Physics 566: Quantum Optics I

Problem Set 1, Solutions

Problem 1. Natural Light

The "natural light" emitted by stars or thermal lamp has a "finite coherence time". The amplitude and phase of the electric field of the wave are stochastic variables, specified by a probability distribution. One source of the randomness of the the field is collisions between the dipole emitters, which randomizes the phase and means the wave trains only have finite coherence lengths.

(a) Let $P_s(t)$ = "survival probabilty" = probability that a dipole oscillates for time t w/o a collision.

Let $\gamma = \frac{1}{\tau_0}$ = rate of collisions $\Rightarrow$ γdt = Probability of collision in infinitesimal dt

Under the "Markoff Approximation," the probability of a collision at any time t is completely independent of the "history" of the trajectory, i.e. the collision probability is uncorrelated between t and $t+dt$

$$\text{Probability of surviving to } t+dt = (\text{Prob. of surviving to } t) \times (\text{Prob of no collision } t \to t+dt)$$

$$P_s(t+dt) = \left(P_s(t) \right) \times \left(1 - \gamma dt \right)$$

$$\Rightarrow \boxed{\frac{1}{P_s}\frac{dP_s}{dt} = -\gamma \Rightarrow P_s(t) = e^{-\gamma t}}$$

(b) Let $p(t)dt$ = Probability of surviving ~~for time~~ a time t and then colliding in the interval $t \to t+dt$.

$$\Rightarrow \boxed{p(t)dt = \left(e^{-\gamma t}\right)\left(\gamma dt\right) = e^{-t/\tau_0}\frac{dt}{\tau_0}}$$

The collision cross section σ_0 define the rate of collisions, by definition:

$$\text{Rate of collision between particles} = \underbrace{n\bar{v}_{rel}}_{\text{flux of incident particles = density} \times \text{rel. velocity}}\sigma_0$$

In thermal equilibrium, each particle has a mean thermal speed $\overline{v^2} = \frac{3kT}{m}$ (equipartition) Maxwell - Boltzmann

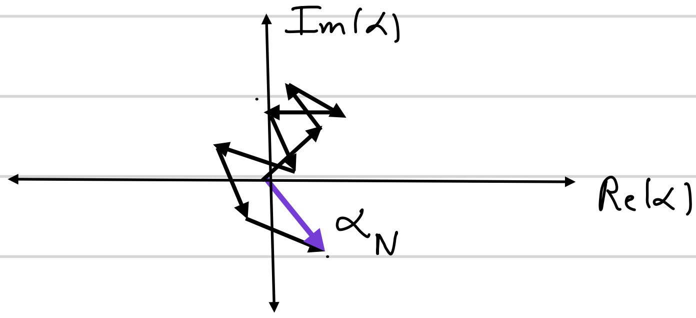
$$\Rightarrow \overline{v_{rel}^2} = \langle(\vec{v}_1 - \vec{v}_2)^2\rangle = \langle v_1^2\rangle + \langle v_2^2\rangle = \frac{6k_BT}{m} \Rightarrow \boxed{\bar{v}_{rel} = \frac{6kT}{m}}$$

(c) The total electric field produced by a collection of N-oscillators with random phases:

$$E(t) = \sum_{i=1}^{N} E_0\, e^{-i\omega t} e^{i\phi(t)} = E_0 e^{-i\omega t} \underbrace{\sum_{i=1}^{N} e^{i\phi(t)}}_{\alpha(t)} = E_0\, e^{-i\omega t} a(t)\, e^{i\varphi(t)}$$

$\alpha(t)$ ← random complex number

The complex amplitude $\alpha(t)$ can be view as the end point of a random walk in 2D



A random walker has a Gaussian probability distribution of being away from the origin.

$$P(\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)) = \left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{[\mathrm{Re}(\alpha)]^2}{2\sigma^2}}\right]\left[\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{[\mathrm{Im}(\alpha)]^2}{2\sigma^2}}\right]$$

Here the walker moves with fixed radius ⇒ After N-steps $\langle|\alpha|^2\rangle = \langle(\mathrm{Re}(\alpha))^2\rangle + \langle(\mathrm{Im}(\alpha))^2\rangle$
$= 2\sigma^2 = N$

$$\Rightarrow \boxed{P(\alpha) = \frac{1}{\pi N}\, e^{-\frac{|\alpha|^2}{N}}}$$

This probability distribution describes a field with random phase and mean zero amplitude.

