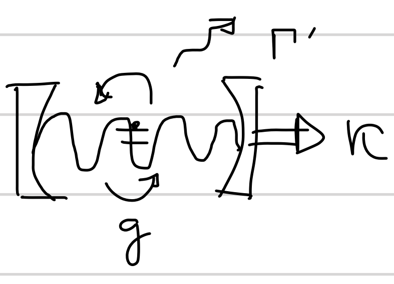
Physics 566 - Quantum Optics II
Problem Set 10 - Solutions

Problem 1: The Purcell Effect

The Purcell Effect is named after Ed Purcell, one of the great AMO physicists of the mid 20th century, who showed how the existence of boundary conditions, e.g. those associated with mirrors in an E+M cavity, could affect the spontaneous emission of atoms in a seminal paper, Phys. Rev 69, 681 (1946).

It is not a surprise that boundary conditions effect radiating systems - we are familiar with this classically. The microscopic charges in the material that make up the boundary react to a neighboring dipole. The details of this microscopic response of summarized by the boundary conditions. The interesting part is that this applies at the quantum level. The modes of the quantum electromagnetic field must satisfy the boundary conditions. This affect spontaneous emission.

We seek the rate of emission of an atom into a given cavity mode. This only makes sense in the "weak coupling regime" in which the rate of decay of photons due to absorption in and/or transmission through the cavity mirrors is much faster than the rate of coherent Rabi oscillations between the atom of cavity mode



$$\hbar g = \sqrt{\frac{2\pi\hbar\omega_k}{V}}\ \vec{u}_c(\vec{R})\cdot \vec{d}_{eg}$$

