Physics 566: Quantum Optics I
Problem Set #8
Due Thursday, November 9, 2023

Problem 1: Chaotic Light (25 Points)
Natural light arising from, e.g., stars, is not "coherent" in contrast to the light from a laser. The phase of the field fluctuates, and is only well correlated for a short "coherence time." One source of those fluctuations is random collisions between the radiators.

(a) Let $P_s(t)$ be the "survival probability," i.e., the probability that the molecule freely oscillates and survives a time *t* without a collision. Under the assumption that the time of the next collision is independent of the previous (such as random process is said to be *Markovian* – there is no "memory" of the previous trajectory), show that

$P_s(t) = e^{-\gamma t}$, where $\gamma = 1/\tau_0$ is the rate of collisions, and τ_0 is the average time between collisions.

(b) Show that the probability that the oscillator free oscillates for time *t* and then suffers a collision between times *t* and *t+dt*

$$p(t)dt = e^{-t/\tau_0}\frac{dt}{\tau_0}.$$

Use the kinetic theory of gases to show that $1/\tau_0 = n\sigma_0\overline{v}_{rel}$, where *n* is the density of molecules, s$_0$ is the collision cross section and $\overline{v}_{rel}$ is the average relative speed of the molecules. What is $\overline{v}_{rel}$ for a gas in thermal equilibrium?

(c) The electric field produced by each of the oscillators will has a random phase, $E_i(t) = E_0 e^{-i\omega t}e^{i\phi_i(t)}$. The total field $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t}e^{i\phi_i(t)} = E_0 e^{-i\omega t}\alpha(t) = E_0 e^{-i\omega t}a(t)e^{i\varphi(t)}$, where $\alpha(t)$ is the random complex amplitude, $a(t) = |\alpha(t)|$ and φ is overall the random phase. Argue that for *N* large, the probability of a given complex amplitude is Gaussian distributed in amplitude, and independent of phase

$$p(\alpha(t)) = \frac{1}{\pi N}e^{-|\alpha(t)|^2/N}$$

(d) Argue that under the *ergodic assumption* (the random signal samples different values according to the given probability distribution, so ensemble averages equal time averages), the two-time correlation function is $\left\langle E^*(t)E(t+\tau)\right\rangle = \Gamma(\tau) = NE_0^2 e^{-i\omega\tau}e^{-\tau/\tau_0}$.
Hint: Use the fact that $\left\langle e^{i(\phi_i(t+\tau)-\phi_j(t))}\right\rangle = 0$ unless *i*=*j* and also that there is no collision between t and $t+\tau$, and thus $\langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\rangle = \text{probability not to collide in time } \tau$.

(e) The electric field, on average is zero, but there are fluctuations around the average. This implies that the intensity $I(t) = |E(t)|^2$ fluctuates. Using $E(t) = \sum_{i=1}^{N} E_0 e^{-i\omega t}e^{i\phi_i(t)}$, show that
$\langle I(t)\rangle = NE_0^2$, $\left\langle (I(t))^2\right\rangle = 2\langle I\rangle^2 \Rightarrow \Delta I = \langle I\rangle$, and generally the probability distribution of intensities is

$$P[I(t)] = \frac{1}{\langle I\rangle}e^{-\frac{I(t)}{\langle I\rangle}} \Rightarrow \left\langle (I(t))^n\right\rangle = n!\langle I\rangle^n$$

Problem 2: Wiener-Khinchin Theorem (25 Points)

Consider a real function $f(t)$ (this could be a deterministic or random process). Defining the Fourier transform in our usual way, $\tilde{f}(\omega) = \int_{-\infty}^{\infty} f(t)e^{+i\omega t}\,dt$; this exists if $f(t)$ is square integrable. In this case according to Parceval's theorem $\int_{-\infty}^{\infty} f^2(t)dt = \int_{-\infty}^{\infty} \left|\tilde{f}(\omega)\right|^2 \frac{d\omega}{2\pi}$ and $\left|\tilde{f}(\omega)\right|^2$ is known as the spectral density.

(a) Let $C(\tau) = \int_{-\infty}^{\infty} f(t)f(t+\tau)dt$ (autocorrelation function). Show that for $f(t)$ real

$$\left|\tilde{f}(\omega)\right|^2 = \int_{-\infty}^{\infty} C(\tau)e^{i\omega\tau}\,d\tau\,, \quad C(\tau) = \int_{-\infty}^{\infty} \left|\tilde{f}(\omega)\right|^2 e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

That is, the spectrum density is the Fourier transform of the autocorrelation function and vice versa. This a form of the *Wiener-Khinchin Theorem*.