(c) We seek the auto-correlation function at two different times for $E(t) = \sum_{i=1}^{N} E_0\, e^{-i\omega_0 t} e^{i\phi_j(t)}$

$$\langle E^*(t)\, E(t+\tau)\rangle = \sum_i E_0^2\, e^{-i\omega_0\tau}\langle e^{i(\phi_i(t+\tau)-\phi_i(t))}\rangle + \sum_{i\neq j} E_0^2\, e^{-i\omega_0\tau}\langle e^{-i\phi_i(t)} e^{+i\phi_j(t+\tau)}\rangle$$

Because the oscillators are uncorrelated $\langle e^{-i\phi_i(t)} e^{+i\phi_j(t+\tau)}\rangle_{i\neq j} = \langle e^{-i\phi_i(t)}\rangle\langle e^{i\phi_j(t+\tau)}\rangle = 0$

Aside: $\langle e^{-i(\phi_i(t+\tau)-\phi_i(t))}\rangle = 0$ unless the oscillator does not collide in time τ.
$= 1 -$ Probality that the oscillator collides in time interval τ
$= 1 - \int_0^\tau p(t)\,dt = e^{-\tau/\tau_0}$ (the same for all oscillators)

$$\boxed{\Rightarrow \langle E^*(t)\, E(t+\tau)\rangle = N E_0^2\, e^{-i\omega_0 t} e^{-\tau/\tau_0}}$$

(d) While the mean electric field of natural light is zero, there are fluctuations. This implies that there is an average intensity and fluctuations in intensity

$$\boxed{\langle I(t)\rangle = \langle E^*(t)\,E(t)\rangle \underset{\substack{\uparrow \\ \text{from part (c)}}}{=} N E_0^2 \text{ (intensities add for incoherent oscillators).}}$$

Generally, we can derive the probability distribution $P(I(t))$ using $P(\alpha(t))$:

$$I(t) = |E(t)|^2 = E_0^2 |\alpha(t)|^2 \Rightarrow P(I(t))\,dI = P(I(\alpha(t)))\,d^2\alpha = P(|\alpha|)\,2\pi|\alpha|\,d|\alpha|$$

$$\Rightarrow P(I(t)) = 2\pi|\alpha|\left(\frac{dI}{d|\alpha|}\right)^{-1} P(|\alpha|) = 2\pi|\alpha|\left(2E_0^2|\alpha(t)|\right)^{-1}\frac{1}{N\pi}e^{-\frac{|\alpha|^2}{N}} = \frac{1}{NE_0^2}e^{-\frac{E_0^2|\alpha(t)|^2}{E_0^2 N}}$$

$$\Rightarrow \boxed{P(I(t)) = \frac{1}{\langle I\rangle}e^{-\frac{I(t)}{\langle I\rangle}}}$$

We can calculate the moments of this distribution:

$$\boxed{\langle I^n\rangle = \int_0^\infty I^n \frac{e^{-I/\langle I\rangle}}{\langle I\rangle}\,dI = n!\,\langle I\rangle^n}$$

In particular $\langle I^2\rangle = 2\langle I\rangle^2 \Rightarrow \Delta I^2 = \langle I^2\rangle - \langle I\rangle^2 = \boxed{\langle I\rangle^2 \Rightarrow \Delta I = \langle I\rangle}$

For natural light, the fluctuations in intensity are equal to the mean.

