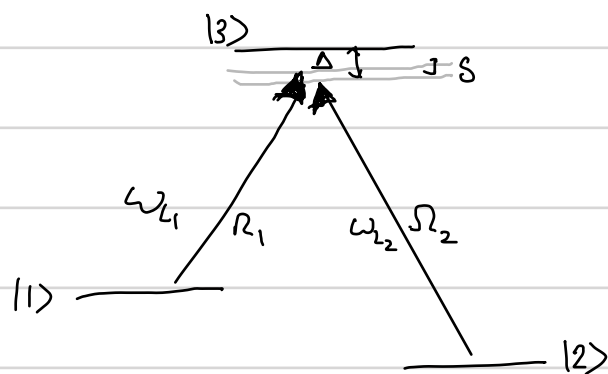


Physics 566: Quantum Optics I
 Problem Set #5: Solutions

Problem 1: Λ -transitions and the Master Equation



$$\Delta = \Delta_1 = \omega_{L1} - \frac{E_3 - E_1}{\hbar}$$

$$\Delta_2 = \omega_{L2} - \frac{E_3 - E_2}{\hbar}$$

$$\delta = \Delta_1 - \Delta_2 = \omega_{L1} - \omega_{L2} - \frac{E_2 - E_1}{\hbar}$$

(a) The Hamiltonian $\hat{H} = \hat{H}_A - \hat{H}_{AL}$ in the Schrödinger Picture:

$$\hat{H}_A = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| + E_3 |3\rangle\langle 3|, \quad \hat{H}_{AL} = \frac{\hbar\Omega_1}{2} (|3\rangle\langle 1| e^{-i\omega_{L1}t} + |1\rangle\langle 3| e^{+i\omega_{L1}t}) \\ + \frac{\hbar\Omega_2}{2} (|3\rangle\langle 2| e^{-i\omega_{L2}t} + |2\rangle\langle 3| e^{+i\omega_{L2}t})$$

We go to a "rotating frame" by making a unitary transformation $\hat{U} = \sum_{j=1}^3 e^{-i\lambda_j t} |j\rangle\langle j|$

$$\text{In the rotating frame: } \hat{H}_{RF} = \hat{U}^\dagger \hat{H} \hat{U} + \frac{\hbar}{-i} \frac{\partial \hat{U}^\dagger}{\partial t} \hat{U} = \hat{U}^\dagger \hat{H} \hat{U} - \sum_j \hbar \lambda_j |j\rangle\langle j|$$

$$\Rightarrow \hat{H}_{RF} = (E_1 - \hbar\lambda_1) |1\rangle\langle 1| + (E_2 - \hbar\lambda_2) |2\rangle\langle 2| + (E_3 - \hbar\lambda_3) |3\rangle\langle 3| \\ + \frac{\hbar\Omega_1}{2} (|3\rangle\langle 1| e^{-i[\omega_{L1} - (\lambda_3 - \lambda_1)]t} + \text{h.c.}) + \frac{\hbar\Omega_2}{2} (|3\rangle\langle 2| e^{-i(\omega_{L2} - (\lambda_3 - \lambda_2))t} + \text{h.c.})$$

We choose the rotating frame to make \hat{H}_{RF} time independent

$$\Rightarrow \lambda_3 - \lambda_1 = \omega_{L1}, \quad \lambda_3 - \lambda_2 = \omega_{L2}$$

This is two equations for three unknowns. We can arbitrarily set the zero of energy.

$$\Rightarrow \text{Choose } \lambda_1 = E_1/\hbar \Rightarrow \lambda_3 = \omega_{L1} + E_1/\hbar, \quad \lambda_2 = -\omega_{L2} + \lambda_3 = \omega_{L1} - \omega_{L2} + E_1/\hbar$$

$$\Rightarrow \hat{H}_{RF} = -\hbar\delta |2\rangle\langle 2| - \hbar\Delta |3\rangle\langle 3| + \frac{\hbar\Omega_1}{2} (|3\rangle\langle 1| + |1\rangle\langle 3|) + \frac{\hbar\Omega_2}{2} (|3\rangle\langle 2| + |2\rangle\langle 3|)$$

