Physics 566: Quantum Optics I
Problem Set 6
Due: Thursday, October 19, 2023

Problem 1: Momentum and Angular Momentum in the E&M Field (25 points)

From classical electromagnetic field theory, we know that conservation laws require that the field carry momentum and angular momentum

$$\mathbf{P} = \int d^3x \left(\frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right), \quad \mathbf{J} = \int d^3x \left(\mathbf{x} \times \frac{\mathbf{E}(\mathbf{x}) \times \mathbf{B}(\mathbf{x})}{4\pi c} \right).$$

(a) Show that when these quantities become field operators, the momentum operator becomes, $\hat{\mathbf{P}} = \sum_{\mathbf{k},\lambda} \hbar \mathbf{k} \hat{a}^\dagger_{\mathbf{k},\lambda} \hat{a}_{\mathbf{k},\lambda}$; interpret.

(b) Show that $\mathbf{J} = \mathbf{J}_{orb} + \mathbf{J}_{spin}$

where $\mathbf{J}_{spin} = \dfrac{1}{4\pi c} \displaystyle\int d^3x \, (\mathbf{E}(\mathbf{x}) \times \mathbf{A}(\mathbf{x}))$, this is spin angular momentum.

$\mathbf{J}_{orb} = \dfrac{1}{4\pi c} \displaystyle\int d^3x \, E_i(\mathbf{x}) (\mathbf{x} \times \nabla) A_i(\mathbf{x})$, orbital angular momentum.

(c) Show that $\hat{\mathbf{J}}_{spin} = \hbar \sum_{\mathbf{k}} \left(\hat{a}^\dagger_{\mathbf{k},+} \hat{a}_{\mathbf{k},+} - \hat{a}^\dagger_{\mathbf{k},-} \hat{a}_{\mathbf{k},-} \right) \mathbf{e}_{\mathbf{k}}$. Interpret this.

Extra credit: Show $\hat{\mathbf{J}}_{orb} = \sum_{\mathbf{k},\mathbf{k}'} \sum_{\lambda} \hat{a}^\dagger_{\mathbf{k}'\lambda} \left(i\hbar \nabla_{\mathbf{k}} \delta(\mathbf{k} - \mathbf{k}') \times \mathbf{k} \right) \hat{a}_{\mathbf{k}\lambda}$. Interpret this.

(d) The spin of the photon has magnitude S=1, yet there are only two helicity states. Thus, we can map the spin angular momentum onto the Bloch (Poincaré) sphere for S=1/2, via

$$\hat{\mathbf{J}}_{spin} = \hat{J}_x \mathbf{e}_x + \hat{J}_y \mathbf{e}_y + \hat{J}_z \mathbf{e}_z,$$

$$\text{with } J_z = \frac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z+} - \hat{a}^\dagger_{z-} \hat{a}_{z-}), \quad J_x = \frac{\hbar}{2} (\hat{a}^\dagger_{z+} \hat{a}_{z-} + \hat{a}^\dagger_{z-} \hat{a}_{z+}), \quad J_y = \frac{\hbar}{2i} (\hat{a}^\dagger_{z+} \hat{a}_{z-} - \hat{a}^\dagger_{z-} \hat{a}_{z+}),$$

where $(\hat{a}_{z+}, \hat{a}_{z-})$ are the mode operators for positive and negative helicity operators relative to a *space fixed* quantization axis.

Show that these operators satisfy the SU(2) commutation algebra for angular momentum. This relationship is known as the "Schwinger representation" of angular momentum.

(e) The mean values of $\hat{J}_x, \hat{J}_y, \hat{J}_z$ are the "Stokes parameters" in classical optics, equivalent to the components of the Bloch vector of the Poincaré sphere. Explain the relationship between these operators and the Pauli operators using "second quantization".