Physics 566 - Quantum Optics I
Problem Set 9 - Solutions

Problem 1: Cavity QED

(a) Each atom is prepared in $|e\rangle$ is sent through the cavity in vacuum. The joint atom-cavity system evolves according to the Jaynes-Cummings Hamiltonian. Since we begin in the 1-excitation subspace,

$$|\Psi_{AF}(t)\rangle = \cos(g_0 t)\,|e,0\rangle + i\sin(g_0 t)\,|g,1\rangle$$

The probability of finding the atom in the ground state is $\sin^2(g_0 t)$ – This is vacuum Rabi flopping

This is the ideal case, neglecting any damping or inhomogeneity.

Note: When $g_0 t = \pi$, we deterministically deposit one photon in the cavity.

(b) We now send pairs of atoms through the cavity. The first is prepared in $|e\rangle_1$ and after at time $2g_0 = \pi$, $|\Psi_{AF}\rangle = i\,|g\rangle_1 \otimes |1\rangle$. After a time T the second atom sent through the cavity. Because the photon has a finite lifetime in the cavity, the probability amplitude to have one photon in the cavity decays, by the effective Hamiltonian

$$\Rightarrow |1\rangle_F \to e^{-\frac{\Gamma_c t}{2}|1\rangle\langle 1|}\,|1\rangle_F = e^{-\frac{\Gamma_c t}{2}}\,|1\rangle_F = e^{-\frac{t}{2\tau_c}}\,|1_F\rangle, \quad \text{where } \tau_c = \text{life time}$$

If we condition on measuring the first atom in $|g\rangle$, then we know, right at that time, we have one photon in the cavity. After time T, the cavity decays back to the vacuum, and a second atom, now prepared in $|g\rangle$ is sent through the cavity for another π-pulse.

The resulting joint evolution of atom-2 + field $\Rightarrow e^{-T/2\tau_c}\,|e\rangle_2 \otimes |0\rangle$

$$\Rightarrow \underbrace{P_{e_2|g_1}} = \left|e^{-T/2\tau_c}\right|^2 = e^{-T/\tau_c}$$

Conditional probability of measuring atom-2 in $|e\rangle$ having measured atom-1 in $|g\rangle$.

(c) We use the Ramsey interferometer to measure coherence in the atom. Here we study the transfer of quantum coherence between two atoms mediated by a cavity.
The evolution of the first atom + cavity:

$$|e\rangle_1 \otimes |0\rangle \underset{R_1}{\Rightarrow} \left(\frac{|e\rangle_1 + i|g\rangle_1}{\sqrt{2}}\right) \otimes |0\rangle = \frac{1}{\sqrt{2}}\left(|e\rangle_1 \otimes |0\rangle + |g\rangle_1 \otimes |0\rangle\right) \underset{C}{\Rightarrow} |g\rangle_1 \otimes \left(\frac{|1\rangle + i|0\rangle}{\sqrt{2}}\right)$$

(Note, the Jaynes-Cummings π-pulse swapped the qubit between the atom & cavity.)

Now we send the second atom through the cavity after time T. Ignoring, for the moment the cavity decay, the ideal evolution is

$$|g\rangle_2 \otimes \left[\frac{|1\rangle + i|0\rangle}{\sqrt{2}}\right] = \frac{|g\rangle_2 \otimes |1\rangle + i|g\rangle_2 |0\rangle}{\sqrt{2}} \underset{C}{\Rightarrow} \left(\frac{|e\rangle_2 + i|g\rangle_2}{\sqrt{2}}\right) \otimes |0\rangle$$

We have thus swapped the coherence from atom-1 to the cavity and then from the cavity into atom-2.

The second Ramsey pulse now measures the coherence in atom 2. From lecture 7 we showed that we the second $\pi/2$ pulse is phase-shifted relative to the first, the Ramsey fringe will be

