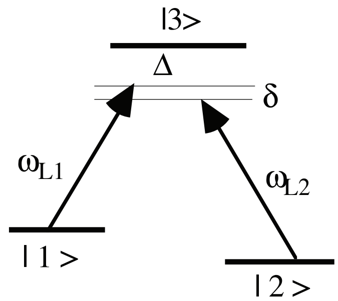
Physics 566: Quantum Optics I
Problem Set 5

Due: October 5, 2023

Problem 1: Λ–Transitions and the master equation (25 Points)

Consider a three-level atom in the so-called "lambda" configuration (because it looks like the Greek letter Λ):



Levels $|1\rangle$ and $|2\rangle$ are connected to level $|3\rangle$ on two dipole-allowed transitions driven by lasers at frequencies ω_{L1} and ω_{L2} respectively. Laser-1 is detuned from resonance by Δ. Difference between the detunings of lasers 1 and 2 is $\delta = (\omega_{L1} - \omega_{L2}) - (E_2 - E_1)/\hbar$ **.**

(a) The Hamiltonian for this system (in the RWA) is $H = H_A + H_{AL}$, where

$$H_A = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + E_3|3\rangle\langle 3| \,,$$

$$H_{AL} = \frac{\hbar\Omega_1}{2}\left(e^{-i\omega_{L1}t}|3\rangle\langle 1| + e^{i\omega_{L1}t}|1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(e^{-i\omega_{L2}t}|3\rangle\langle 2| + e^{i\omega_{L2}t}|2\rangle\langle 3|\right).$$

where $\Omega_{1,2}$ are the two Rabi frequencies. Because there are two laser frequencies, the usual unitary transformation to the frame rotating at ω_L does not apply. However, one can perform a unitary transformation that makes H time independent. Define a "rotating frame": $|\psi\rangle_{RF} = U^\dagger|\psi\rangle$, $H_{RF} = U^\dagger H U + i\hbar\frac{\partial U^\dagger}{\partial t}U$, where $U = \sum_{j=1}^{3} e^{-i\lambda_j t}|j\rangle\langle j|$.

Show that for appropriate choice of λ_j we can transform H to,

$$\hat{H}_{RF} = -\hbar\delta|2\rangle\langle 2| - \hbar\Delta|3\rangle\langle 3| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right).$$

(b) Suppose that level-3 decays to level-1 at a rate Γ_{31} and level-2 with rate Γ_{32}, and the total decay rate from level-3 is $\Gamma = \Gamma_{31} + \Gamma_{32}$. The effective non-Hermitian Hamiltonian is $\hat{H}_{eff} = \hat{H} - i\hbar\frac{\Gamma}{2}|3\rangle\langle 3|$. The (trace preserving) dynamics of the density operator for the system is described by the master equation,

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{eff}\right) + \mathcal{L}_{feed}[\hat{\rho}], \quad \text{where}$$

$$\mathcal{L}_{feed}[\hat{\rho}] = \Gamma_{31}|1\rangle\langle 3|\hat{\rho}|3\rangle\langle 1| + \Gamma_{32}|2\rangle\langle 3|\hat{\rho}|3\rangle\langle 2|.$$

Show that the matrix elements evolve according to:

$$\dot{\rho}_{11} = \Gamma_{31}\rho_{33} - i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right),$$

$$\dot{\rho}_{22} = \Gamma_{32}\rho_{33} - i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right),$$

$$\dot{\rho}_{33} = -\Gamma\rho_{33} + i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right) + i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right),$$

$$\dot{\rho}_{23} = -i\left(\Delta - \delta - i\frac{\Gamma}{2}\right)\rho_{23} - i\frac{\Omega_2}{2}\left(\rho_{33} - \rho_{22}\right) + i\frac{\Omega_1}{2}\rho_{21},$$