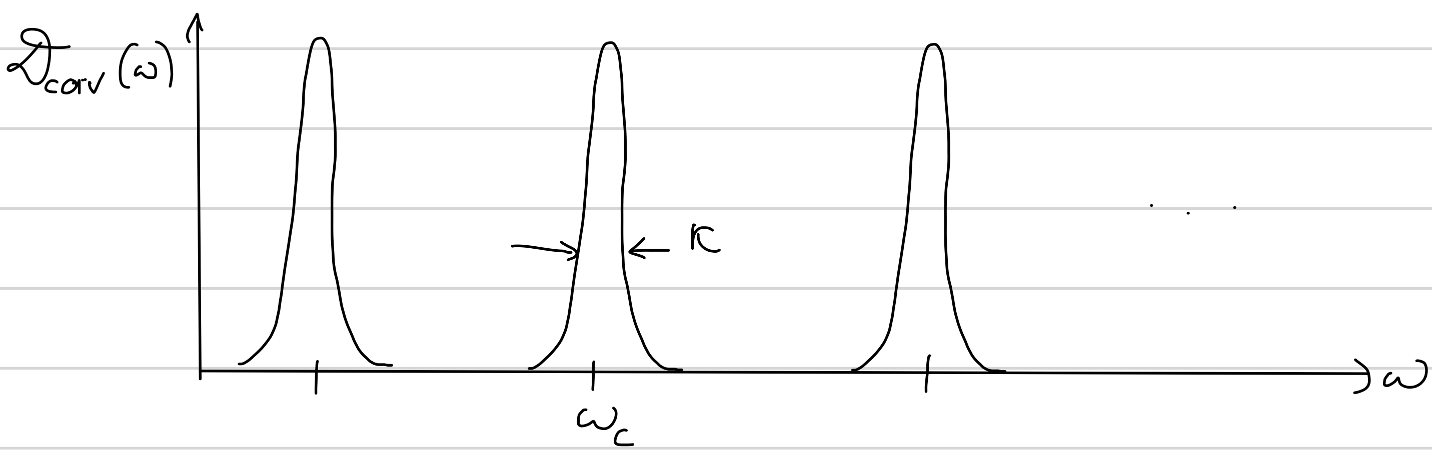
Cavity mode @ atom's position

Vacuum Rabi freq

$$\kappa = \frac{1}{\tau_{\text{cavity}}} \gg g$$

Γ' = spontaneous emission rate in to all other modes

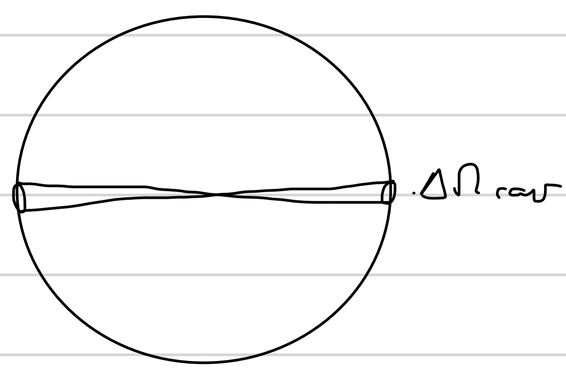
(a) One way to calculate the Purcell effect is to use Fermi's Golden Rule. We seek the rate of spontaneous emission into a given cavity mode. This notion only makes sense in the weak coupling regime, otherwise photons emitted into the cavity will get reabsorbed in a coherent Rabi oscillation; spontaneous emission is irreversibly decay. The effect of cavity is to modify the density of states of electromagnetic modes available for the atom in which it can radiate. The density of states in directions captured by the cavity has resonance peaks



Near a particular cavity resonance ω_c, the density of state is Lorentzian

$$\mathcal{D}_{cav}(\omega) = \frac{\kappa/2\pi}{(\omega-\omega_c)^2 + \kappa^2/4}$$

Note, this is the density of states around the frequency band near ω_c, and for modes propagating within the solid angle subtended by the cavity.



Emission into solid angles $4\pi - \Delta\Omega_{cav}$ near the atomic resonance defines Γ'. Generally $\Delta\Omega \ll 4\pi$, i.e. the cavity modes are "paraxial." In this case, $\Gamma' \approx \Gamma_{free}$, the free-space spontaneous emission rate.

With this background, we can write the spontaneous emission rate into the cavity mode using Fermi's Golden Rule

$$\Gamma_{cav} = \frac{2\pi}{\hbar^2} |\langle \psi_{final} | \hat{H}_{int} | \psi_{initial} \rangle|^2 \mathcal{D}_{cav}(\omega_{eg})$$

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where $|\psi_{initial}\rangle = |e,0\rangle$, $|\psi_{final}\rangle = |g,1\rangle$ ← one photon in the cavity

$$\hat{H}_{int} = \hbar g \left(\hat{a}_c^{\dagger}\hat{\sigma}_+ + \hat{a}_c\hat{\sigma}_-\right)$$

$$\Rightarrow \Gamma_{cav} = \frac{2\pi}{\hbar^2}(\hbar g)^2 \mathcal{D}_{cav}(\omega_{eg}) = \left(\frac{4\pi^2\omega_{eg}}{\hbar V}\right)\left|\vec{u}_{cav}(\vec{R})\cdot\vec{d}_{eg}\right|^2 \mathcal{D}_{cav}(\omega_{eg})$$

Now $\Gamma_{free} = 2\pi\,\overline{|g(\omega_{eg})|^2}\,\mathcal{D}_{free}(\omega_{eg})$ is the free-space spontaneous emission rate

where $\overline{|g(\omega_{eg})|^2} = \frac{16\pi^2\omega_{eg}}{3\hbar V}|d_{eg}|^2$ is the free space coupling constant, averaged over all directions of propagation and polarizations, $\mathcal{D}_{free}(\omega_{eg}) = \frac{V}{(2\pi)^3}\frac{\omega_{eg}^2}{c^3} = \left(\frac{V}{\lambda_{eg}^3}\right)\frac{1}{\omega_{eg}}$ is the free-space density of states.

