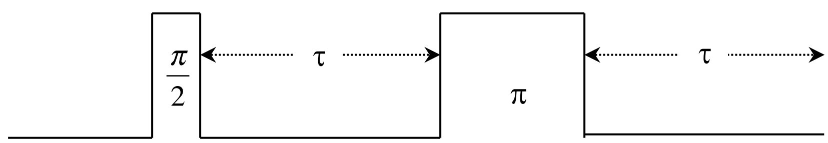
where $P(\uparrow_z, t|\Delta, \Omega)$ is the probability of finding spin-up for a fixed Rabi frequency and detuning (assume initially spin down). Show that for very small inhomogeneity, $\delta << \Omega, |\Delta_0|$

$$P(\uparrow_z, t) \approx \frac{\Omega^2}{2\Omega_{tot}^2}\left[1 - \cos(\Omega_{tot} t)\exp\left(-\frac{t^2}{2T_2^{*2}}\right)\right],$$ where $\Omega_{tot} = \sqrt{\Omega^2 + \Delta_0^2}$.

$1/T_2^* = \delta|\Delta_0|/\Omega_{tot}$ is the inhomogeneous linewidth, resulting from dephasing on the timescale of T_2^*. What is this linewidth in the limiting cases, $|\Delta_0|/\Omega \ll 1$ and $|\Delta_0|/\Omega \gg 1$?

(c) Plot this solution for different ratios of δ/Ω, taking $|\Delta_0|/\Omega = 1$.

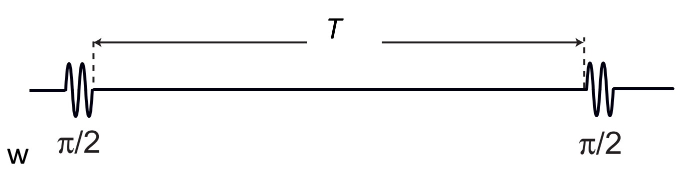
(d) Spin echo: Though inhomogeneous broadening will cause a decay of the ensemble averaged coherence; it is not a truly irreversible process. A way to see this is through the phenomenon of "spin echo". Consider an ensemble of spins sitting in the inhomogeneous bias B-field, as in part (a). Now consider the following pulse sequence.



The $\pi/2$-pulse about the *x*-axis to bring all spins onto the *y*-axis of the Bloch sphere. For a time τ, the spins randomly precess about the z-axis and the ensemble dephases. The π-pulse about the *x*-axis acts to time reverse the process. An "echo" signal will be seen at a time τ later when the spins "refocus" and returns to its initial. **Explain this process using this Bloch sphere.**

Problem 2: Ramsey fringes and the measurement of T2 times (25 Points)

(a) We seek to measure the coherence of between the computational basis states of a qubit $\{|0\rangle, |1\rangle\}$. Consider a two-pulse Ramsey sequence: A fast $\pi/2$ pulse around *x*-axis with detuning Δ, free evolution for a time T, a second fast $\pi/2$ pulse around *x*-axis at the same detuning Δ.



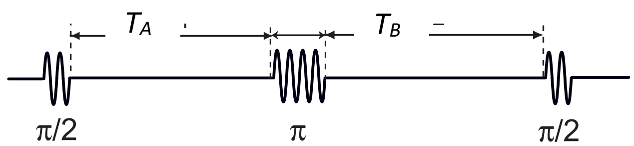
During the free evolution, in the absence of decay, the qubit will precess around the z-axis of the Bloch sphere. In the rotating frame, it precess at the frequency $\omega_0 - \omega = -\Delta$, where w is the frequency of the driving pulse, and ω_0 is the qubit resonance frequency. In reality, the coherence ρ_{01} decays exponentially with rate $1/T_2$. Show that, given the qubit initially in $|1\rangle$, the probability to find $|0\rangle$ after the sequence is

$$P_0 = \frac{1}{2}\left[1 + \cos(\Delta T)e^{-T/T_2}\right]$$

Explain this using the evolution on the Bloch sphere. Plot this for $\Delta/2\pi = 1$ MHz and T_2=25 μs, for T=0 to 25 μs.

(b) Suppose now that in addition to homogeneous decay, there is inhomogeneous decay T_2^*. Suppose that if the pulses are tuned to frequency ω, the probability the detuning seen by the qubit is Gaussian distributed, $p(\Delta) = e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}} / \sqrt{2\pi\delta^2}$, where Δ_0 is the mean detuning and $\delta = 1/T^*_2$ is the spread in detunings. Calculate the probability P_0 in the same two-pulse Ramsey sequence of part (a) for $T_2^* = 5$ μs. Comment on the result.

(c) Now consider a three-pulse Hahn spin-echo Ramsey sequence. : a fast π/2 pulse around *y*-axis with detuning Δ, free evolution for a time T_A a "time reverse" fast pulse around *x,* free evolution for a time T_B, and then a second fast $\pi/2$ pulse around *y*-axis at the same detuning Δ.