Problem 2: Wiener-Khinchin Theorem

(a) We begin with a function whose Fourier transform exists (i.e. a square normalizable function). $\tilde{f}(\omega) \equiv \int dt\, f(t)\, e^{+i\omega t}$, $f(t) = \int \frac{d\omega}{2\pi} \tilde{f}(\omega)\, e^{-i\omega t}$ (take real)

Autocorrelation function over all time: $C(\tau) \equiv \int_{-\infty}^{\infty} d\tau\, f(t) f(t+\tau)\, dt$

This is nothing but the convolution of f with itself. It then follows

$$C(\tau) = \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\, \tilde{f}(\omega)\, \tilde{f}(\omega')\, e^{-i\omega t}\, e^{-i\omega'(t+\tau)} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{2\pi}\, \tilde{f}(\omega)\, \tilde{f}(\omega') \underbrace{\int_{-\infty}^{\infty} dt\, e^{-i(\omega+\omega')t}}_{2\pi\,\delta(\omega+\omega')} e^{+i\omega'\tau}$$

$$\Rightarrow C(\tau) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, \tilde{f}(\omega)\, \tilde{f}(-\omega)\, e^{+i\omega\tau} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, \tilde{f}(\omega)\, \tilde{f}^*(\omega)\, e^{-i\omega t} \quad \text{when } f(t) \text{ is real}$$

$$\Rightarrow \boxed{C(\tau) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, |\tilde{f}(\omega)|^2\, e^{+i\omega\tau} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, |\tilde{f}(-\omega)|^2\, e^{-i\omega\tau} = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, |\tilde{f}(\omega)|^2\, e^{-i\omega\tau}}$$

inverse $\boxed{|\tilde{f}(\omega)|^2 = \int_{-\infty}^{\infty} d\tau\, C(\tau)\, e^{+i\omega\tau}}$

(b) For a stationary process, the spectral density (or power spectral density) is defined in terms of the time averaged power per frequency interval.

$$S(\omega) \equiv \lim_{T\to\infty} \frac{1}{T} \left| \int_{-T/2}^{T/2} f(t)\, e^{-i\omega t}\, dt \right|^2$$

The autocorrelation function is the time-average $G(\tau) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} f(t)\, f(t+\tau)\, dt$

Let $f_T(t) \equiv \Theta_T(t)\, f(t) = \begin{cases} f(t) & -\frac{T}{2} \le t \le T/2 \\ 0 & \text{otherwise} \end{cases}$ $\Rightarrow$ $S(\omega) = \lim_{T\to\infty} \frac{1}{T} |\tilde{f}_T(\omega)|^2$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, S(\omega)\, e^{-i\omega\tau} = \lim_{T\to\infty}\frac{1}{T}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, |\tilde{f}_T(\omega)|^2\, e^{-i\omega\tau} = \lim_{T\to\infty}\int_{-\infty}^{\infty} dt\, f_T(t)\, f_T(t+\tau) \quad (\text{as in part a})$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt\, f(t)\, f_T(t+\tau) = \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)\, f(t+\tau)\, \Theta_T(t+\tau)\, dt$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} f(t)\, f(t+\tau)\, dt \quad \text{since } \Theta_T(t+\tau) = 1 \text{ in integrand as } T\to\infty$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\, S(\omega)\, e^{-i\omega\tau} = G(\tau), \qquad S(\omega) = \int d\tau\, G(\tau)\, e^{+i\omega\tau}}$$

(c) Let $f(t)$ be an ergodic, stationary process

Then $G(\tau) = \lim_{T\to\infty} \int_{-\infty}^{\infty} f(t) f(t+\tau)\, dt \underset{\text{ergodic}}{=} \langle f(t) f(t+\tau)\rangle \underset{\text{stationary}}{=} \langle f(0) f(\tau)\rangle$ ← ensemble average

$$\langle \tilde{f}^*(\omega) \tilde{f}(\omega')\rangle = \left\langle \left(\int_{-\infty}^{\infty} dt\, f(t)\, e^{i\omega t}\right)^* \left(\int_{-\infty}^{\infty} dt'\, f(t')\, e^{i\omega' t'}\right)\right\rangle$$
$$= \int_{-\infty}^{\infty} dt \int_{-\infty}^{\infty} dt'\, e^{i(\omega' t' - \omega t)} \underbrace{\langle f(t) f(t')\rangle}_{G(t'-t)\ \text{(stationary)}}$$