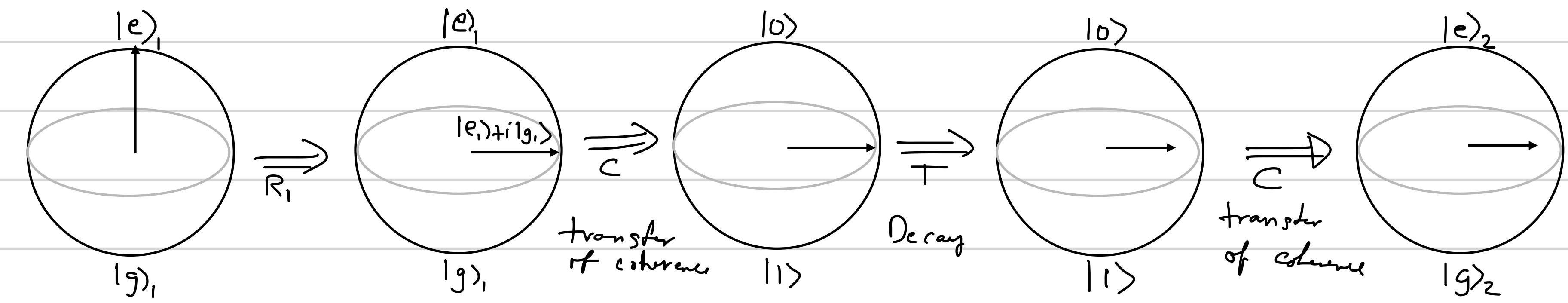
$$P_{e_2|g_1} = \frac{1}{2}\left(1 + \cos\phi\, e^{-T/T_2}\right) \quad \text{where } T_2 = \text{coherence decay time}$$

Here $T_2 = \frac{\tau_c}{2} = \frac{1}{2}$ life time in cavity since according to the effective Hamiltonian

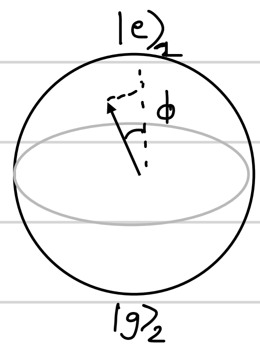
$$\frac{|1\rangle + i|0\rangle}{\sqrt{2}} \Rightarrow e^{-T/2\tau_c}\frac{|1\rangle + i|0\rangle}{\sqrt{2}}$$ for the cavity coherence, which is the transfered to the atom.

$$\boxed{\Rightarrow P_{e_2|g_1} = \frac{1}{2}\left(1 + \cos\phi\, e^{-T/2\tau_c}\right)}$$

Note, in contrast to part (a), the decay is half as fast because the Ramsey fringe depends on T_2, where as the probability of decay in (a) depends on T_1 : $T_2 = \frac{1}{2}T_1$ for lifetime decay.



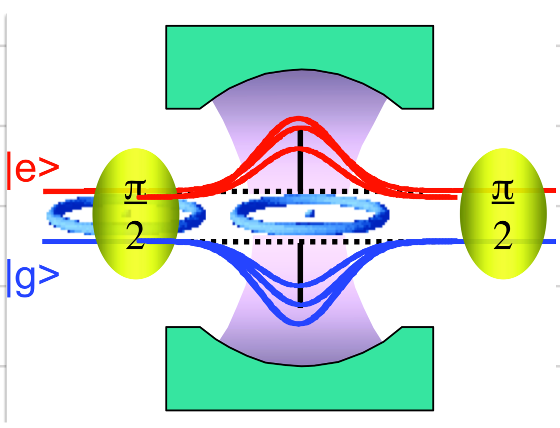
Read out coherence in atom-2 $\xRightarrow{R_{2,\phi}}$



For a review of these experiments by the Haroche group see: S. Haroche, M. Brune, and J. M. Raimond, Phil Trans R. Lond A. $\underline{355}$, 2367 (1997).

(d) We now consider an off-resonant interaction between the atom and cavity mode. In this circumstance, there will not be emission or absorption of "real photons" into the cavity, but instead a kind of "virtual emission and absorption." This leads to a light shift, or joint phase shift on the atom/cavity system. The amount of light shift will depend on the number of photons in the cavity, so this apparatus allows us to measure the # of photons in the cavity without absorbing any photons $\Rightarrow$ Quantum Nondemolition (QND) measurement.

The evolution is assumed to be adiabatic $\Rightarrow$ The system follows the dressed eigenstate as sketched in the figure below (credit to Prof. Haroche).

