Phys 566, Quantum Optics

Problem Set #6 Solutions

Problem 1: Momentum and Angular Momentum in Field

$$\hat{\vec{P}} = \int d^3x \left(\frac{\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x})}{4\pi c} \right), \qquad \hat{\vec{J}} = \int d^3x \left(\vec{x} \times \frac{\hat{\vec{E}}(\vec{x}) \times \hat{\vec{B}}(\vec{x})}{4\pi c} \right)$$

Quantized field

$$\hat{\vec{A}} = \underbrace{\sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar c^2}{V\omega_k}}\, \vec{\epsilon}_{\vec{k},\lambda}\, \hat{a}_{k,\lambda}\, e^{i\vec{k}\cdot\vec{x}}}_{\hat{\vec{A}}^{(+)}} + \underbrace{h.c.}_{\vec{A}^{(-)}}$$

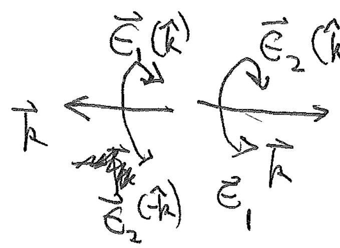
$$\hat{\vec{E}} = \underbrace{+i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \vec{\epsilon}_{\vec{k},\lambda}\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}_{k,\lambda}}_{\vec{E}^{(+)}} + \underbrace{h.c.}_{\vec{E}^{(-)}}$$

$$\hat{\vec{B}} = \underbrace{+i \sum_{\vec{k},\lambda} \sqrt{\frac{2\pi\hbar\omega_k}{V}}\, \hat{k}\times\vec{\epsilon}_{\vec{k},\lambda}\, e^{i\vec{k}\cdot\vec{x}}\, \hat{a}_{k,\lambda}}_{\vec{B}^{(+)}} + \underbrace{h.c.}_{\vec{B}^{(-)}}$$

Here I have used the basis of circularly polarized plane waves with orthogonality

$$\vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k}',\lambda'} = \delta_{\hat{k}\hat{k}'}\, \delta_{\lambda\lambda'}$$

Note also $\quad \vec{\epsilon}_{-\vec{k},\lambda_1} = \vec{\epsilon}_{\vec{k},\lambda_2}$



(1a)

Let us plug the mode decomposition into $\hat{\vec{P}}$

$$\Rightarrow \hat{\vec{P}} = \int d^3x \left(\frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} + \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(+)}}{4\pi c} + h.c. \right)$$

Consider first term:

$$\int d^3x \, \frac{\hat{\vec{E}}^{(+)} \times \hat{\vec{B}}^{(-)}}{4\pi c} = \sum_{\vec{k}\lambda, \vec{k}'\lambda'} \frac{1}{4\pi c} \left(2\pi\hbar \sqrt{c\omega_k \omega_{k'}} \right) \vec{\epsilon}_{\vec{k},\lambda} \times \left(\hat{k}' \times \vec{\epsilon}^{\,*}_{\vec{k}',\lambda'} \right) \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{= \delta^{(3)}_{\vec{k}\vec{k}'}} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k}',\lambda'}$$

$$= \sum_{\vec{k}} \sum_{\lambda,\lambda'} \frac{\hbar}{2} k \underbrace{\left[\vec{\epsilon}_{\vec{k},\lambda} \times \left(\hat{k} \times \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} \right) \right]}_{\hat{k} \left(\underbrace{\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}^{\,*}_{\vec{k},\lambda'}} \right) - \vec{\epsilon}^{\,*}_{\vec{k},\lambda'} \left(\underbrace{\hat{k} \cdot \vec{\epsilon}_{\vec{k},\lambda}}_{=0 \text{ (transversality)}} \right)} \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda'}$$

$$\Rightarrow \int d^3x \, \frac{\vec{E}^{(+)} \times \vec{B}^{(-)}}{4\pi c} = \frac{1}{2} \sum_{\vec{k},\lambda} \hbar \vec{k} \, \hat{a}_{\vec{k},\lambda} \hat{a}^\dagger_{\vec{k},\lambda}$$