Change variables: $\tau = t' - t$, $T = \frac{t+t'}{2}$, Jacobian $dt_1 dt_2 = dt' dT$

$$\Rightarrow \langle \tilde{f}^*(\omega) \tilde{f}(\omega')\rangle = \int_{-\infty}^{\infty} dT\, e^{i(\omega'-\omega)T} \int_{-\infty}^{\infty} d\tau\, e^{i(\omega+\omega')\frac{\tau}{2}} G(\tau) = 2\pi\, \delta(\omega-\omega') \int_{-\infty}^{\infty} e^{i\omega\tau} G(\tau)\, d\tau$$

$$\boxed{\Rightarrow \langle \tilde{f}^*(\omega) \tilde{f}(\omega')\rangle = 2\pi\, S(\omega)\, \delta(\omega-\omega'), \quad S(\omega) = \int_{-\infty}^{\infty} d\tau\, e^{i\omega\tau} \langle f(0) f(\tau)\rangle \equiv \text{Spectral density}}$$

(d) Fourier transform of the real signal $\tilde{E}(\omega) = \int_{-\infty}^{\infty} dt\, E(t)\, e^{-i\omega t}$

$$E(t) = \underbrace{\int_0^{\infty} \frac{d\omega}{2\pi}\, \tilde{E}(\omega)\, e^{-i\omega t}}_{E^{(+)}(t)} + \underbrace{\int_0^{\infty} \frac{d\omega}{2\pi}\, \tilde{E}^*(\omega)\, e^{+i\omega t}}_{E^{(-)}(t)}$$

Analytic signal: $\tilde{E}_c(t) = 2E_c^{(+)}(t) = 2\int_0^{\infty} \frac{d\omega}{2\pi}\, \tilde{E}(\omega)\, e^{-i\omega t}$, $\qquad E(t) \equiv \mathrm{Re}(\tilde{E}(t)) = \frac{1}{2}(\tilde{E}(t) + \tilde{E}^*(t))$

Complex correlation function $\Gamma(\tau) = \langle E^{(-)}(0)\, E^{(+)}(\tau)\rangle$ (for stationary/ergodic process)

$$\Gamma(\tau) = \int_0^{\infty} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{d\omega'}{\pi} \underbrace{\langle \tilde{E}(\omega) \tilde{E}^*(\omega')\rangle}_{2\pi\,\delta(\omega-\omega')\,S(\omega)} e^{-i\omega'\tau} = \int_0^{\infty} S(\omega)\, e^{-i\omega\tau} \frac{d\omega}{2\pi}$$

$$\Gamma(\tau) + \Gamma^*(\tau) = 2\,\mathrm{Re}(\Gamma(\tau)) = \int_0^{\infty} S(\omega)\, e^{-i\omega\tau} \frac{d\omega}{2\pi} + \int_0^{\infty} \underset{S(-\omega)\ \text{since}\ G(\tau)\ \text{real}}{\underset{\shortparallel}{S^*(\omega)}}\, e^{+i\omega\tau} \frac{d\omega}{2\pi}$$

$$\boxed{\Rightarrow \mathrm{Re}(\Gamma(\tau)) = \frac{1}{2}\int_{-\infty}^{\infty} S(\omega)\, e^{-i\omega\tau} \frac{d\omega}{2\pi}, \quad S(\omega) = 2\int_{-\infty}^{\infty} d\tau\, \mathrm{Re}(\Gamma(\tau))\, e^{+i\omega\tau}}$$