(b) Given: Level $|3\rangle$ decays to $|1\rangle$ and $|2\rangle$ with rates Γ_{31} and Γ_{32} respectively. The total decay rate is $\Gamma = \Gamma_{31} + \Gamma_{32}$, so the effective (non-Hermitian) Hamiltonian that includes decay of $|3\rangle$ according to an imaginary part of the excited state eigenvalue: $\hat{H}_{\text{eff}} = \hat{H} - i\frac{\Gamma}{2}|3\rangle\langle 3|$. The Master equation that describes the evolution of the density operator:

$$\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} (\hat{H}_{\text{eff}} \hat{\rho} - \hat{\rho} \hat{H}_{\text{eff}}^\dagger) + \mathcal{L}_{\text{feed}}[\hat{\rho}]$$

$$\mathcal{L}_{\text{feed}}[\hat{\rho}] = \underbrace{\Gamma_{31} \rho_{33} |1\rangle\langle 1|}_{\text{Feeding level } |1\rangle} + \underbrace{\Gamma_{32} \rho_{33} |2\rangle\langle 2|}_{\text{Feeding level } |2\rangle}$$

We can find the equations of motion for the density matrix most easily using the non-Hermitian Schrödinger equation: $\frac{\partial}{\partial t} |\psi\rangle = -\frac{i}{\hbar} \hat{H}_{\text{eff}} |\psi\rangle$, which includes the decay, but not feeding.

$$|\psi\rangle = c_1 |1\rangle + c_2 |2\rangle + c_3 |3\rangle$$

$$\Rightarrow \dot{c}_1 = -\frac{i}{\hbar} \langle 1 | \hat{H}_{\text{eff}} | \psi \rangle = -i\frac{\Omega_1}{2} c_3; \quad \dot{c}_2 = -\frac{i}{\hbar} \langle 2 | \hat{H}_{\text{eff}} | \psi \rangle = i\delta c_2 - \frac{i}{2} \Omega_2 c_3;$$

$$\dot{c}_3 = -\frac{i}{\hbar} \langle 3 | \hat{H}_{\text{eff}} | \psi \rangle = (i\Delta - \frac{\Gamma}{2}) c_3 - \frac{i}{2} \Omega_1 c_1 - \frac{i}{2} \Omega_2 c_2.$$

$$\rho_{\alpha\beta} = c_\alpha c_\beta^*$$

$$\dot{\rho}_{\alpha\beta} = \dot{c}_\alpha c_\beta^* + c_\alpha \dot{c}_\beta^* + \langle \alpha | \mathcal{L}_{\text{feed}}[\hat{\rho}] | \beta \rangle$$

$$\frac{d}{dt} \rho_{11} = \dot{c}_1 c_1^* + c_1 \dot{c}_1^* + \frac{d}{dt} \rho_{11}|_{\text{feed}} = -i\frac{\Omega_1}{2} (\rho_{31} - \rho_{13}) + \Gamma_{31} \rho_{33}$$

$$\frac{d}{dt} \rho_{22} = \dot{c}_2 c_2^* + c_2 \dot{c}_2^* + \frac{d}{dt} \rho_{22}|_{\text{feed}} = -\frac{i}{2} \Omega_2 (\rho_{32} - \rho_{23}) + \Gamma_{32} \rho_{33}$$

$$\frac{d}{dt} \rho_{33} = \dot{c}_3 c_3^* + c_3 \dot{c}_3^* = -\Gamma \rho_{33} - \frac{i}{2} \Omega_1 (\rho_{13} - \rho_{31}) - \frac{i}{2} \Omega_2 (\rho_{23} - \rho_{32})$$

$$\frac{d}{dt} \rho_{23} = \dot{c}_2 c_3^* + c_2 \dot{c}_3^* = i(\delta - \Delta + i\frac{\Gamma}{2}) \rho_{23} - \frac{i}{2} \Omega_2 (\rho_{33} - \rho_{22}) + \frac{i}{2} \Omega_1 \rho_{21}$$

