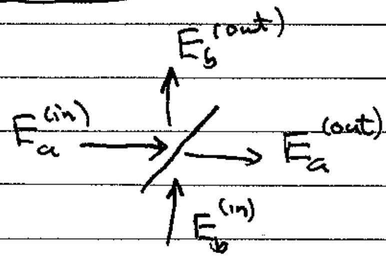
Physics 566

Problem Set # 7 : Solutions

The beam splitter and other linear transformations



$$\begin{bmatrix} E_a^{(out)} \\ E_b^{(out)} \end{bmatrix} = \underbrace{\begin{bmatrix} t & r \\ r & t \end{bmatrix}}_{\text{"S-matrix" } S} \begin{bmatrix} E_a^{(in)} \\ E_b^{(in)} \end{bmatrix}$$

(a) Unitarity of the S-matrix: $S^\dagger S = \mathbb{1}$

$$S S^\dagger = \begin{bmatrix} t & r \\ r & t \end{bmatrix} \begin{bmatrix} t^* & r^* \\ r^* & t^* \end{bmatrix} = \begin{bmatrix} |t|^2 + |r|^2 & tr^* + t^* r \\ t^* r + t r^* & |t|^2 + |r|^2 \end{bmatrix}$$

$\Rightarrow \boxed{|t|^2 + |r|^2 = 1}$ $\qquad Re(t r^*) = 0$

$\qquad\qquad\qquad\qquad \Rightarrow \boxed{Arg(t) - Arg(r) = \pm \pi/2}$

Let $T = |t|^2$

$\Rightarrow t = \sqrt{T}\, e^{i\phi_t} \qquad r = i\sqrt{1-T}\, e^{i\phi_t}$

ϕ_t depends on details on beamsplitter

for $\phi_t = 0$

$$\boxed{\begin{aligned} E_a^{(out)} &= \sqrt{T}\, E_a^{(in)} + i\sqrt{1-T}\, E_b^{(in)} \\ E_b^{(out)} &= \sqrt{T}\, E_b^{(in)} + i\sqrt{1-T}\, E_a^{(in)} \end{aligned}}$$

(b) Quantized mode: $E_a \Rightarrow \hat{a}$ $\quad E_b \Rightarrow \hat{b}$

Suppose no field is injected into port "b"

Classically $\quad E_a^{(out)} = \sqrt{T}\, E_a^{(in)}$

Quantum analog $\quad \hat{a}^{(out)} = \sqrt{T}\, \hat{a}^{(in)}$?

<u>No</u> $\quad \left[\hat{a}^{(out)}, \hat{a}^{(out)\dagger}\right] = T\left[\hat{a}^{(in)}, \hat{a}^{(in)\dagger}\right] = T \leq 1$

(c) ~~So~~ the uncertainty principle would be <u>violated</u>.

The problem is that we allowed <u>attenuation of vacuum fluctuations</u>. Formally, we violated unitarity in the transformation between input and output:

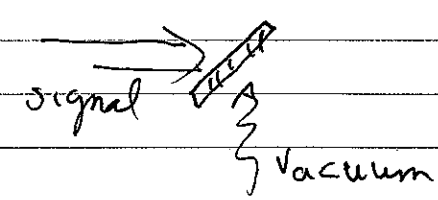
$$\hat{a}^{(out)} = \hat{S}^{\dagger}\hat{a}^{(in)}\hat{S} = \underbrace{\sqrt{T}}_{t}\,\hat{a}^{(in)} + \underbrace{i\sqrt{1-T}}_{r}\,\hat{b}^{(in)}$$

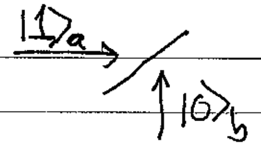
$$\Rightarrow \left[\hat{a}^{(out)}, \hat{a}^{(out)\dagger}\right] = |t|^2\left[\hat{a}^{(in)}, \hat{a}^{(in)\dagger}\right] + |r|^2\left[\hat{b}^{(in)}, \hat{b}^{(in)\dagger}\right]$$

$$+\, t r^* \underbrace{\left[\hat{a}^{(in)}, \hat{b}^{(in)\dagger}\right]}_{0} + t^* r \underbrace{\left[\hat{a}^{(in)\dagger}, \hat{b}^{(in)}\right]}_{0}$$

$$\Rightarrow \boxed{\left[\hat{a}^{(out)}, \hat{a}^{(out)\dagger}\right] = |t|^2 + |r|^2 = 1}$$