This "Ramsey interferometer" is equivalent to a Mach Zender interferometer, as we have studied. The initial state is |1> (the ground state). Show that the probability to state in the excited state

$$P_0 = \frac{1}{2}\left(1 - \cos[\Delta_0(T_A - T_B)]e^{-\delta^2(T_A-T_B)^2/2}e^{-(T_A+T_B)/T_2}\right)$$

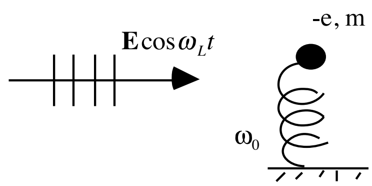
,

and plot for $T_A = 10\mu s$, as a function of T_B=0 to 25 μs.

Problem 3: The ac-Stark effect (25 points)

Suppose an atom is perturbed by a monochromatic electric field oscillating at frequency ω_L $\mathbf{E}(t) = E_z \cos(\omega_L t)\mathbf{e}_z$ (such as from a linearly polarized laser), rather than the dc-field studied in class. We know that such field can be absorbed and cause transitions between the energy levels; we will systematically study this effect later in the semester. The laser will also cause a *shift* of energy levels of the unperturbed states, known alternatively as the "ac-Stark shift", the "light shift", and sometimes the "Lamp shift" (don't you love physics humor). In this problem, we will look at this phenomenon is the simplest case that the field is near to resonance between the ground state $|g\rangle$ and some excited state $|e\rangle$, $\omega_L \approx \omega_{eg} \equiv (E_e - E_g)/\hbar$, so that we can ignore all other energy levels in the problem (the "two-level atom" approximation).

(i) The classical picture. Consider first the "Lorentz oscillator" model of the atom – a charge on a spring – with natural resonance ω_0.



The Hamiltonian for the system is $H = \dfrac{p^2}{2m} + \dfrac{1}{2}m\omega_0^2 z^2 - \mathbf{d}\cdot\mathbf{E}(t)$, where $d = -ez$ is the dipole.

(a) Ignoring damping of the oscillator, use Newton's Law to show that the induced dipole moment is

$$\mathbf{d}_{induced}(t) = \alpha\mathbf{E}(t) = \alpha E_z \cos(\omega_L t),$$

where $\alpha = \dfrac{e^2/m}{\omega_0^2 - \omega_L^2} \approx \dfrac{-e^2}{2m\omega_0\Delta}$ is the polarizability with $\Delta \equiv \omega_L - \omega_0$ the "detuning".

Note: classically, the shift in the energy is

$$\Delta H = -\frac{1}{2}d_{ind}(t)E(t) = -\frac{1}{2}\alpha E^2(t)\text{, or a time average value } \overline{H} = -\frac{1}{4}\alpha E_z^2$$

The factor of ½ arises because the dipole is *induced.*

(ii) Quantum picture. We consider the two-level atom described above. As we will derive later, the Hamiltonian for this system can be written in a time independent form (equivalent to the time-averaging done in the classical case)

$$\hat{H} = \hat{H}_{atom} + \hat{H}_{\text{int}},$$

where $\hat{H}_{atom} = -\hbar\Delta|e\rangle\langle e|$ is the "unperturbed" atomic Hamiltonian, and

$\hat{H}_{\text{int}} = \frac{\hbar\Omega}{2}(|e\rangle\langle g| + |g\rangle\langle e|)$ is the dipole-interaction with $\hbar\Omega \equiv \langle e|\mathbf{d}|g\rangle\cdot\mathbf{E}$ (the Rabi frequency).

(b) Find the *exact* energy eigenvalues and eigenvectors for this simple two-dimensional Hilbert space and plot the levels as a function of Δ. These are known as the atomic "dressed states."

(c) Expand your solution in (b) to lowest nonvanishing order in Ω to find the perturbation to the energy levels. Under what condition is this expansion valid?

(d) Confirm your answer to (c) using perturbation theory. Find also the mean induced dipole moment (to lowest order in perturbation theory), and from this show that the atomic polarizability, defined by $\langle \mathbf{d} \rangle = \alpha \mathbf{E}$ is $\alpha = \frac{-|\langle e|\mathbf{d}|g\rangle|^2}{\hbar\Delta}$, so that the second order perturbation to the ground state is $E_g^{(2)} = -\frac{1}{4}\alpha E_z^2$ as in part (a).

(e) Show that the ratio of the polarizability calculated classical in (a) and the quantum expression in (d) has the form

$$f \equiv \frac{\alpha_{quantum}}{\alpha_{classical}} = \frac{|\langle e|z|g\rangle|^2}{\left(\Delta z^2\right)_{SHO}},$$ where $\left(\Delta z^2\right)_{SHO}$ the SHO zero point variance.

This ratio is known as the oscillator strength.

Lessons:

- In lowest order perturbation theory an atomic resonance look just like a harmonic oscillator, with a correction factor given by the oscillator strength.
- Harmonic perturbations cause energy level shifts as well as absorption and emission.