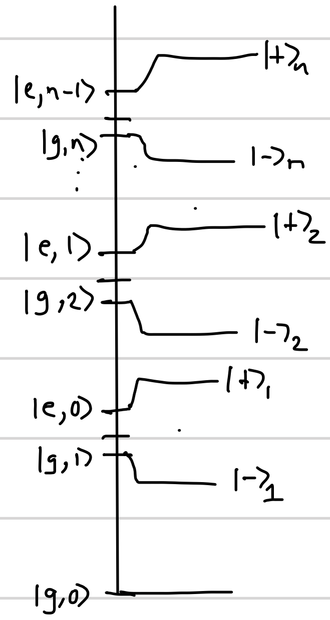


(Show for red detuning)

Recall the dressed energy levels with N_T-excitations $E_\pm(N_T) = \frac{\hbar\omega_c}{2}(N_T - \frac{1}{2}) \pm \frac{\hbar}{2}\sqrt{\Delta^2 + N_T(2g)^2}$

Dressed States: $|\pm_{N_T}\rangle = \cos\frac{\theta_{N_T}}{2}\,|g, N_T\rangle \pm \sin\frac{\theta_{N_T}}{2}\,|e, N_T\rangle$: $\tan\theta_{N_T} = \frac{-\sqrt{N_T}\,2g_0}{\Delta}$

Consider red detuning. The Jaynes-Cummings Ladder



The Bare state $|g,n\rangle$ adiabatically connects to $|-\rangle_n$
The Bare state $|e,n\rangle$ addiabatically connects to $|+\rangle_{n+1}$

The energy shift of these levels leads to a a phase shift:

$$\Delta E_{g,n} = E_{gn}^{dressed} - E_{gn}^{bare}, \quad \Delta E_{en} = E_{en}^{dressed} - E_{en}^{bare}$$

$$E_{gn}^{bare} = -\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c \Rightarrow \Delta E_{gn} = -\frac{\hbar\Delta}{2} - \frac{\hbar}{2}\sqrt{\Delta^2 + n(2g)^2}$$

$$E_{en}^{bare} = +\frac{\hbar\omega_{eg}}{2} + n\hbar\omega_c \Rightarrow \Delta E_{en} = +\frac{\hbar\Delta}{2} + \frac{\hbar}{2}\sqrt{\Delta^2 + (n+1)(2g)^2}$$

Apply the Ramsey $\frac{\pi}{2}$-pulse to the atom, and then passing it through the cavity with $|n\rangle$ photons, the evolution of atom-cavity system:

$$|e\rangle \otimes |n\rangle \Rightarrow \left(\frac{|e\rangle + i|g\rangle}{\sqrt{2}}\right) \otimes |n\rangle = \frac{|e,n\rangle + i|g,n\rangle}{\sqrt{2}} \Rightarrow \frac{e^{-i\delta_{e,n}}|e,n\rangle + i e^{-i\delta_{g,n}}|g,n\rangle}{\sqrt{2}}$$

where $\delta_{e,n} = \frac{\Delta E_{en} t}{\hbar} = \left(\frac{1}{2}\sqrt{\Delta^2 + (n+1)(2g)^2} + \frac{\Delta}{2}\right)t$

$\delta_{g,n} = \frac{\Delta E_{gn} t}{\hbar} = \left(\frac{1}{2}\sqrt{\Delta^2 + n(2g)^2} - \frac{\Delta}{2}\right)t$

$$\equiv \left(\frac{|e\rangle + i e^{i\Phi_n}|g\rangle}{\sqrt{2}}\right) \otimes |n\rangle \quad \text{(up to an overall phase)}$$

where $\Phi_n = \delta_{e,n} - \delta_{g,n}$ = Rotation angle of atom in Bloch sphere.

$\Rightarrow$ We can measure Φ_n in the Ramsey interferometer.

By applying the second $\frac{\pi}{2}$-pulse in R_2, with relative phase ϕ, the Ramsey fringe

$$P_e(\phi) = \cos^2(\phi - \Phi_n)$$

The photon # is then encoded in the phase of the Ramsey fringe!
See Gleyzes et al., Nature 446, 297 (2007).

Problem 2: Collapse and revival in the Jaynes-Cummings model

We consider a two-level atom coupled to a single mode of a high-Q cavity on resonance. The field is initially prepared in a coherent state and the atom in the ground state: $|\Psi(0)\rangle = |g\rangle \otimes |\alpha\rangle$. The joint atom-field state then evolves according to the Jaynes-Cummings Hamiltonian (neglecting here any spontaneous emission into other modes, or cavity losses.