One way to interpret this is that although we do not input a signal into port-b, vacuum fluctuations <u>always</u> enter that port



(d) 

Input single photon into mode-a and nothing in mode-b

$$\Rightarrow |\Psi_{in}\rangle = |1\rangle_a \otimes |0\rangle_b = \hat{a}^{(in)\dagger}|0,0\rangle \leftarrow \text{total vacuum}$$

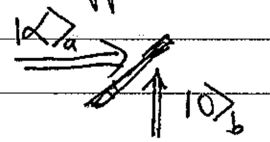
$$|\Psi_{out}\rangle = \hat{S}|\Psi_{in}\rangle = \hat{a}^{(out)\dagger}|0,0\rangle$$

$$= \left(t\,\hat{a}^{(in)\dagger} + r\,\hat{b}^{(in)\dagger}\right)|0,0\rangle$$

$$= t|1,0\rangle + r|0,1\rangle$$

$$\boxed{|\Psi_{out}\rangle = t|1\rangle_a \otimes |0\rangle_b + r|0\rangle_a \otimes |1\rangle_b}$$

(e) Now suppose we inject a coherent state



$$|\Psi_{in}\rangle = |\alpha\rangle_a \otimes |0\rangle_b$$

$$= \hat{D}_a^{(in)}(\alpha)|0,0\rangle$$

$\uparrow$ displacement operator)

$$\hat{D}_a^{(in)}(\alpha) \equiv \exp\{\alpha\,\hat{a}^{(in)} + \alpha^*\,\hat{a}^{(in)\dagger}\}$$

$$\Rightarrow |\Psi_{out}\rangle = \hat{S}|\Psi_{in}\rangle = \hat{S}\hat{D}_a^{(in)}\hat{S}^\dagger\underbrace{\hat{S}|0,0\rangle}_{=|0,0\rangle}$$

$$= \exp\{\alpha\,\hat{S}\hat{a}^{(in)}\hat{S}^\dagger + \alpha^*\,\hat{S}\hat{a}^{(in)\dagger}\hat{S}\}\,|0,0\rangle$$

$$= \exp\{\alpha\,\hat{a}^{(out)} + \alpha^*\,\hat{a}^{(out)\dagger}\}|0,0\rangle$$

$$= \exp\{\alpha(t\,\hat{a}^{(in)} + r\,\hat{b}^{(in)}) + \alpha^*(t^*\hat{a}^{(in)\dagger} + r^*\hat{b}^{(in)\dagger})\}$$

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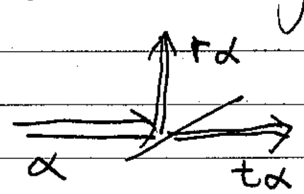
Since $\hat{a}^{(in)}$ and $\hat{b}^{(in)}$ modes commute

$$|\Psi_{out}\rangle = \hat{D}_a^{(in)}(t\alpha)\,\hat{D}_b^{(in)}(r\alpha)\,|0,0\rangle$$

$$\Rightarrow \boxed{|\Psi_{out}\rangle = |t\alpha\rangle_a \otimes |r\alpha\rangle_b}$$

This is the classically expected result



This is in contrast to the state

$$t|\alpha_a\rangle \otimes |0\rangle_b + r|0\rangle_a \otimes |\alpha\rangle_b$$

which is a "Schrödinger cat" state. This state describes a superposition of two macroscopic outcomes: the entire beam is transmitted (with probability $|t|^2$) or the entire beam is reflected (with probability $|r|^2$). This is a very nonclassical transformation, not accomplished by the linear beam splitter. The coherent state is basically a a many photon copy of the single photon state. Each photon acts independently and randomly takes the transmitted or reflected path. The Poisson Statistics are preserved.