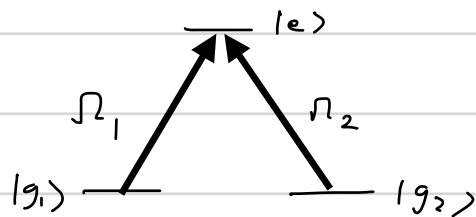
$$\frac{d}{dt} \rho_{13} = \dot{c}_1 c_3^* + c_1 \dot{c}_3^* = (-i\Delta - \frac{\Gamma}{2}) \rho_{13} - \frac{i}{2} \Omega_1 (\rho_{33} - \rho_{11}) + \frac{i}{2} \Omega_2 \rho_{12}$$

$$\frac{d}{dt} \rho_{12} = \dot{c}_1 c_2^* + c_1 \dot{c}_2^* = -i\delta \rho_{12} - i\frac{\Omega_1}{2} \rho_{32} + \frac{i}{2} \Omega_2 \rho_{13}$$

These equations determine the full dynamics of the three-level system, including decay and refueling of population by optical pumping. However, when the saturation parameter is low, often one can obtain the important physics solely from the non-Hermitian Schrödinger equation.

Problem 2: Dark States

We consider the 3-level "lambda system" with two ground states resonantly coupled to an excited state.



In the rotating frame: $\hat{H}_{RF} = \frac{\hbar\Omega_1}{2} (|g_1\rangle\langle e| + |e\rangle\langle g_1|) + \frac{\hbar\Omega_2}{2} (|g_2\rangle\langle e| + |e\rangle\langle g_2|)$

(a) The "dressed states" are the eigenstates of the coupled atom-laser system

The Hamiltonian matrix: $\hat{H}_{RF} = \frac{\hbar}{2} \begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix}$ In the ordered basis $\{|g_1\rangle, |g_2\rangle, |e\rangle\}$

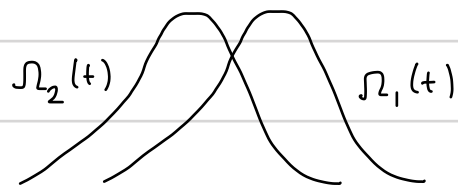
The eigenvectors/eigenvalues of this matrix are:

$$|\psi_{\text{dark}}\rangle = \frac{-\Omega_2|1\rangle + \Omega_1|2\rangle}{\sqrt{\Omega_1^2 + \Omega_2^2}}, \quad E_{\text{dark}} = 0 \quad (\text{Dark state})$$

$$|\psi_{\text{coup}, \pm}\rangle = \frac{1}{\sqrt{2}} (|\psi_{\text{Bright}}\rangle \pm |e\rangle), \quad E_{\text{coup}, \pm} = \pm \frac{\hbar}{2} \sqrt{\Omega_1^2 + \Omega_2^2} \quad (\text{Coupled states})$$

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 $(\Omega_1|1\rangle + \Omega_2|2\rangle) / \sqrt{\Omega_1^2 + \Omega_2^2}$

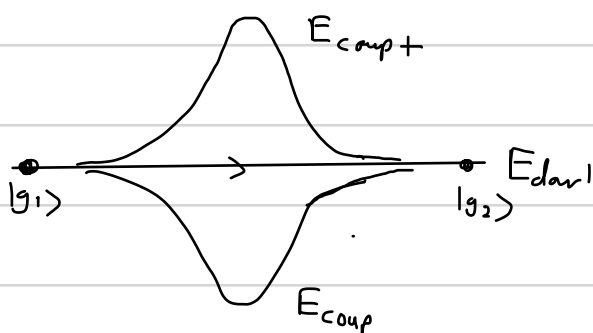
(b) One can transfer population from $|g_1\rangle$ to $|g_2\rangle$ using the "counter-intuitive pulse sequence".



First apply field that couples $|2\rangle \rightarrow |3\rangle$.