(a) To evolve the state, we can first decompose the initial state into the eigenstates of J.C. Hamiltonian: $\{ |+(n)\rangle = \frac{1}{\sqrt{2}}(|e\rangle|n-1\rangle + |g\rangle|n\rangle) \; ; \; |-(n)\rangle = \frac{1}{\sqrt{2}}(|e\rangle|n-1\rangle - |g\rangle|n\rangle)\}$
with eigenvalues: $E_{\pm(n)} = n\hbar\omega_c \pm \hbar\sqrt{n}\, g \leftarrow$ coupling constant

Within the 2D-subspace, the time evolution operator is

$$\hat{U}_n = e^{-iE_{+(n)}t/\hbar}|+(n)\rangle\langle+(n)| + e^{-iE_{-(n)}t/\hbar}|-(n)\rangle\langle-(n)|$$

$$= e^{-in\omega_c t}\left[e^{-i\sqrt{n}gt}|+(n)\rangle\langle+(n)| + e^{+i\sqrt{n}gt}|-(n)\rangle\langle-(n)|\right] \qquad n>0$$

$$\Rightarrow \hat{U}_n |g\rangle\otimes|n\rangle = e^{-in\omega_c t}\left[\cos(\sqrt{n}gt)|g\rangle\otimes|n\rangle + i\sin(\sqrt{n}gt)|e\rangle\otimes|n-1\rangle\right]$$

$$|\Psi(0)\rangle = |g\rangle\otimes|\alpha\rangle = |g\rangle\otimes\sum_n c_n|n\rangle = \sum_{n=0}^{\infty} c_n |g\rangle\otimes|n\rangle \quad \text{where} \quad c_n = \frac{\alpha^n}{\sqrt{n!}}e^{-|\alpha|^2/2}$$

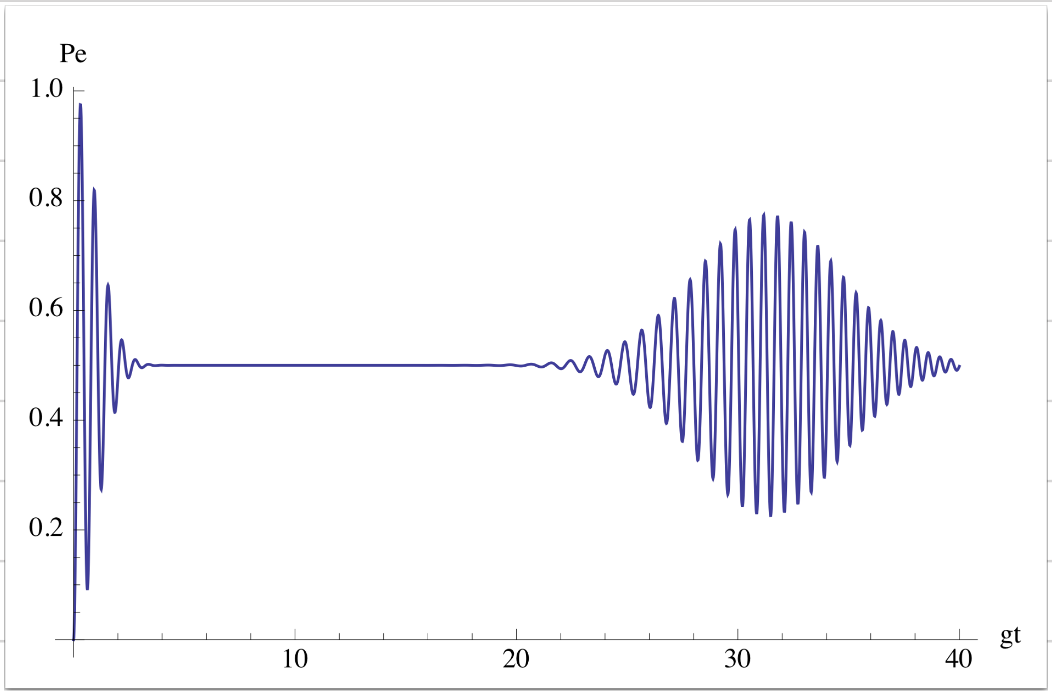
$$= c_0|g\rangle\otimes|0\rangle + \sum_{n=1}^{\infty} c_n |g\rangle\otimes|n\rangle$$

$$\boxed{\begin{aligned}\Rightarrow |\Psi(t)\rangle &= \bigoplus_n \hat{U}_n|\Psi(0)\rangle = \\ &= c_0|g\rangle\otimes|0\rangle + \sum_{n=1}^{\infty} c_n e^{-in\omega_c t}\left[\cos(\sqrt{n}gt)|g\rangle\otimes|n\rangle + i\sin(\sqrt{n}gt)|e\rangle\otimes|n-1\rangle\right]\end{aligned}}$$