(g) Linear optics $\Rightarrow$ linear transformation of modes

$$E_k^{(out)} = \sum_{k'} \underbrace{\mathcal{U}_{kk'}}_{\text{unitary matrix}} E_{k'}^{(in)}$$

Quantum mechanically, $\hat{a}_k^{(out)} = \hat{S}\,\hat{a}_k^{(in)}\,\hat{S}^\dagger = \sum_{k'} \mathcal{U}_{kk'}\,\hat{a}_{k'}^{(in)}$

Suppose we start in an arbitrary multimode coherent state

$$|\psi_{in}\rangle = \hat{D}^{(in)}(\{\alpha_k\})|0\rangle = \prod_k \hat{D}^{(in)}(\alpha_k)|0\rangle$$

$$= \prod_k \exp\left(\alpha_k \hat{a}_k^{(in)\dagger} - \alpha_k^* \hat{a}_k^{(in)}\right) = \exp\left[\sum_k\left(\alpha_k \hat{a}_k^{(in)\dagger} - \alpha_k^* \hat{a}_k^{(in)}\right)\right]$$

$$|\psi_{out}\rangle = \hat{S}|\psi_{in}\rangle = \exp\left[\sum_k\left(\alpha_k \hat{a}_k^{(out)\dagger} - \alpha_k^* \hat{a}_k^{(out)}\right)\right]|0\rangle$$

$$= \exp\left[\sum_{k,k'}\left(\alpha_k \mathcal{U}_{kk'}^* \hat{a}_{k'}^{(in)\dagger} - \alpha_k^* \mathcal{U}_{kk'} \hat{a}_{k'}^{(in)}\right)\right]|0\rangle$$

$$= \exp\left[\sum_{k'}\left\{\left(\sum_k \mathcal{U}_{k'k}\,\alpha_k\right)\hat{a}_{k'}^{(in)\dagger} - \left(\sum_k \mathcal{U}_{k'k}\,\alpha_k\right)^* \hat{a}_{k'}^{(in)}\right\}\right]|0\rangle$$

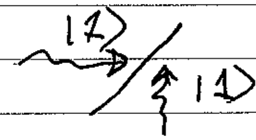
where I have used $\mathcal{U}_{kk'}^* = \mathcal{U}_{k'k}$ for unitary matrix

$$\Rightarrow \quad |\psi_{out}\rangle = \exp\left[\sum_{k'}\left(\tilde{\alpha}_{k'}^{(out)}\,\hat{a}_{k'}^{(in)\dagger} - \tilde{\alpha}_{k'}^{(out)*}\,\hat{a}_{k'}^{in}\right)\right]|0\rangle$$

$$\boxed{\begin{array}{c} \text{where } \tilde{\alpha}_{k'} = \sum_k \mathcal{U}_{k'k}\,\alpha_k \\[1em] \Rightarrow \quad |\psi_{out}\rangle = \hat{D}(\{\tilde{\alpha}_k\})|0\rangle \end{array}}$$

(h) A non-classical input leads to nonclassical phenomena, even for linear transformations

Suppose $|\psi^{(in)}\rangle = |1\rangle_a \otimes |1\rangle_b$, two "mode matched" ~~single photons~~ incident simultaneously on a beam splitter:



$$\Rightarrow |\psi^{(in)}\rangle = \hat{a}^{\dagger}_{in}\hat{b}^{\dagger}_{in}|0\rangle$$

$$\Rightarrow |\psi^{(out)}\rangle = \hat{a}^{\dagger}_{out}\hat{b}^{\dagger}_{out}|0\rangle = \frac{1}{\sqrt{2}}(\hat{a}^{\dagger} - i\hat{b}^{\dagger})(\hat{b}^{\dagger} - i\hat{a}^{\dagger})|0\rangle$$

$$= \frac{1}{\sqrt{2}}\left(\hat{a}^{\dagger}\hat{b}^{\dagger} - i\hat{a}^{\dagger 2} - i\hat{b}^{\dagger 2} + (-i)^2\hat{b}^{\dagger}\hat{a}^{\dagger}\right)|0\rangle$$