(c) The power spectrum is defined by the spectral density $S(\omega)$. Consider "natural light" e.g. light from a thermal lamp, collision broadened. We found in class, the auto-correlation function between the complex amplitudes

$$\langle E^{(-)}(0)\, E^{(+)}(\tau)\rangle = \frac{I_0}{4}\, e^{\left(-i\omega_0\tau - \frac{|\tau|}{\tau_c}\right)}, \quad \text{where } \omega_0 = \text{central frequency}, \quad \frac{1}{\tau_c} = \text{Collision rate}$$

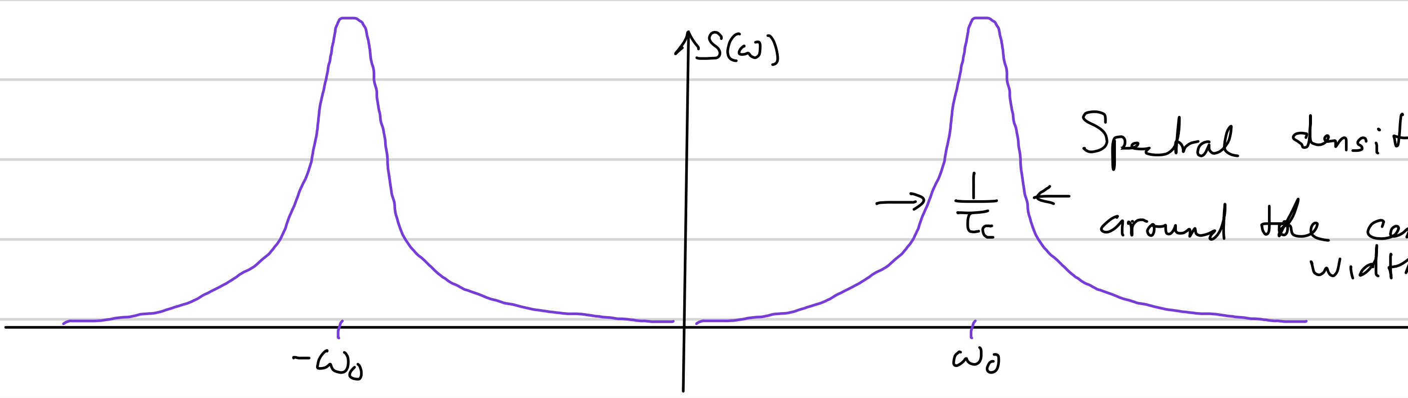
I_0 = intensity at $\tau = 0$

$\Rightarrow$ Spectral density: $S(\omega) = \frac{I_0}{2}\int_{-\infty}^{\infty} d\tau\, \cos(\omega_0\tau)\, e^{-\frac{|\tau|}{\tau_c}}\, e^{+i\omega\tau} = \frac{I_0}{4}\int_{-\infty}^{\infty} d\tau \left[e^{+i(\omega-\omega_0)t - \frac{|\tau|}{\tau_0}} + e^{i(\omega+\omega_0)t - \frac{|\tau|}{\tau_c}}\right]$

Aside: $\int_{-\infty}^{\infty} d\tau\, e^{i\Omega\tau - \frac{|\tau|}{\tau_c}} = \int_0^{\infty} d\tau\, e^{\left(-i\Omega - \frac{1}{\tau_c}\right)\tau} + \int_{-\infty}^{0} d\tau\, e^{\left(-i\Omega + \frac{1}{\tau_c}\right)\tau}$