Now, by similar steps

$$\int d^3x \, \frac{\vec{E}^{(+)} \times \vec{B}^{(+)}}{4\pi c} = \sum_{\vec{k}} \sum_{\lambda,\lambda'} \frac{\hbar k}{2} \left(\underbrace{\vec{\epsilon}_{\vec{k},\lambda} \times \left(-\hat{k} \times \vec{\epsilon}_{-\vec{k},\lambda'} \right)}_{-\hat{k} \left(\underbrace{\vec{\epsilon}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{-\vec{k},\lambda'}}_{=0 \text{ (see page 4)}} \right) + \vec{\epsilon}_{-\vec{k},\lambda'} \left(\underbrace{\hat{k} \cdot \vec{\epsilon}_{\vec{k},\lambda}}_{0} \right)} \right)$$

$$= 0$$

Thus,

$$\hat{\vec{P}} = \int d^3x \, \frac{\vec{E}^{(+)} \times \vec{B}^{(\,)}}{4\pi c} + h.c.$$

$$= \sum_{\vec{k},\lambda} \hbar \vec{k} \left(\frac{\hat{a}_{\vec{k},\lambda} \hat{a}^{+}_{\vec{k},\lambda} + \hat{a}^{+}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}}{2} \right)$$

$$= \sum_{\vec{k},\lambda} \hbar \vec{k} \, \hat{a}^{+}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda} + \frac{\hbar}{2} \sum_{\lambda} \cancel{\left(\sum_{\vec{k}} \vec{k} \right)} \rightarrow 0$$

no mean momentum!

$$\Rightarrow \boxed{\hat{\vec{P}} = \sum_{\vec{k},\lambda} \hbar \vec{k} \, \hat{a}^{+}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}}$$

Neat! in the plane wave decomposition, we see that the momentum in the field decompose into sum of photon momenta $\hbar k$ times the number operator

$$\hat{N}_{\vec{k}} = \sum_{\lambda} \hat{a}^{+}_{\vec{k},\lambda} \hat{a}_{\vec{k},\lambda}$$ counting the number of photons with wave vector $\vec{k}$.

(1b) Total angular momentum in field:

$$\hat{\vec{J}} = \int d^3x \; \vec{x} \times \hat{\vec{P}}(\vec{x})$$

where $\hat{\vec{P}}(\vec{x}) = \frac{1}{4\pi c} \left(\hat{\vec{E}} \times \hat{\vec{B}} \right)$ = momentum density

Lets massage these equations a bit.

$$(\hat{\vec{E}} \times \hat{\vec{B}})_i = \epsilon_{ijk} E_j B_k \quad \text{(summation convention)}$$

$$= \epsilon_{ijk} E_j \epsilon_{k\ell m} \partial_\ell A_m$$

$$= (\delta_{i\ell}\delta_{jm} - \delta_{im}\delta_{j\ell}) E_j \partial_\ell A_m$$

$$= E_\ell \partial_i A_\ell - E_\ell \partial_\ell A_i$$

Now $\vec{J} = \int d^3x \; \vec{x} \times \vec{P}(\vec{x})$

$$\Rightarrow \quad J_j = \epsilon_{jki} \int d^3x \; x_k P_i$$

$$= \epsilon_{jki} \frac{1}{4\pi c} \int d^3x \left[E_\ell (x_k \partial_i) A_\ell - (x_k E_\ell)(\partial_\ell A_i) \right]$$

$$= \frac{1}{4\pi c} \int d^3x \; E_\ell (\vec{x} \times \vec{\nabla})_j A_\ell$$

$$+ \frac{1}{4\pi c} \int d^3x \, \epsilon_{jki} \underbrace{\partial_\ell (x_k E_\ell)}_{[\delta_{\ell k} + \vec{\nabla} \cdot \vec{E}]} A_i \quad \text{(integration by parts)}$$

$\vec{\nabla} \cdot \vec{E} \to 0$ in free space

$$\Rightarrow J_j = \frac{1}{4\pi c}\left(\int d^3x \; E_\ell \, (\vec{x}\times\vec{\nabla})_j \, A_\ell + \int d^3x \; (\vec{E}\times\vec{A})_j \right)$$

$$\Rightarrow \vec{J} = \vec{J}_{orb} + \vec{J}_{spin}$$

$$\boxed{\begin{aligned} \vec{J}_{orb} &= \frac{1}{4\pi c}\int d^3x \; E_\ell \, (\vec{x}\times\vec{\nabla}) \, A_\ell \\ \vec{J}_{spin} &= \frac{1}{4\pi c}\int d^3x \; (\vec{E}\times\vec{A}) \end{aligned}}$$

(1c)

Let us expand these terms in the plane wave basis:

$$\vec{J}_{orb} = \left(\frac{1}{4\pi c}\int d^3x \; E_\ell^{(-)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)} + h.c. \right) + \left(\frac{1}{4\pi c}\int d^3x \; E_\ell^{(+)} (\vec{x}\times\vec{\nabla}) \vec{A}_\ell^{(+)} + h.c. \right)$$

