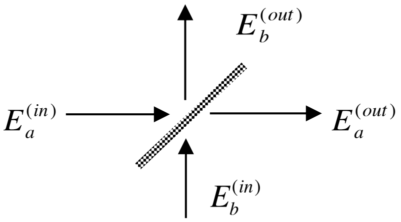
Physics 566, Quantum Optics
Problem Set #7
Due: Thursday Nov. 2, 2022

Problem1: The beam splitter and other linear transformations (25 points)

Consider a symmetric beam splitter



In the first weeks of lecture, we showed that the pair $\left(E_a^{(out)}, E_b^{(out)}\right)$ is related to $\left(E_a^{(in)}, E_b^{(in)}\right)$ through a unitary "scattering matrix"

$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

where $|t|^2 + |r|^2 = 1$, $Arg(t) = Arg(r) \pm \frac{\pi}{2}$, so that a possible transformation is,

$$E_a^{(out)} = \sqrt{T}\, E_a^{(in)} + i\sqrt{1-T}\, E_b^{(in)}, \quad E_b^{(out)} = \sqrt{T}\, E_b^{(in)} + i\sqrt{1-T}\, E_a^{(in)}, \text{ where } T = |t|^2 .$$

Classically, if we inject a field only into one input port, leaving the other empty, the field in that mode will become attenuated, e.g., $E_a^{(out)} = \sqrt{T}\, E_a^{(in)} < E_a^{(in)}$.

(a) Consider now the quantized theory for these two modes, $E_a \Rightarrow \hat{a}$, $E_b \Rightarrow \hat{b}$. Suppose again that a field is injected only into the "a-port". Show that

$$\hat{a}^{(out)} = \sqrt{T}\hat{a}^{(in)} \text{ is inconsistent with the quantum uncertainty.}$$

(b) In order to preserve the proper commutation relations we cannot ignore *vacuum fluctuations* entering the unused port. Show that if the "in" and "out" creation operators are related by the scattering matrix,

$$\begin{bmatrix} \hat{a}^{(out)\dagger} \\ \hat{b}^{(out)\dagger} \end{bmatrix} = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} \hat{a}^{(in)\dagger} \\ \hat{b}^{(in)\dagger} \end{bmatrix}, \text{ the commutator is preserved.}$$

(c) Suppose a single photon is injected into the a-port, so that the "in-state" is $\left|\psi^{(in)}\right\rangle = |1\rangle_a \otimes |0\rangle_b$. The "out-state" is $\left|\psi^{(out)}\right\rangle = \hat{S}\left|\psi^{(in)}\right\rangle$ where $\hat{S}$ is the "scattering operator", defined so that $\hat{S}\hat{a}^{(in)\dagger}\hat{S}^\dagger = \hat{a}^{(out)\dagger}$ and $\hat{S}\hat{b}^{(in)\dagger}\hat{S}^\dagger = \hat{b}^{(out)\dagger}$.

$$\text{Show that } \left|\psi^{(out)}\right\rangle = t|1\rangle_a \otimes |0\rangle_b + r|0\rangle_a \otimes |1\rangle_b .$$

(d) Suppose a coherent state is injected into the a-port $\left|\psi^{(in)}\right\rangle = |\alpha\rangle_a \otimes |0\rangle_b$. Which is the output, $\left|\psi^{(out)}\right\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b$ or $\left|\psi^{(out)}\right\rangle = r|\alpha\rangle_a \otimes |0\rangle_b + t|0\rangle_a \otimes |\alpha\rangle_b$? Explain the difference between these.

(e) A general linear optical system consisting, e.g., of beam-splitters, phase shifters, mirrors, etalons, etc. can be described by a unitary transformation on the modes

$$E_k^{(out)} = \sum_{k'} u_{kk'} E_{k'}^{(in)} .$$

In the quantum description the mode operators transform by the scattering transformation $\hat{a}_k^{(out)} = \hat{S}\hat{a}_k^{(in)}\hat{S}^\dagger = \sum_{k'} u_{kk'}\hat{a}_{k'}^{(in)}$, where $u_{kk'}$ is a unitary matrix.