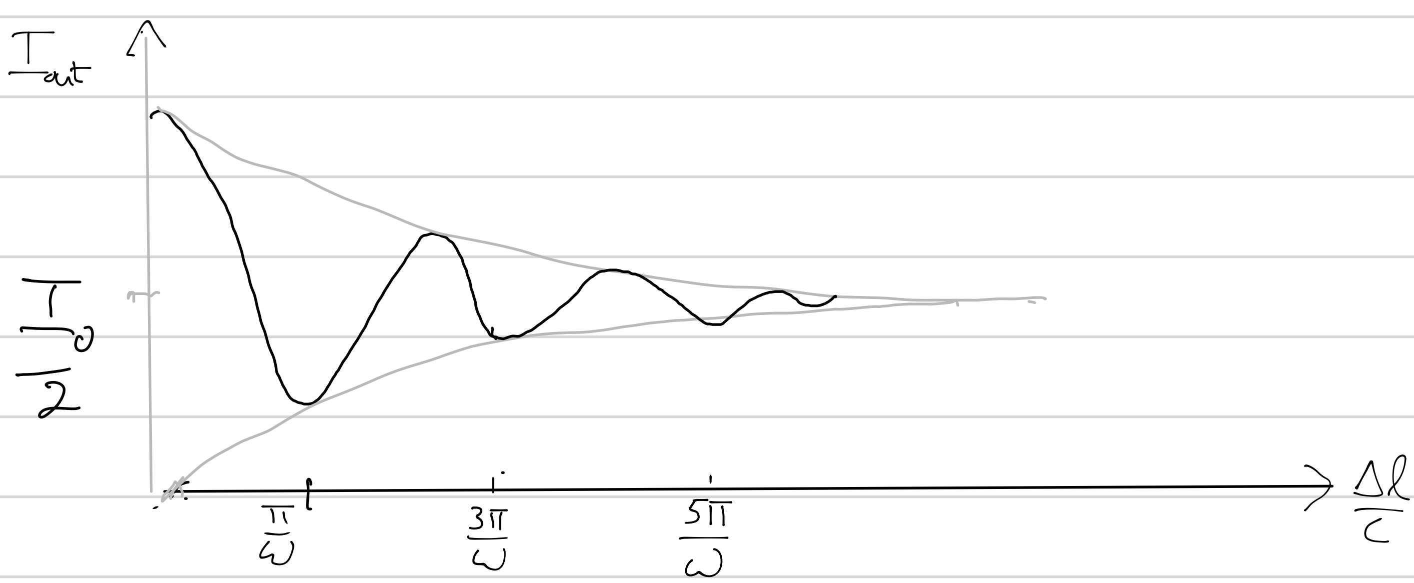
$$= \frac{1}{-i\Omega - \frac{1}{\tau_c}} + \frac{-1}{-i\Omega + \frac{1}{\tau_c}} = \frac{2/\tau_c}{\Omega^2 + (1/\tau_c)^2}$$

$$\Rightarrow S(\omega) = \frac{I_0}{2}\left[\frac{1/\tau_c}{(\omega-\omega_0)^2 + (1/\tau_c)^2} + \frac{1/\tau_c}{(\omega+\omega_0) + (1/\tau_c)^2}\right]$$



Spectral density is Lorentzian around the central freq, width $\cong \frac{1}{\tau_c}$

The output intensity of the interferometer is the Fourier transform of the spectral density $\Rightarrow$ For a Lorentzian power spectral density, fringes decay exponentially.



Problem 3: Bose statistics and photon correlations

(a) Suppose we have an n-photon Fock state in a given temporal mode. We seek the m-point correlation function, corresponding to simultaneously detecting m photons in the mode

$$G^{(m)}(0) = \langle n| :\hat{n}^m: |n\rangle = \langle n| \hat{a}^{\dagger m} \hat{a}^m |n\rangle = \sum_{n'} \langle n| \hat{a}^{\dagger m} |n'\rangle\langle n'| \hat{a}^m |n\rangle$$

$$= |\langle n-m| \hat{a}^{\dagger m} |n\rangle|^2 = |\sqrt{n(n-1)\cdots(n-m)}|^2 = \frac{n!}{(n-m)!} = m! \binom{n}{m}$$

We can interpret this in terms of the Bose statistics of identical particles as follows. We seek to detect m-photons out of n-photons. There are $\binom{n}{m} = \frac{n!}{m!\,(n-m)!}$ ways of doing so. But since these particles are bosons, the are $n!$ different permutation that we can chose to get the same result. This is the multiphoton interference.