Consider

$$\frac{1}{4\pi c}\int d^3x \; E_\ell^{(-)} (\vec{x}\times\vec{\nabla}) A_\ell^{(+)}$$

$$= \sum_{\vec{k}\vec{k}',\lambda\lambda'} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}} \; \hat{a}^{\dagger}_{\vec{k}',\lambda'} \hat{a}_{\vec{k}\lambda} \; \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \cdot \vec{\epsilon}_{\vec{k}',\lambda'} \quad \left(\text{Summing over } \ell\right)$$

$$\underbrace{\int \frac{d^3x}{V} \, e^{-i\vec{k}'\cdot\vec{x}} \left(\vec{x}\times -i\vec{\nabla}\right) e^{i\vec{k}\cdot\vec{x}}}_{\mathcal{d}}$$

Aside: $\mathcal{I} = \int \frac{d^3x}{V} e^{-i\vec{k}'\cdot\vec{x}} (\vec{x} \times -i\vec{\nabla}) e^{i\vec{k}\cdot\vec{x}}$

$$= \int \frac{d^3x}{V} e^{-i\vec{k}'\cdot\vec{x}} (\vec{x} \times \vec{k}) e^{i\vec{k}\cdot\vec{x}}$$

$$= \left[\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}} \vec{x}\right] \times \vec{k}$$

$$= +i\vec{\nabla}_{\vec{k}'} \underbrace{\left[\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}\right]}_{\delta_{\vec{k},\vec{k}'}} \times \vec{k}$$

$$\Rightarrow \frac{1}{4\pi c}\int d^3x\, \hat{E}^{(-)}_{\ell} (\vec{x}\times\vec{\nabla}) \hat{A}^{(+)}_{\ell}$$

$$= \sum_{\substack{\vec{k},\vec{k}',\\ \lambda,\lambda'}} \frac{\hbar}{2}\sqrt{\frac{\omega_k}{\omega_{k'}}}\, \hat{a}^{\dagger}_{\vec{k}'\lambda'} \hat{a}_{\vec{k},\lambda}\, \vec{\epsilon}^{\,*}_{\vec{k}\lambda}\cdot\vec{\epsilon}_{\vec{k}'\lambda'} \; +i\nabla_{\vec{k}'}\delta_{(\vec{k}',\vec{k})} \times \vec{k}$$

Though this is a finite sum, we are ultimately interested in the limit $V \to \infty$, when the sum goes to an integral and we can perform integration by parts.

$$\Rightarrow = \frac{1}{2}\sum_{\vec{k},\lambda} \left(+i\vec{\nabla}_{\vec{k}}\, \hat{a}_{\vec{k},\lambda}\right)^{\dagger} \times \hbar\vec{k}\, \hat{a}_{\vec{k},\lambda}$$

having used $\vec{\epsilon}_{\vec{k}\lambda}\cdot\vec{\epsilon}_{\vec{k}\lambda'} = \delta_{\lambda\lambda'}$

after integrating over $\vec{k}'$ with the delta funct.

Adding the conjugate term: $E_\ell^{(+)}(\vec{x}\times\vec{\nabla})A_\ell^{(-)}$

$$\vec{J}_{orbital} = \frac{1}{2}\sum_{\vec{k},\lambda}\Big[(i\vec{\nabla}_{\vec{k}}\hat{a}_{\vec{k},\lambda})^{\dagger}\times\hbar\vec{k}\,\hat{a}_{\vec{k},\lambda} + \hbar\vec{k}\,\hat{a}^{\dagger}_{\vec{k}\lambda}\times(i\vec{\nabla}_{\vec{k}}\hat{a}_{\vec{k},\lambda})\Big]$$

$$= \sum_{\vec{k},\lambda}(i\vec{\nabla}_{\vec{k}}\hat{a}_{\vec{k}\lambda})^{\dagger}\times\hbar\vec{k}\,\hat{a}_{\vec{k}\lambda}$$

$$+ \underbrace{\sum_{\vec{k},\lambda}\hbar\vec{k}\; i\vec{\nabla}_{\vec{k}}(\hat{a}^{\dagger}_{\vec{k}\lambda}\hat{a}_{\vec{k}\lambda})}_{\text{Integration by parts} \Rightarrow 0}$$

$$\boxed{\Rightarrow \vec{J}_{orbital} = \sum_{\vec{k},\lambda}(i\vec{\nabla}_{\vec{k}}\hat{a}_{\vec{k}\lambda})^{\dagger}\times\hbar\vec{k}\,\hat{a}_{\vec{k}\lambda}}$$

or equivalent

$$\boxed{= \sum_{\vec{k},\vec{k}',\lambda}\hat{a}^{\dagger}_{\vec{k}',\lambda}\left[i\vec{\nabla}_{\vec{k}'}\delta(\vec{k}-\vec{k}')\times\hbar\vec{k}\right]\hat{a}_{\vec{k}\lambda}}$$