For a formally stationary process, $\tilde{f}(\omega)$ does not exist. In that case we have to be a little more careful. One defines the time average power $P = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f^2(t)dt = \int_{-\infty}^{\infty} S(\omega)\frac{d\omega}{2\pi}$, where $S(\omega)$ is the power spectral density. If follows that $S(\omega) = \lim_{T\to\infty}\frac{1}{T}\left|\int_{-T/2}^{T/2} f(t)e^{i\omega t}\,dt\right|^2$

(b) Show that the general form of the Wiener-Khinchin Theorem is

$$S(\omega) = \int G(\tau)e^{+i\omega\tau}d\tau\,, \text{ where } G(\tau) \equiv \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)f(t+\tau)dt = \int S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi}$$

(c) Now let $f(t)$ be an *ergodic and stationary* random process. Show that

$\left\langle \tilde{f}^*(\omega)\tilde{f}(\omega')\right\rangle = 2\pi S(\omega)\delta(\omega-\omega')$, where angle brackets is the ensemble average.

Next, note that for a real function $\tilde{f}(-\omega) = \tilde{f}^*(\omega)$, Thus

$$f(t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t} = \underbrace{\int_{0}^{\infty}\frac{d\omega}{2\pi}\tilde{f}(\omega)e^{-i\omega t}}_{f^{(+)}(t)} + \underbrace{\int_{0}^{\infty}\frac{d\omega}{2\pi}\tilde{f}^*(\omega)e^{+i\omega t}}_{f^{-}(t)}.$$

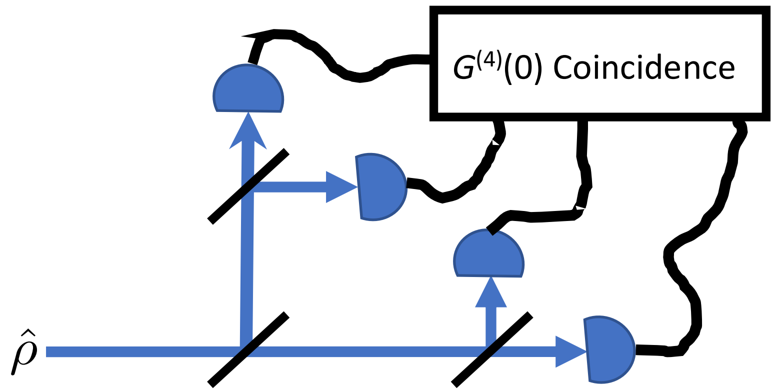
(d) Consider the complex correlation function that determines temporal coherence in a standard interferometer for an ergodic, stationary process, $\Gamma(\tau) = \left\langle E^{(-)}(0)E^{(+)}(\tau)\right\rangle$. Show that

$$\text{Re}\left[\Gamma(\tau)\right] = \frac{1}{2}\int_{-\infty}^{\infty} S(\omega)e^{-i\omega\tau}\frac{d\omega}{2\pi}\,,\ S(\omega) = 2\int_{-\infty}^{\infty}\text{Re}\left[\Gamma(\tau)\right]e^{+i\omega\tau}\,d\tau$$

(e) What is the power spectrum of natural light arising from a collision broadened natural source of light source?

Problem 3: Bose statistics and photon correlations – number basis (20 points).

Consider the coincidence count of *m*-photons in a given temporal mode $G^{(m)}(0) = \langle : \hat{n}^m : \rangle$.
For example, the measurement of the four-point correlation function is depicted below.



a) Suppose the input state an *n*-photon Fock state

$$G^{(m)}(0) = \langle n| : \hat{n}^m : |n\rangle = m! \binom{n}{m} = \frac{n!}{(n-m)!}.$$

Interpret this in terms of m-particle interference of identical bosons.

b) For a general state show that

$$G^{(m)}(0) = \sum_n P_n \frac{n!}{(n-m)!}, \quad \text{where } P_n = \langle n|\hat{\rho}|n\rangle$$

c) For a coherent state, $P_n = \frac{\langle \hat{n} \rangle^n}{n!} e^{-\langle \hat{n} \rangle}$, use b) to show that $G^{(m)}(0) = \langle \hat{n} \rangle^m$.

d) For a thermal state, $P_n = \frac{\langle \hat{n} \rangle^n}{(1 + \langle \hat{n} \rangle)^{n+1}}$, use b) to show that $G^{(m)}(0) = m! \langle \hat{n} \rangle^m$.