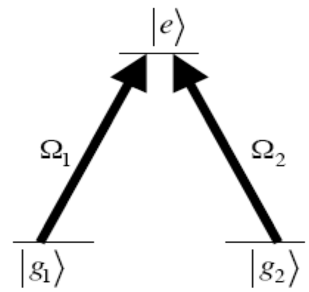
$$\dot{\rho}_{13} = -i\left(\Delta - i\frac{\Gamma}{2}\right)\rho_{13} - i\frac{\Omega_1}{2}\left(\rho_{33} - \rho_{11}\right) + i\frac{\Omega_2}{2}\rho_{12},$$

$$\dot{\rho}_{12} = -i\delta\,\rho_{12} + i\frac{\Omega_2}{2}\rho_{13} - i\frac{\Omega_1}{2}\rho_{32}$$

These equations describe the full dynamics, including optical pumping (refeeding) and saturation. They're pretty complicated to solve. Often, we can obtain good approximation and physical insight by using solely non-Hermitian Schrödinger evolution, $\frac{\partial}{\partial t}|\psi\rangle = -\frac{i}{\hbar}\hat{H}_{eff}|\psi\rangle$, as we studied in class. When possible, one should do this.

Problem 2: Dark states (25 points)

Let us consider again a three level "lambda system"



The two ground states are resonantly coupled to the excited state, each with a different Rabi frequency. Taking the two ground states as the zero of energy, then in the RWA (and in the rotating frame) the Hamiltonian is

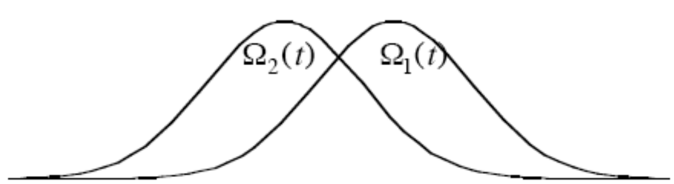
$$\hat{H}_{RF} = \frac{\hbar}{2}\left[\Omega_1\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \Omega_2\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)\right]$$

(a) Find the "dressed states" of this system (i.e. the eigenstates and eigenvalues of the total atom laser system). You should find that one of these states has a **zero** eigenvalue,

$$|\psi_{Dark}\rangle = \frac{\Omega_2|g_1\rangle - \Omega_1|g_2\rangle}{\sqrt{\Omega_1^2 + \Omega_2^2}}$$

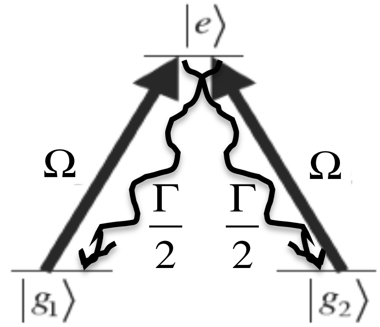
This particular superposition is called a "**dark state**" or uncoupled state because the laser field does not couple it to the excited state.

(b) Adiabatic transfer through the "nonintuitive" pulse sequence. Suppose we want to transfer population from $|g_1\rangle$ to $|g_2\rangle$. A robust method is to use adiabatic passage, *always staying in the local dark state*. This can then be *on resonance*. If we apply a slowly varying pulse $\Omega_2(t)$ overlapped, but followed by $\Omega_1(t)$ shown below, we accomplish this transfer



Sketch the dressed state eigenvalues a function of time. Explain the conditions necessary to achieve the adiabatic transfer.

(c) As discussed in lecture, when including spontaneous emission of the excited state, the atom will "relax" to the dark state. This is known as coherent population trapping (CPT). In class we solved this under the conditions of adiabatic elimination. Let's return to this here, under the condition of strong coupling. Consider, for simplicity, the case that $\Omega_1 = \Omega_2 = \Omega$, and the atom decays with equal rates to the two ground sublevels:



Write the master equation in the basis $\left\{ |D\rangle = \frac{|g_1\rangle - |g_2\rangle}{\sqrt{2}}, |B\rangle = \frac{|g_1\rangle + |g_2\rangle}{\sqrt{2}}, |e\rangle \right\}$, where $|D\rangle$ is the dark-state and $|B\rangle$ is the bright-state. Show that

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{\dagger} \right) + \mathcal{L}_{feed}[\hat{\rho}]$$

$$\hat{H}_{eff} = -i\frac{\hbar\Gamma}{2}|e\rangle\langle e| + \frac{\hbar\sqrt{2}\Omega}{2}\left(|e\rangle\langle B| + |B\rangle\langle e| \right), \quad \mathcal{L}_{feed}[\hat{\rho}] = \frac{\Gamma}{2}\langle e|\hat{\rho}|e\rangle\left(|B\rangle\langle B| + |D\rangle\langle D| \right)$$

Comment on this representation.