Thus,
$$\frac{\Gamma_{cav}}{\Gamma_{free}} = \frac{3}{8\pi}\frac{\left|\vec{u}_{cav}(\vec{R})\cdot\vec{d}_{eg}\right|^2}{|\vec{d}_{eg}|^2}\frac{\mathcal{D}_{cav}(\omega_{eg})}{\mathcal{D}_{free}(\omega_{eg})}$$

$$= \frac{3}{16\pi^2}\frac{\lambda_{eg}^3}{V}\frac{\omega_{eg}}{\kappa}\frac{\kappa^2}{(\omega_{eg}-\omega_c)^2+\kappa^2/4}\frac{\left|\vec{u}_{cav}(\vec{R})\cdot\vec{d}_{eg}\right|^2}{|\vec{d}_{eg}|^2}$$

$$\Rightarrow \boxed{\Gamma_{cav} = \Gamma_{Free}\,\mathcal{F}\,\frac{\kappa^2/4}{(\omega_{eg}-\omega_c)^2+\kappa^2/4}\frac{\left|\vec{u}_{cav}(\vec{R})\cdot\vec{d}_{eg}\right|^2}{|\vec{d}_{eg}|^2}}$$

where $\mathcal{F} = \frac{3}{4\pi^2}\frac{\lambda_{eg}^3}{V}Q$ = "Purcell factor", $Q = \frac{\omega_{eg}}{\kappa}$ = Cavity "quality factor"

For $Q \gg 1$ (e.g. $10^3 - 10^6$) and V not to large compared to λ^3, $\mathcal{F} \gg 1$

(b) We the cavity is tuned to the atomic resonance $\omega_c = \omega_{eg}$, and we the atom is positioned so that $|\vec{u}_{cav}\cdot\vec{d}_{eg}| = |\vec{d}_{eg}| \Rightarrow \Gamma_{cav} = \mathcal{F}\,\Gamma_{free}$, $\mathcal{F} \gg 1 \Rightarrow$ on resonance the decay rate into the cavity mode is enhanced. When

$$\Gamma_{cav} \approx \Gamma_{free}\,\mathcal{F}\,\frac{\kappa^2}{4\omega_c^2} = \Gamma_{free}\frac{3}{4\pi^2}\frac{\lambda^3}{V}Q\frac{1}{4Q^2} = \Gamma_{free}\frac{3}{16\pi^2 Q}\frac{\lambda^3}{V}:$$ emission into the cavity is <u>suppressed</u>

(c) We can recover this solution by explicitly solving the atom-photon dynamics in the Born-Markov (Wigner-Weisskopf) approximation.

We treat decay of the atom and cavity into other modes using a nonHermitian Hamiltonian
$$\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma'}{2}\hat{\sigma}_+\hat{\sigma}_- - i\frac{\hbar\kappa}{2}\hat{a}_c^\dagger\hat{a}_c$$

In the interaction picture the atom-cavity coupling Hamiltonian is
$$\hat{H}^{(I)}_{int} = \hbar g\left(\hat{a}_c\hat{\sigma}_+ e^{-i(\omega_c-\omega_{eg})t} + \hat{a}_c^\dagger\hat{\sigma}_- e^{+i(\omega_c-\omega_{eg})t}\right)$$

The interaction couples states only in 2D subspaces. We take $|\psi(0)\rangle = |e,0\rangle \Rightarrow$ (in the interaction picture) $|\psi(t)\rangle = c_{g,1}(t)|g,1\rangle + c_{e,0}(t)|e,0\rangle$

$$\Rightarrow \boxed{\begin{aligned} \dot{c}_{g1} &= \frac{-i}{\hbar}\langle g,1|\hat{H}^{(I)}_{eff}|\psi\rangle = -\frac{\kappa}{2}c_{g1} - igc_{e,0}\,e^{+i(\omega_c-\omega_{eg})t} \\ \dot{c}_{e,0} &= \frac{-i}{\hbar}\langle e,0|\hat{H}^{(I)}_{eff}|\psi\rangle = -\frac{\Gamma'}{2}c_{e,0} - igc_{g,1}\,e^{-i(\omega_c-\omega_{eg})t} \end{aligned}}$$