overall phase

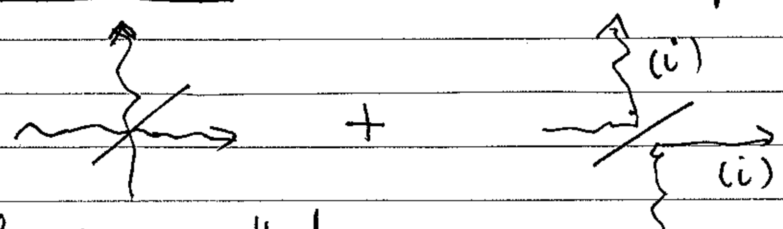
$$= \frac{-i}{\sqrt{2}}(\hat{a}^{\dagger 2} + \hat{b}^{\dagger 2})|0\rangle + \underbrace{\frac{1}{\sqrt{2}}\cancel{(\hat{a}^{\dagger}\hat{b}^{\dagger} - \hat{b}^{\dagger}\hat{a}^{\dagger})|0\rangle}}_{\Rightarrow 0}$$

since $[\hat{a}^{\dagger}, \hat{b}^{\dagger}] = 0$

$$\Rightarrow \boxed{|\psi^{(out)}\rangle = \frac{1}{\sqrt{2}}\left(|2\rangle_a \otimes |0\rangle_b + |0\rangle_a \otimes |2\rangle_b\right)}$$

Thus both photons go off <u>together</u>.

We see that there is <u>destructive interference</u> for the two processes below



both transmitted

both reflected. Each picks up $\frac{\pi}{2}$ phase shift, causing destructive interference.

Problem 2: Boson Algebra

(a) Completeness of coherent states: $|\alpha\rangle = \sum_n c_n |n\rangle$, $c_n = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}$

$$\int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n,m} \int \frac{d^2\alpha}{\pi} e^{-|\alpha|^2} \frac{(\alpha^*)^n (\alpha)^m}{\sqrt{n!\,m!}} |m\rangle\langle n| = \sum_{n,m} \int_0^\infty r\,dr\, e^{-r^2} \frac{r^{n+m}}{\sqrt{n!\,m!}} \int_0^{2\pi} \frac{d\phi}{\pi} e^{i(m-n)\phi} |m\rangle\langle n|$$

Let $\alpha = re^{i\phi}$

$$= \sum_{n,m} \int_0^\infty r\,dr\, e^{-r^2} \frac{r^{n+m}}{\sqrt{n!\,m!}} (2\delta_{nm}) |m\rangle\langle n| = \sum_n \frac{2}{n!} \underbrace{\left(\int_0^\infty e^{-r^2} r^{2n+1} dr \right)}_{n!/2} |m\rangle\langle n|$$

$$\Rightarrow \int \frac{d^2\alpha}{\pi} |\alpha\rangle\langle\alpha| = \sum_{n=0}^\infty |n\rangle\langle n| = \hat{\mathbb{1}} \checkmark$$

(b) Weyl-Heisenberg group:

Using $e^{\hat{A}} e^{\hat{B}} = e^{\hat{A}+\hat{B}} e^{\frac{1}{2}[\hat{A},\hat{B}]}$ where $[\hat{A},\hat{B}]$ commutes with $\hat{A}$ and $\hat{B}$

$$\hat{D}(\alpha)\hat{D}(\beta) = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} e^{\beta\hat{a}^\dagger - \beta^*\hat{a}} = e^{(\alpha+\beta)\hat{a}^\dagger - (\alpha+\beta)^*\hat{a}} e^{\frac{1}{2}\underbrace{[\alpha\hat{a}^\dagger - \alpha^*\hat{a},\, \beta\hat{a}^\dagger - \beta^*\hat{a}]}_{\alpha\beta^* - \alpha^*\beta}}$$

$$\Rightarrow \boxed{\hat{D}(\alpha)\hat{D}(\beta) = \hat{D}(\alpha+\beta)\, e^{i\,\mathrm{Im}(\alpha\beta^*)}}$$

$$\boxed{\begin{aligned}\langle\alpha|\beta\rangle &= \langle 0|\hat{D}^\dagger(\alpha)\hat{D}(\beta)|0\rangle = \langle 0|\hat{D}(-\alpha)\hat{D}(\beta)|0\rangle = \langle 0|\hat{D}(\beta-\alpha)|0\rangle\, e^{-i\,\mathrm{Im}(\alpha\beta^*)} \\ &= e^{-\frac{1}{2}|\alpha-\beta|^2} e^{-i\,\mathrm{Im}(\alpha\beta^*)} \qquad \text{Note } |\langle\alpha|\beta\rangle|^2 = e^{-|\alpha-\beta|^2} \to 0 \text{ when } |\alpha-\beta| \gg 1\end{aligned}}$$