(d) Show that the equations of motion for the density matrix in this basis are

$$\dot{\rho}_{ee} = -\Gamma\rho_{ee} + i\frac{\sqrt{2}\Omega}{2}\left(\rho_{eB} - \rho_{Be} \right), \quad \dot{\rho}_{BB} = +\frac{\Gamma}{2}\rho_{ee} - i\frac{\sqrt{2}\Omega}{2}\left(\rho_{eB} - \rho_{Be} \right),$$

$$\dot{\rho}_{eB} = -\frac{\Gamma}{2}\rho_{eB} - i\frac{\sqrt{2}\Omega}{2}\left(\rho_{BB} - \rho_{ee} \right), \quad \dot{\rho}_{DD} = +\frac{\Gamma}{2}\rho_{ee},$$

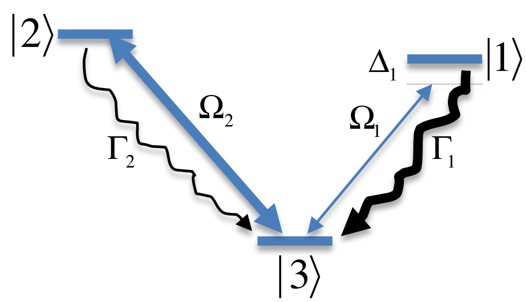
$$\dot{\rho}_{eD} = -\frac{\Gamma}{2}\rho_{eD} - i\frac{\sqrt{2}\Omega}{2}\rho_{BD}, \quad \dot{\rho}_{BD} = -i\frac{\sqrt{2}\Omega}{2}\rho_{eD}$$

and the steady state solution is $\hat{\rho}^{s.s.} = |D\rangle\langle D|$, i.e., the system relaxes to the dark-state.

Epilogue: The relaxation to the dark state is somewhat mysterious from the equations of motion since a spontaneous decay of along one of the two paths sketched above CANNOT land us in the dark state – we land in $|g_1\rangle$ or $|g_2\rangle$. Actually, we relax to the dark state when we DO NOT see a spontaneous decay. Not seeing spontaneous emission is information too. We'll return to this later when we study "quantum trajectories."

Problem 3: Autler-Townes (Extra Credit -25 Points)

Consider a 3-level atom, with one ground state, coupled through two different laser fields to two excited states in a "V-configuration".



For the transition, $|3\rangle \leftrightarrow |1\rangle$, the laser weakly excites the atom, and can be detuned from resonance. The transition $|3\rangle \leftrightarrow |2\rangle$ is tuned on resonance and can be highly excited. The goal of this problem is to study the effect of the strong $|3\rangle \leftrightarrow |2\rangle$ coupling on the optical response on the $|3\rangle \leftrightarrow |1\rangle$ transition.

The density matrix for the atom evolves according to the Master Equation,

$$\frac{d}{dt}\hat{\rho} = -\frac{i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}^{\dagger}_{eff}\right) + (\Gamma_1\rho_{11} + \Gamma_2\rho_{22})|3\rangle\langle 3|,$$

where the "effective" non-Hermitian Hamiltonian in the RWA is,

$$\hat{H}_{eff} = -\hbar\left(\Delta_1 + \frac{i}{2}\Gamma_1\right)|1\rangle\langle 1| - \frac{i}{2}\hbar\Gamma_2|2\rangle\langle 2| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right).$$