(d) As we did in free space, we can formally integrate

$$\Rightarrow c_{g1}(t) = \cancel{c_{g1}(0)}^{\,0}e^{-\frac{\kappa t}{2}} - ig\int_0^t dt'\, e^{-\frac{\kappa}{2}(t-t')}\, e^{i(\omega_c-\omega_{eg})t'}\, c_{e,0}(t')$$

$$\Rightarrow \dot{c}_{e,0} = -\frac{\Gamma'}{2}c_{e,0} - g^2\int_0^{t\,\to\infty\ \text{(Markov approx)}} dt'\, e^{[-i(\omega_c-\omega_{eg})-\frac{\kappa}{2}](t-t')}\, c_{e,0}(t') \quad c_{e,0}(t') \Rightarrow c_{e,0}(t) \text{ (Born - Approximation)}$$

$$\Rightarrow \dot{c}_{e,0} \approx -\frac{\Gamma'}{2}c_{e,0} - g^2\frac{1}{+i(\omega_c-\omega_{eg})+\frac{\kappa}{2}}\,c_{e,0}(0)$$

$$= \Bigg(\underbrace{-\frac{\Gamma'}{2} - \pi g^2\frac{\kappa/2\pi}{(\omega_{eg}-\omega_c)^2+\kappa^2/4}}_{\Gamma_{cav}/2} \; \underbrace{-ig^2\frac{(\omega_c-\omega_{eg})}{(\omega_c-\omega_{eg})^2+\frac{\kappa^2}{4}}}_{\delta E_{cav}/\hbar}\Bigg)c_{e,0}$$

$\frac{\delta E_{cav}}{\hbar} \Leftarrow$ Lamb shift change due to cavity.

Problem 2: The relative role of vacuum fluctuations + radiation reaction in spontaneous emission

(a) The Hamiltonian for the two-level atom interacting with the quantized field in the RWA:

$$\hat{H} = \frac{\hbar\omega_{eg}}{2}\hat{\sigma}_z + \sum_{\vec{k},\mu}\hbar\omega_k \hat{a}^\dagger_{\vec{k}\mu}\hat{a}_{\vec{k}\mu} + \sum_{\vec{k},\mu}\hbar\left(g_{\vec{k}\mu}\hat{\sigma}_+\hat{a}_{\vec{k}\mu} + g^*_{\vec{k}\mu}\hat{a}^\dagger_{\vec{k}\mu}\hat{\sigma}_-\right)$$

Note: Here, in the Heisenberg picture, all operators are evaluated at the same time and commute.
Heisenberg equations of motion: (using $[\hat{a}_{\vec{k}\mu},\hat{a}^\dagger_{\vec{k}'\mu'}] = \delta_{\vec{k}\vec{k}'}\delta_{\mu\mu'}$, $[\hat{\sigma}_+,\hat{\sigma}_-] = \hat{\sigma}_z$, $[\hat{\sigma}_z,\hat{\sigma}_\pm] = \pm 2\hat{\sigma}_\pm$

$$\frac{d}{dt}\hat{a}_{\vec{k}\mu} = -\frac{i}{\hbar}[\hat{a}_{\vec{k}\mu},\hat{H}] = -i\omega_k\hat{a}_{\vec{k}\mu} - ig^*_{\vec{k}\mu}\hat{\sigma}_-$$

$$\frac{d}{dt}\hat{\sigma}_+ = -\frac{i}{\hbar}[\hat{\sigma}_+,\hat{H}] = i\omega_{eg}\hat{\sigma}_+ - i\sum_{\vec{k}\mu} g^*_{\vec{k}\mu}\hat{a}^\dagger_{\vec{k}\mu}\hat{\sigma}_z = i\omega_{eg}\hat{\sigma}_+ - i\sum_{\vec{k}\mu} g^*_{\vec{k}\mu}\overbrace{\left[s(\hat{a}^\dagger_{\vec{k}\mu}\hat{\sigma}_z) + (1-s)\hat{a}^\dagger_{\vec{k}\mu}\hat{\sigma}_z\right]}^{\text{adding + subtracting } s g^*_{\vec{k}\mu}\hat{a}^\dagger_{k\mu}\hat{\sigma}_z}$$

$$= i\omega_{eg}\hat{\sigma}_+ - i\sum_{\vec{k}\mu} g^*_{\vec{k}\mu}\left[s(\hat{a}^\dagger_{\vec{k}\mu}\hat{\sigma}_z) + (1-s)\hat{\sigma}_z\hat{a}^\dagger_{\vec{k}\mu}\right] \qquad \text{since } [\hat{\sigma}_z(t),\hat{a}^\dagger_{\vec{k}\mu}(t)] = 0$$