(c) $$\mathrm{Tr}(\hat{D}^\dagger(\alpha)\hat{D}(\beta)) = \int \frac{d^2\gamma}{\pi} \langle\gamma|\hat{D}^\dagger(\alpha)\hat{D}(\beta)|\gamma\rangle = \int \frac{d^2\gamma}{\pi} \langle\gamma|\hat{D}(\beta-\alpha)|\gamma\rangle\, e^{\frac{\beta\alpha^* - \alpha^*\beta}{2}}$$

Aside: $$\langle\gamma|\hat{D}(\alpha-\beta)|\gamma\rangle = e^{-\frac{1}{2}|\alpha-\beta|^2} \langle\gamma| e^{(\alpha-\beta)\hat{a}^\dagger} e^{-(\alpha-\beta)^*\hat{a}} |\gamma\rangle = e^{-\frac{1}{2}|\alpha-\beta|^2} e^{(\alpha-\beta)\gamma^* - (\alpha-\beta)^*\gamma}$$

$$\Rightarrow \boxed{\mathrm{Tr}(\hat{D}^\dagger(\alpha)\hat{D}(\beta)) = e^{-\frac{1}{2}|\alpha-\beta|^2} e^{\frac{\beta\alpha^* - \alpha^*\beta}{2}} \underbrace{\int \frac{d^2\gamma}{\pi} e^{(\alpha-\beta)\gamma^* - (\alpha-\beta)^*\gamma}}_{\delta^{(2)}(\alpha-\beta)} = \delta^{(2)}(\alpha-\beta)} \checkmark$$

(d) • $\langle 0|\hat{D}(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle 0|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}$ ✓

• $\langle\alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = e^{-\frac{1}{2}|\alpha|^2}\langle\alpha_1|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|\alpha_1\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha\alpha_1^* - \alpha^*\alpha_2}$

Alternatively $\langle\alpha_1|\hat{D}(\alpha)|\alpha_2\rangle = \langle 0|\hat{D}(-\alpha_1)\hat{D}(\alpha)\hat{D}(\alpha_2)|0\rangle$

$= \langle 0|\hat{D}(-\alpha_1)\hat{D}(\alpha+\alpha_2)|0\rangle\, e^{i\,\mathrm{Im}(\alpha\alpha_2^*)} = \langle 0|\hat{D}(\alpha+\alpha_2-\alpha_1)|0\rangle\, e^{-i\,\mathrm{Im}(\alpha_1^*(\alpha+\alpha_2))}\, e^{i\,\mathrm{Im}(\alpha\alpha_2^*)}$

$= e^{-\frac{1}{2}|\alpha+\alpha_2-\alpha_1|^2}\, e^{i\,\mathrm{Im}(\alpha\alpha_2^* - \alpha_1\alpha^* - \alpha_1\alpha_2^*)}$ ✓

• $\langle n|\hat{D}(\alpha)|n\rangle = e^{-|\alpha|^2/2}\langle n|e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}}|n\rangle = e^{-|\alpha|^2/2}\sum_m \frac{(-1)^m(|\alpha|^2)^m}{(m!)^2}\underbrace{\langle n|(\hat{a}^\dagger)^m(\hat{a})^m|n\rangle}_{\text{only equal \# } \hat{a}^\dagger \text{ \& } \hat{a}}$

Now $\langle n|(\hat{a}^\dagger)^m\hat{a}^m|n\rangle = \|\hat{a}^m|n\rangle\|^2 = n(n-1)\cdots(n-m) = \frac{n!}{(n-m)!}$ $m \le n$ 0, otherwise

$$\Rightarrow \boxed{\langle n|\hat{D}(\alpha)|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{m=0}^{n}\binom{n}{m}\frac{(-1)^m}{m!}(|\alpha|^2)^m = e^{-\frac{|\alpha|^2}{2}}\mathcal{L}_n(|\alpha|^2)}$$

Laguerre polynomial.