(a) Show that the equations of motion of the density matrix elements are,

$$\dot{\rho}_{11} = -\Gamma_1\rho_{11} - \frac{i}{2}\Omega_1\left(\rho_{31} - \rho_{13}\right), \quad \dot{\rho}_{22} = -\Gamma_2\rho_{22} - \frac{i}{2}\Omega_2\left(\rho_{32} - \rho_{23}\right),$$

$$\rho_{11} + \rho_{22} + \rho_{33} = 1$$

$$\dot{\rho}_{13} = \left(i\Delta_1 - \frac{\Gamma_1}{2}\right)\rho_{13} - \frac{i}{2}\Omega_1\left(\rho_{33} - \rho_{11}\right) + \frac{i}{2}\Omega_2\rho_{12},$$

$$\dot{\rho}_{23} = -\frac{\Gamma_2}{2}\rho_{23} - \frac{i}{2}\Omega_2\left(\rho_{33} - \rho_{22}\right) + \frac{i}{2}\Omega_1\rho_{21},$$

$$\dot{\rho}_{12} = \left(i\Delta_1 - \frac{\Gamma_1 + \Gamma_2}{2}\right)\rho_{12} + \frac{i}{2}\Omega_2\rho_{13} - \frac{i}{2}\Omega_1\rho_{32},$$

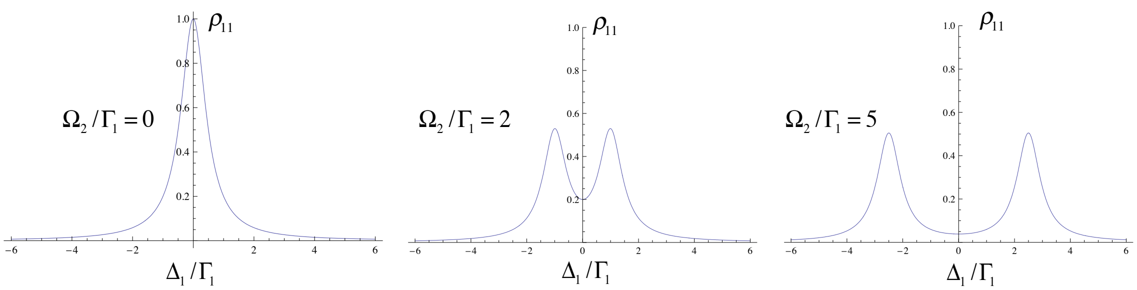
(b) Under the assumption of weak excitation of $|1\rangle$, in order to find the response on the $|3\rangle \leftrightarrow |1\rangle$ transition, we need only retain terms to first order in Ω_1. Coherence ρ_{12} is dominated by the strong field, Ω_2. Further we will assume $\Gamma_2 << \Gamma_1$. Show that in steady state in these approximations,

$$\rho_{11} = -\frac{\Omega_1}{\Gamma_1}\,\mathrm{Im}(\rho_{13})\,, \quad \rho_{13} \approx -\frac{\Omega_1}{(2\Delta_1 + i\Gamma_1)}\rho_{33} + \frac{\Omega_2}{(2\Delta_1 + i\Gamma_1)}\rho_{12}\,, \quad \rho_{12} \approx \frac{\Omega_2}{(2\Delta_1 + i\Gamma_1)}\rho_{13}\,.$$

(c) Put all this together to show that the population excited into $|1\rangle$ is

$$\rho_{11} \approx -\frac{\Omega_1^2}{\Gamma_1}\rho_{33}\,\mathrm{Im}\left[\frac{2\Delta_1 + i\Gamma_1}{(2\Delta_1 + i\Gamma_1)^2 - \Omega_2^2}\right]$$

Plots ρ_{11} as a function of Δ_1, normalized in units of $\frac{\Omega_1^2}{\Gamma_1^2}\rho_{33}$, for different coupling strengths on the auxiliary transition, Ω_2, are shown below.



For no coupling, we see the familiar Lorentzian lineshape. As the coupling increases so that $\Omega_2 > \Gamma_1$, we see the line split into a doublet known as the Autler-Townes splitting.

A more intuitive understanding of the origin of the Autler-Townes doublet is to think about the "dressed states" of atom+laser. The strong laser field on the $|3\rangle \leftrightarrow |2\rangle$ dresses the atom. The weak laser then probes the absorption from the dressed states.

(d) Find the dressed states of the $|3\rangle \leftrightarrow |2\rangle$ two-level system, coupled on resonance (diagonalize of the Hermitian part of the 2×2 Hamiltonian matrix in the $|3\rangle,|2\rangle$ subspace). Based on the eigenvalues and eigenvectors, explain the doublet.

(e) Explain the physical difference between Autler-Townes splitting and EIT. Under what conditions are they equivalent?