The probability to find the atom in the excited state, irrespective of photon number n is $P_e(t) = \sum_{n=0}^{\infty} |\langle e,n|\Psi(t)\rangle|^2$ (sum over n)

$$\boxed{\Rightarrow P_e(t) = \sum_{n=1}^{\infty}|c_n|^2\sin^2(\sqrt{n}gt) = \sum_{n=1}^{\infty}\frac{\bar{n}^n}{n!}e^{-\bar{n}}\sin^2(\sqrt{n}gt)}$$

(b) Shown is a numerical plot of $P_e(t)$ for $\bar{n}=25$ as a function of $0 \le gt \le 40$.

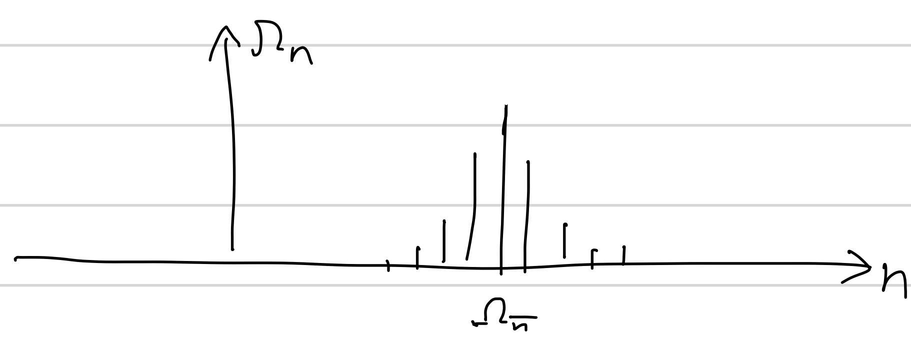


This is the famous Jaynes-Cummings "collapse and revival." The collapse is excepted due to a spread in the Rabi-frequencies $\Omega_n = \sqrt{n}\,2g$, $(\Delta n^2 = \bar{n}$, Poisson number fluctuations).
Given a spread in photon number, $\Delta n = \sqrt{\bar{n}}$, the spread in Rabi frequencies is

$$\Delta\Omega \sim |\Omega_{\bar{n}+\sqrt{\bar{n}}} - \Omega_{\bar{n}-\sqrt{\bar{n}}}| = 2g\sqrt{\bar{n}+\sqrt{\bar{n}}} - 2g\sqrt{\bar{n}-\sqrt{\bar{n}}} \approx 2g\sqrt{\bar{n}}\left(1+\frac{1}{2\sqrt{\bar{n}}}\right) - 2g\sqrt{\bar{n}}\left(1-\frac{1}{2\sqrt{\bar{n}}}\right)$$
$$\approx 2g \quad \text{for } \bar{n} \gg 1$$

$\Rightarrow$ Collapse time $t_c \sim \frac{1}{\Delta\Omega} \sim \frac{1}{2g}$ (as seen in the plot)

(c) The revival, by contrast, is a much more subtle effect that is a unique signature of quantum fluctuations of the cavity field. That is the Rabi frequencies are associated with a discrete set photons, rather than the continuum of intensities. Thus the probability $P_e(t)$ is a Fourier sum rather than an integral.



At times such that $(\Omega_{\bar{n}} - \Omega_{\bar{n}+1}) t_m \cong 2\pi m$ we expect a "rephasing" and thus a revival of the oscillations.

$$\Rightarrow t_m \cong \frac{2\pi m}{2g\sqrt{\bar{n}} - 2g\sqrt{\bar{n}+1}} \cong \frac{\pi m}{g\sqrt{\bar{n}}} \frac{1}{\left(1-\left(1-\frac{1}{2\bar{n}}\right)\right)} = \frac{2\pi m \sqrt{\bar{n}}}{g}$$

$\Rightarrow$ First revival: $\boxed{t_1 \cong \frac{2\pi\sqrt{\bar{n}}}{g} \Rightarrow g t_1 \cong 2\pi\sqrt{\bar{n}} \cong 31.4,}$ as seen in the plot.

<u>Note:</u> The revival is not perfect because the sum is <u>infinite</u>. A finite Fourier series will always have perfect revivals.