Show that if we start with a multimode coherent state $\left|\psi^{(in)}\right\rangle = \left|\left\{\alpha_k^{(in)}\right\}\right\rangle$, the output state is also a coherent state, $\left|\psi^{(out)}\right\rangle = \left|\left\{\alpha_k^{(out)}\right\}\right\rangle$, with $\alpha_k^{(out)} = \sum_{k'} u_{kk'}\alpha_{k'}^{(in)}$.

(f) The previous part highlights how linear transformations are essentially classical. This was true for input with exactly one photon or for coherent states. However, this is not true for more general inputs. Suppose we send one photon into *both ports*, of a 50-50 beam-splitter T=1/2, $\left|\psi^{(in)}\right\rangle = |1\rangle_a \otimes |1\rangle_b$. Show that the output state is,

$$\left|\psi^{(out)}\right\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_a|0\rangle_b + |0\rangle_a|2\rangle_b\right).$$

This says that the two photons both going to port-a or to port-b, but never one in port-a and one in port-b. This is an effect of Bose-Einstein quantum statistics. Explain in terms of destructive interference between indistinguishable processes.

Problem 2: Boson Algebra (25 points)

This problem is to give you some practice manipulating the boson algebra.

(a) Prove the (over) completeness integral for coherent states

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \hat{1} \text{ (Hint: Expand in number states).}$$

This basis is over-complete since as the coherent states are not orthonormal (see next part).

(b) Prove the group property of the displacement operator

$$\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\exp\{i\,\mathrm{Im}(\alpha\beta^*)\}$$

$$\text{and thus } \langle\alpha|\beta\rangle = e^{-\frac{|\alpha-\beta|^2}{2}} e^{-i\mathrm{Im}(\alpha\beta^*)}$$

(c) Show that the displacement operator has the following matrix elements

Vacuum: $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-|\alpha|^2/2}$

Coherent states: $\langle \alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-|\alpha+\alpha_2-\alpha_1|^2/2} e^{i\,\mathrm{Im}\left(\alpha\alpha_2^*-\alpha_1\alpha^*-\alpha_1\alpha_2^*\right)}$

Fock states: $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\mathsf{L}_n\left(|\alpha|^2\right)$, where L_n is the Laguerre polynomial of order n

$$L_n(x) = \sum_{m=0}^{n} \binom{n}{m} \frac{(-1)^m}{m!} x^m$$

Problem 3: Thermal Light (25 points)

Consider a single mode field in thermal equilibrium at temperature *T*, Boltzmann factor $\beta = 1/k_B T$. The state of the field is described by the "canonical ensemble",
$\hat{\rho} = \frac{1}{Z} e^{-\beta \hat{H}}$, $\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a}$ is the Hamiltonian and $Z = Tr\left(e^{-\beta \hat{H}}\right)$ is the partition function.

(a) Remind yourself of the basic properties by deriving the following:

- $\langle n \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}$ (the Planck spectrum)
- $P_n = \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}}$ (the Bose-Einstein distribution).

(b) Make a list-plot of P_n for both the thermal state and the coherent state on the same graph as a function of *n,* for each of the following: $\langle n \rangle = 0.1, 1, 10, 100$.

(c) Use the Bose-Einstein distribution to show that for a thermal state

$$\Delta n^2 = \langle n \rangle^2 + \langle n \rangle$$

(d) Show that the Glauber-Sudharshan distribution of this state, $P(\alpha) = \frac{1}{\pi \langle n \rangle} e^{-|\alpha|^2/\langle n \rangle}$,

satisfies $\int d^2\alpha\, P(\alpha) |\alpha\rangle\langle\alpha| = \sum_n \frac{\langle n \rangle^n}{\left(1 + \langle n \rangle\right)^{n+1}} |n\rangle\langle n|$. Sketch $P(\alpha)$ in the phase plane.

(e) In class, we studied the m^{th}-order correlation function that gives the average number of *m*-photon coincident counts in a given time interval. For the thermal state we showed that that this could be written for $\langle n \rangle \ll 1$, as

$$G^{(m)}(0) = \langle : \hat{n}^m : \rangle = m! \langle \hat{n} \rangle^m.$$

Using the Bose-Einstein distribution, show that this expression is exact, for any value of $\langle \hat{n} \rangle$. Repeat the calculation and confirm this using the Glauber-P representation. Interpret in terms of intensity fluctuations in "chaotic light."