(b) For a general state $\hat{\rho}$, $G^{(m)}(0) = \mathrm{Tr}(\hat{\rho} :\hat{n}^m:) = \sum_n \langle n| \hat{\rho}\, \hat{a}^{\dagger m} \hat{a}^m |n\rangle$

$$\Rightarrow G^{(m)}(0) = \sum_{n,n'} \langle n|\hat{\rho}|n'\rangle \underbrace{\langle n'| \hat{a}^{\dagger m} \hat{a}^m |n\rangle}_{\langle n| :\hat{n}^m: |n\rangle\, \delta_{nn'}} = \sum_n P_n \frac{n!}{(n-m)!}, \quad P_n = \langle n|\hat{\rho}|n\rangle$$

Thus, the correlation function depends critically on the photon statistics

(c) For a coherent state, $G^{(m)}(0) = \sum_{n=0}^{\infty} e^{-\langle\hat{n}\rangle} \frac{\langle\hat{n}\rangle^n}{n!} \frac{n!}{(n-m)!} = e^{-\langle\hat{n}\rangle} \sum_{n=0}^{\infty} \frac{\langle\hat{n}\rangle^n}{(n-m)!}$

$$\Rightarrow e^{-\langle\hat{n}\rangle} \sum_{n=0}^{\infty} \frac{\langle\hat{n}\rangle^{n-m}}{(n-m)!} \langle\hat{n}\rangle^m = e^{-\langle\hat{n}\rangle} \underbrace{\sum_{n'=0}^{\infty} \frac{\langle\hat{n}\rangle^{n'}}{n'!}}_{e^{+\langle\hat{n}\rangle}} \langle\hat{n}\rangle^m = \langle\hat{n}\rangle^m$$

Thus, for a Coherent state $G^{(m)}(0) = \langle\hat{n}\rangle^m = (G^{(1)}(0))^m$

We see this trivially in the coherent state basis

$$G^{(m)}(0) = \langle\alpha| \hat{a}^{\dagger m} \hat{a}^m |\alpha\rangle = \alpha^{*m} \alpha^m = (|\alpha|^2)^m = \langle\hat{n}\rangle^m$$

(d) For a thermal state $G^{(m)}(0) = \sum_{n=m}^{\infty} \frac{\langle\hat{n}\rangle^n}{(1+\langle\hat{n}\rangle)^{n+1}} \frac{n!}{(n-m)!} = \frac{1}{1+\langle\hat{n}\rangle} \sum_{n=m}^{\infty} \left(\frac{\langle\hat{n}\rangle}{1+\langle\hat{n}\rangle}\right)^n \frac{n!}{(n-m)!}$

<u>Aside:</u> let $z = \frac{\langle\hat{n}\rangle}{1+\langle\hat{n}\rangle}$ $\qquad \frac{d^m}{dz^m} z^n = z^{n-m} \frac{n!}{(n-m)!} \Rightarrow z^m \frac{d}{dz^m} z^n = z^n \frac{n!}{(n-m)!}$

$$\Rightarrow G^{(m)}(0) = \frac{z^m}{1+\langle\hat{n}\rangle} \frac{d^m}{dz^m} \sum_{n=0}^{\infty} z^{n'+m} = \frac{z^m}{1+\langle\hat{n}\rangle} \frac{d^m}{dz^m}\left(\frac{z^m}{1-z}\right) \underset{\text{using Mathematica}}{=} \frac{z^m}{1+\langle\hat{n}\rangle} \frac{m!}{(1-z)^{m+1}}$$

$$\Rightarrow G^{(m)}(0) = m!\,\langle\hat{n}\rangle^m \cancel{\frac{1}{(1+\langle\hat{n}\rangle)^m} \frac{1}{\left(1-\frac{\langle\hat{n}\rangle}{1+\langle\hat{n}\rangle}\right)^m}} \Rightarrow G^{(m)}(0) = m!\,\langle\hat{n}\rangle^m$$ Phew!