This is the "second quantized" form of orbital angular momentum. Recall from wave mechanics

$$\hat{L}_{orbital} = \hat{\vec{x}}\times\hat{\vec{p}} \doteq \left[-i\vec{\nabla}_{\vec{k}'}\delta(\vec{k}-\vec{k}')\right]\times\hbar\vec{k}$$

in momentum space! Thus, if we have an electromagnetic wave packet (beam/pulse) we general carry both orbital and spin angular momentum.

Consider now the spin term

$$\int d^3x \, \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{-i\hbar}{2} \sum_{\vec{k}\vec{k}',\lambda\lambda'} \hat{a}^\dagger_{\vec{k}\lambda} \hat{a}_{\vec{k}'\lambda'} \, \vec{\epsilon}^{\,*}_{\vec{k},\lambda} \times \vec{\epsilon}_{\vec{k}',\lambda'} \underbrace{\int \frac{d^3x}{V} e^{i(\vec{k}-\vec{k}')\cdot\vec{x}}}_{\to\ \delta^{(3)}(\vec{k}-\vec{k}')}$$

$$= \frac{-i\hbar}{2} \sum_{\vec{k}} \left[(\vec{\epsilon}^{\,*}_{\vec{k},+} \times \vec{\epsilon}_{\vec{k},+}) \hat{a}^\dagger_{\vec{k}+} \hat{a}_{\vec{k}+} + (\vec{\epsilon}^{\,*}_{\vec{k},-} \times \vec{\epsilon}_{\vec{k},-}) \hat{a}^\dagger_{\vec{k}-} \hat{a}_{\vec{k}-} \right]$$

Aside $\vec{\epsilon}_{\vec{k},\pm} \equiv \frac{\vec{e}_1 \pm i\vec{e}_2}{\sqrt{2}}$ where $\vec{e}_1$ and $\vec{e}_2$ are two orthonormal vectors with $\vec{e}_1 \times \vec{e}_2 = \hat{k}$

$$\Rightarrow \vec{\epsilon}^{\,*}_{\vec{k},\pm} \times \vec{\epsilon}_{\vec{k},\pm} = \pm i \, \vec{e}_k$$

$$\Rightarrow \int d^3x \, \frac{\vec{E}^{(-)} \times \vec{A}^{(+)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}^\dagger_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^\dagger_{\vec{k},-} \hat{a}_{\vec{k},-}) \vec{e}_{\vec{k}}$$

Now $$\int d^3x \, \frac{\vec{E}^{(+)} \times \vec{A}^{(-)}}{4\pi c} = \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}_{\vec{k}+} \hat{a}^\dagger_{\vec{k},+} - \hat{a}_{\vec{k}-} \hat{a}^\dagger_{\vec{k}-}) \vec{e}_k$$

$$= \frac{\hbar}{2} \sum_{\vec{k}} (\hat{a}^\dagger_{\vec{k}+} \hat{a}_{\vec{k}+} - \hat{a}^\dagger_{\vec{k}-} \hat{a}_{\vec{k}-}) \vec{e}_k$$ (commutator cancels)

Finally note: $\vec{\epsilon}_{k,\pm} \times \vec{\epsilon}_{k,\pm} = 0$

$$\Rightarrow \int d^3x \, \vec{E}^{(+)} \times \vec{A}^{(+)} = \int d^3x \, \vec{E}^{(-)} \times \vec{A}^{(-)} = 0$$

Thus
$$\boxed{\vec{J}_{spin} = \hbar \sum_{\vec{k}} \left(\hat{a}^{\dagger}_{\vec{k},+} \hat{a}_{\vec{k},+} - \hat{a}^{\dagger}_{\vec{k},-} \hat{a}_{\vec{k},-} \right) \vec{e}_k}$$

Each photon has intrinsic "spin" angular momentum. In the circularly polarized, plane wave basis, the photon has a definite helicity, ~~carrying~~ carry one hbar of angular momentum along (opposite to) the direction of propagation $\vec{e}_k$ for positive (negative) handed polarization.