As we turn that field off turn of $|1\rangle \rightarrow |3\rangle$

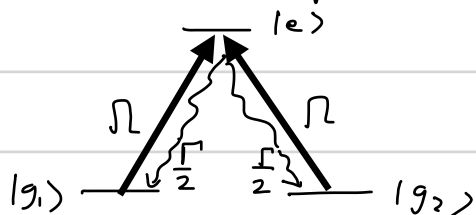
The counter-intuitive pulse sequence achieves the transfer from $|g_1\rangle \rightarrow |g_2\rangle$ by adiabatic following of the dark state. The dressed eigenvalues as a function of time appear as:



At $t=0$ $|\psi\rangle = |g_1\rangle$. As we slowly turn on Ω_2 , $|\psi\rangle$ is in the dark state (the eigenstate of \hat{H} with $\Omega_1=0$). If we turn on and off the fields slowly, so the transition is adiabatic, then $|\psi(t)\rangle$ follows $|\psi_{\text{Dark}}(t)\rangle = \frac{-\Omega_2(t)|2\rangle - \Omega_1(t)|1\rangle}{\sqrt{\Omega_1^2(t) + \Omega_2^2(t)}}$. In particular, when

$\Omega_1 = \Omega_2$ $|\psi_{\text{Dark}}\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$. At the end of pulses $|\psi(t)\rangle = |\psi_{\text{Dark}}(t \rightarrow \infty)\rangle = |2\rangle$.

(c) We now consider spontaneous decay. We simplify to the case $\Omega_1 = \Omega_2 = \Omega$; $\Gamma_{31} = \Gamma_{32} = \frac{\Gamma}{2}$



In the basis $\{|D\rangle, |B\rangle, |e\rangle\}$ where $|D\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$, $|B\rangle = \frac{|1\rangle + |2\rangle}{\sqrt{2}}$, $|e\rangle$
 $|1\rangle = \frac{1}{\sqrt{2}}(|D\rangle + |B\rangle)$; $|2\rangle = \frac{1}{\sqrt{2}}(|B\rangle - |D\rangle)$

$$\Rightarrow \hat{H} = \frac{\sqrt{2}\hbar\Omega}{2} (|B\rangle\langle e| + |e\rangle\langle B|) \quad \left(\begin{array}{l} \text{effective 2-level coupling between} \\ |B\rangle \text{ and } |e\rangle \text{ w/ Rabi frequency } \sqrt{2}\Omega \\ \text{The Dark State is decoupled} \end{array} \right)$$

the effective Hamiltonian $\hat{H}_{\text{eff}} = \hat{H} - i\frac{\hbar\Gamma}{2}|e\rangle\langle e|$

The "Feeding term" in the Master Equation: $\mathcal{L}_{\text{feed}}[\hat{\rho}] = \frac{\Gamma}{2} \rho_{ee} (|1\rangle\langle 1| + |2\rangle\langle 2|) = \frac{\Gamma}{2} \rho_{ee} (|B\rangle\langle B| + |D\rangle\langle D|)$

(d) The Master equation: $\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar} (\hat{H}_{\text{eff}} \hat{\rho} - \hat{\rho} \hat{H}_{\text{eff}}) + \mathcal{L}_{\text{feed}}[\hat{\rho}]$

Non-Hermitian Schrödinger Eqn: $\frac{\partial}{\partial t} |\psi\rangle = \frac{-i}{\hbar} \hat{H}_{\text{eff}} |\psi\rangle$

$$\Rightarrow \dot{c}_e = -\frac{\Gamma}{2} c_e - i\frac{\sqrt{2}\Omega}{2} c_B, \quad \dot{c}_B = -i\frac{\sqrt{2}\Omega}{2} c_e, \quad \dot{c}_D = 0$$

$$\Rightarrow \dot{\rho}_{ee} = \dot{c}_e c_e^* + c_e \dot{c}_e^* = -\Gamma \rho_{ee} - i\frac{\sqrt{2}\Omega}{2} (\rho_{Be} - \rho_{eB})$$

$$\dot{\rho}_{BB} = \dot{c}_B c_B^* + \dot{c}_B^* c_B + \langle B | \mathcal{L}_{\text{feed}}[\hat{\rho}] | B \rangle = +\frac{\Gamma}{2} \rho_{ee} - i\frac{\sqrt{2}\Omega}{2} (\rho_{eB} - \rho_{Be})$$

$$\dot{\rho}_{DD} = \dot{c}_D c_D^* + c_D \dot{c}_D^* + \langle D | \mathcal{L}_{\text{feed}}[\hat{\rho}] | D \rangle = \frac{\Gamma}{2} \rho_{ee}$$

$$\dot{\rho}_{eB} = \dot{c}_e c_B^* + c_e \dot{c}_B^* = -\frac{\Gamma}{2} \rho_{eB} + i\frac{\sqrt{2}\Omega}{2} (\rho_{ee} - \rho_{BB})$$

$$\dot{\rho}_{eD} = \dot{c}_e c_D^* + c_e \dot{c}_D^* = -\frac{\Gamma}{2} \rho_{eD} - i\frac{\sqrt{2}\Omega}{2} \rho_{BD}$$

$$\dot{\rho}_{BD} = \dot{c}_B c_D^* + c_B \dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2} \rho_{eD}$$

The steady-state solution: $\dot{\rho}_{BB} = \frac{\Gamma}{2} \rho_{ee} = 0 \Rightarrow \rho_{ee} = 0$

$$\Rightarrow \dot{\rho}_{ee} = i\frac{\sqrt{2}\Omega}{2} (\rho_{eB} - \rho_{Be}) = 0 \Rightarrow \text{Im}(\rho_{eB}) = 0$$

$$\Rightarrow \dot{\rho}_{eB} = -\frac{\Gamma}{2} \rho_{eB} - i\frac{\sqrt{2}\Omega}{2} \rho_{BB} = 0 \Rightarrow \text{Re}(\rho_{eB}) = 0 \text{ and } \rho_{BB} = 0 \text{ (since Real + Imag = 0)}$$

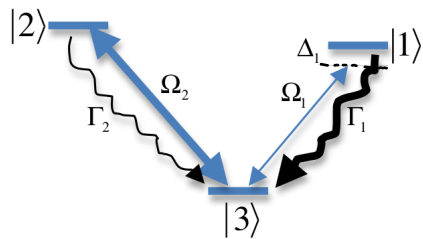
$$\left. \begin{aligned} \dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e \dot{c}_D^* = -\frac{\Gamma}{2} \rho_{eD} + i\frac{\sqrt{2}\Omega}{2} \rho_{BD} = 0 \\ \dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B \dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2} \rho_{eD} = 0 \end{aligned} \right\} \Rightarrow \rho_{eD} = 0, \rho_{BD} = 0$$

\Rightarrow Dynamically ALL elements of $\hat{\rho} \rightarrow 0$ in steady state except ρ_{DD}

\Rightarrow Since the equation is trace preserving, in steady state, $\hat{\rho}_{s.s.} = |D\rangle\langle D|$

Problem 3: Autler-Townes Splitting

We consider a V-configuration:



A strong field on the $|2\rangle \leftrightarrow |3\rangle$ transition "dresses" these states. We then probe these dressed states with a weak probe on the $|3\rangle \leftrightarrow |1\rangle$ transition and measure the absorption as a function of detuning, Δ_1 .

(a) The (trace-preserving) master equation for this 3-level atom is:

$$\frac{d}{dt} \hat{\rho} = -\frac{i}{\hbar} (\hat{H}_{\text{eff}} \hat{\rho} - \hat{\rho} \hat{H}_{\text{eff}}^\dagger) + (\Gamma_1 \rho_{11} + \Gamma_2 \rho_{22}) |3\rangle\langle 3|$$

$$\text{where } \hat{H}_{\text{eff}} = -\frac{\hbar}{2} (\Delta_1 + i\frac{\Gamma_1}{2}) |1\rangle\langle 1| - \frac{i\hbar}{2} \Gamma_2 |2\rangle\langle 2| + \frac{\hbar\Omega_1}{2} (|3\rangle\langle 1| + |1\rangle\langle 3|) + \frac{\hbar\Omega_2}{2} (|3\rangle\langle 2| + |2\rangle\langle 3|)$$