The photon is spin $S=1$, yet there are only two states with definite projection of angular momentum, whereas, we might expect three $(2S+1 = 3)$. This is a very subtle point coming from the fact the the photon is massless. For more details see,

"Photons and Atoms",

(1d) Mapping photon spin onto a two-state Hilbert space

Define $\hat{\vec{J}}_{spin} = \hat{J}_x \vec{e}_x + \hat{J}_y \vec{e}_y + \hat{J}_z \vec{e}_z$

where $\hat{J}_x = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_y = \frac{\hbar}{2i}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+)$

$\hat{J}_z = \frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)$

This is the Schwinger representation of angular momentum connecting the "Boson algebra" $[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$ to the angular momentum algebra $[\hat{J}_i, \hat{J}_j] = i\hbar\,\epsilon_{ijk}\hat{J}_k$

Check: $[\hat{J}_x, \hat{J}_y] = \frac{\hbar^2}{4i}\left([\hat{a}_+^\dagger \hat{a}_-, -\hat{a}_-^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_-]\right)$

$= \frac{\hbar^2}{4i}\Big[2\hat{a}_+^\dagger \hat{a}_+ (\underbrace{[\hat{a}_-^\dagger, \hat{a}_-]}_{=-1}) - 2\hat{a}_-^\dagger \hat{a}_- (\underbrace{[\hat{a}_+^\dagger, \hat{a}_+]}_{=-1})\Big]$

$= i\hbar\left(\frac{\hbar}{2}(\hat{a}_+^\dagger \hat{a}_+ - \hat{a}_-^\dagger \hat{a}_-)\right) = i\hbar\,\hat{J}_z$ ✓

$[\hat{J}_x, \hat{J}_z] = \frac{\hbar^2}{4}\Big([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-]$

$+ [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]\Big)$

$= \frac{\hbar^2}{4}\left(\hat{a}_+^\dagger \hat{a}_-(-1) - \hat{a}_+^\dagger \hat{a}_-(1) + \hat{a}_-^\dagger \hat{a}_+(1) - \hat{a}_-^\dagger \hat{a}_+(-1)\right)$

$= -\frac{\hbar^2}{2}(\hat{a}_+^\dagger \hat{a}_- - \hat{a}_-^\dagger \hat{a}_+) = -i\hbar\,\hat{J}_y$ ✓

$$[\hat{J}_y, \hat{J}_z] = \frac{\hbar^2}{4i}\Big([\hat{a}_+^\dagger \hat{a}_-, \hat{a}_+^\dagger \hat{a}_+] - [\hat{a}_+^\dagger \hat{a}_-, \hat{a}_-^\dagger \hat{a}_-] - [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_+^\dagger \hat{a}_+] + [\hat{a}_-^\dagger \hat{a}_+, \hat{a}_-^\dagger \hat{a}_-]\Big)$$

$$= \frac{\hbar^2}{4i}\left(\hat{a}_+^\dagger \hat{a}_-(-1) - \hat{a}_+^\dagger \hat{a}_-(1) - \hat{a}_-^\dagger \hat{a}_+(1) + \hat{a}_-^\dagger \hat{a}_+(-1)\right)$$

$$= -\frac{\hbar^2}{\cancel{4}2i}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right)$$

$$= i\hbar\left[\frac{\hbar}{2}\left(\hat{a}_+^\dagger \hat{a}_- + \hat{a}_-^\dagger \hat{a}_+\right)\right] = i\hbar \hat{J}_x \quad \checkmark$$

The Schwinger representation is the "second quantized form" of the spin 1/2 operators

$$\hat{J}_x = \frac{\hbar}{2}\left(|+_z\rangle\langle -_z| + |-_z\rangle\langle +_z|\right)$$

$$\hat{J}_y = \frac{\hbar}{2i}\left(|+_z\rangle\langle -_z| - |-_z\rangle\langle +_z|\right)$$

$$\hat{J}_z = \frac{\hbar}{2}\left(|+_z\rangle\langle +_z| - |-_z\rangle\langle -_z|\right)$$

"Second quantize" $|\pm_z\rangle \Longrightarrow \hat{a}_\pm^\dagger$ create spin up or down

$\langle \pm_z| \Rightarrow \hat{a}_\pm$ annihilate spin up or down

Thus, we can easily map the spin angular momentum of the ~~B~~ photon onto the Bloch sphere, also known as the Poincaré sphere as we visited in PS# 1.