Non-unitary evolution of probability amplitudes: $\frac{d}{dt} c_\alpha = -\frac{i}{\hbar} \langle \alpha | \hat{H}_{\text{eff}} | \psi \rangle$, $|\psi\rangle = \sum_{\alpha=1}^3 c_\alpha |\alpha\rangle$

$$\Rightarrow \dot{c}_1 = (i\Delta_1 - \frac{\Gamma_1}{2}) c_1 - i\frac{\Omega_1}{2} c_3, \quad \dot{c}_2 = -\frac{\Gamma_2}{2} c_2 - i\frac{\Omega_2}{2} c_3, \quad \dot{c}_3 = -i\frac{\Omega_1}{2} c_1 - i\frac{\Omega_2}{2} c_2$$

The Master equation: $\dot{\rho}_{\alpha\beta} = \dot{c}_\alpha c_\beta^* + c_\alpha \dot{c}_\beta^* + (\Gamma_1 \rho_{11} + \Gamma_2 \rho_{22}) \delta_{\alpha 3} \delta_{\beta 3}$

$$\Rightarrow \dot{\rho}_{11} = \dot{c}_1 c_1^* + c_1 \dot{c}_1^* = -\Gamma_1 \rho_{11} - i\frac{\Omega_1}{2} (\rho_{31} - \rho_{13}), \quad \dot{\rho}_{22} = \dot{c}_2 c_2^* + c_2 \dot{c}_2^* = -\Gamma_2 \rho_{22} - i\frac{\Omega_2}{2} (\rho_{32} - \rho_{23})$$

$$\dot{\rho}_{33} = -\dot{\rho}_{11} - \dot{\rho}_{22} = +(\Gamma_1 \rho_{11} + \Gamma_2 \rho_{22}) + i\frac{\Omega_1}{2} (\rho_{31} - \rho_{13}) + i\frac{\Omega_2}{2} (\rho_{32} - \rho_{23})$$

$$\dot{\rho}_{13} = \dot{c}_1 c_3^* + c_1 \dot{c}_3^* = (i\Delta_1 - \frac{\Gamma_1}{2}) \rho_{13} - i\frac{\Omega_1}{2} (\rho_{33} - \rho_{11}) + i\frac{\Omega_2}{2} \rho_{12}$$

$$\dot{\rho}_{23} = \dot{c}_2 c_3^* + c_2 \dot{c}_3^* = -\frac{\Gamma_2}{2} \rho_{23} - i\frac{\Omega_2}{2} (\rho_{33} - \rho_{22}) + i\frac{\Omega_1}{2} \rho_{21}$$

$$\dot{\rho}_{12} = \dot{c}_1 c_2^* + c_1 \dot{c}_2^* = (i\Delta_1 - \frac{\Gamma_1 + \Gamma_2}{2}) \rho_{12} + i\frac{\Omega_2}{2} \rho_{13} - i\frac{\Omega_1}{2} \rho_{32}$$

(b) Under the assumption of weak excitation of $|1\rangle$, retain terms only linear in Ω_1 .

The steady state solutions, $\dot{\rho}_{\alpha\beta} = 0$

(Next page)

$$-\Gamma_1 \rho_{11} - \frac{i}{2} \Omega_1 (\rho_{31} - \rho_{13}) = 0$$

$$(i\Delta_1 - \frac{\Gamma_1}{2}) \rho_{13} - \frac{i\Omega_1}{2} (\rho_{33} - \rho_{11}) + \frac{i}{2} \Omega_2 \rho_{12} \approx 0$$

$$(i\Delta_1 - \frac{\Gamma_1}{2}) \rho_{12} + \frac{i}{2} \Omega_2 \rho_{13} - \frac{i}{2} \Omega_1 \rho_{33} \approx 0$$

$$\Rightarrow \rho_{11} = \frac{\Omega_1}{\Gamma_1} \left(\frac{\rho_{13} - \rho_{31}}{2i} \right) = \frac{\Omega_1}{\Gamma_1} \text{Im}(\rho_{13})$$

$$\Rightarrow \rho_{13} \approx \frac{-\Omega_1}{(2\Delta_1 + i\Gamma_1)} \rho_{33} + \frac{\Omega_2}{(2\Delta_1 + i\Gamma_1)} \rho_{21}$$

$$\Rightarrow \rho_{12} \approx \left(\frac{\Omega_2}{(2\Delta_1 + i\Gamma_1)} \right) \rho_{13}$$

(c) Putting this all together:

$$\rho_{13} \approx \frac{-\Omega_1}{2\Delta_1 + i\Gamma_1} \rho_{33} + \frac{\Omega_2^2}{(2\Delta_1 + i\Gamma_1)^2} \rho_{13} \Rightarrow \rho_{13} \approx \frac{-\Omega_1 (2\Delta_1 + i\Gamma_1)}{(2\Delta_1 + i\Gamma_1)^2 - \Omega_2^2} \rho_{33}$$

$$\Rightarrow \rho_{11} = -\frac{\Omega_1^2}{\Gamma_1^2} \rho_{33} \text{Im} \left[\frac{2\Delta_1 + i\Gamma_1}{(2\Delta_1 + i\Gamma_1)^2 - \Omega_2^2} \right]$$

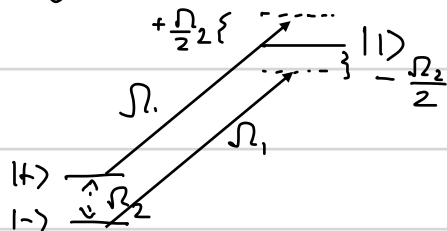
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(d) The Autler-Two splitting is understood in the Dressed-basis of the strongly-coupled $|2\rangle - |3\rangle$ system:

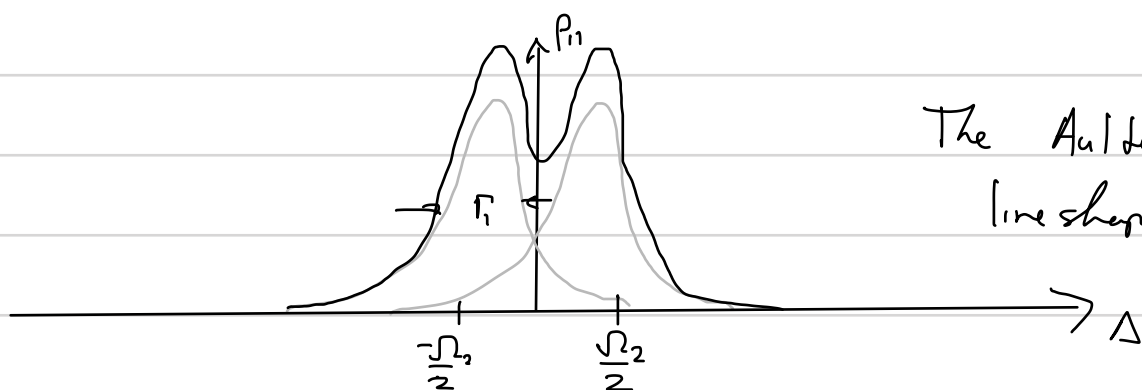
Sketch for $\Delta_1 = 0$



There are two resonances $|1\rangle \rightarrow |1\rangle$
 $|1\rangle \rightarrow |2\rangle$

$$|\pm\rangle = \frac{|2\rangle \pm |3\rangle}{\sqrt{2}}$$

These two resonances are split by the dressed state splitting $\pm \frac{\Omega_2}{2}$. The width of each resonance is Γ_1 .



The Autler-Two lineshape

(c) The difference between EIT and Autler-Townes splitting is that the former is an interference effect arising from the destructive interference of two paths, whereas Autler-Townes is the incoherent sum of two distinguishable paths. In EIT the absorption goes exactly to zero at $\Delta_1 = 0$ whereas in Autler-Townes, it is reduced.

When $\Omega_2 \gg \Gamma_1$, the two paths of EIT are essentially distinguishable - EIT and Autler-Townes absorption spectra look essentially the same in this situation, i.e., two separated Lorentzians, with essentially zero absorption between.