$$\frac{d}{dt}\hat{\sigma}_z = -\frac{i}{\hbar}[\hat{\sigma}_z,\hat{H}] = -2i\sum_{\vec{k}\mu}\left(g_{\vec{k}\mu}\hat{\sigma}_+\hat{a}_{\vec{k}\mu} - g^*_{\vec{k}\mu}\hat{a}^\dagger_{\vec{k}\mu}\hat{\sigma}_-\right) = -2i\sum_{\vec{k}\mu} g_{\vec{k}\mu}\hat{\sigma}_+\hat{a}_{\vec{k}\mu} + \text{h.c.}$$

$$= -2i\sum_{\vec{k}\mu} g_{\vec{k}\mu}\left[s\hat{\sigma}_+\hat{a}_{\vec{k}\mu} + (1-s)\hat{a}_{\vec{k}\mu}\hat{\sigma}_+\right] + \text{h.c.} \quad \text{(using same trick as in (b))}$$

Choose:
$0 \leq s \leq 1$: $s=0$ (Normal order), $s=1$ (anti-normal order), $s=\frac{1}{2}$ (Symmetric order)

(b) We solve formally:

$$\hat{a}_{\vec{k}\mu}(t) = \underbrace{\hat{a}_{\vec{k},\mu}(0)e^{-i\omega_k t}}_{\hat{a}^{free}_{\vec{k}\mu}(t)} \underbrace{-ig^*_{\vec{k}\mu}\int_0^t dt'\, e^{-i\omega_k(t-t')}\hat{\sigma}_-(t')}_{\delta\hat{a}_{\vec{k}\mu}(t)}$$

$$\hat{\sigma}_+(t) = \underbrace{\hat{\sigma}_+(0)e^{i\omega_{eg}t}}_{\hat{\sigma}^{free}_+(t)} - i\underbrace{\int_0^t e^{i\omega_{eg}(t-t')}\sum_{\vec{k}\mu} g^*_{\vec{k}\mu}\hat{a}^\dagger_{\vec{k}\mu}(t')\hat{\sigma}_z(t')}_{\delta\hat{\sigma}_+(t)}$$

$$\hat{\sigma}_z(t) = \underbrace{\hat{\sigma}_z(0)}_{\hat{\sigma}^{free}_z} - \underbrace{2i\int_0^t dt'\sum_{\vec{k}\mu} g_{\vec{k}\mu}\hat{\sigma}_+(t')\hat{a}_{\vec{k}\mu}(t') + \text{h.c.}}_{\delta\hat{\sigma}_z(t)}$$

Now we interate, and keep terms only to first order in $g_{\vec{k}\mu}$ (this is the first Born approximation)

$$\Rightarrow \delta\hat a_{\vec k\mu}(t) = -i g^*_{\vec k\mu}\int_0^t dt'\, e^{-i\omega_k(t-t')}\,\hat\sigma_-^{free}(t') = -i g^*_{\vec k\mu}\underbrace{\int_0^t dt'\, e^{i(\omega_{eg}-\omega_k)(t-t')}}_{\zeta(\omega_{eg}-\omega_k)}\,\hat\sigma_-^{free}(t) \quad \left(\text{since } \hat\sigma_-^{free}(t) = e^{-i\omega_{eg}t}\hat\sigma_-(0)\right)$$

$$\delta\hat\sigma_+(t) = -i\int_0^t e^{i\omega_{eg}(t-t')}\sum_{\vec k\mu} g^*_{\vec k\mu}\,\hat a^{\dagger\, free}_{\vec k\mu}(t')\,\hat\sigma_z^{free}(t') = -i\sum_{\vec k\mu} g^*_{\vec k\mu}\,\zeta(\omega_k-\omega_{eg})\,\hat a^{\dagger\, free}_{\vec k\mu}(t)\,\hat\sigma_z^{free}$$

$$\delta\hat\sigma_z(t) = -2i\int_0^t dt'\sum_{\vec k\mu} g_{\vec k\mu}\,\hat\sigma_+^{free}(t')\,\hat a^{free}_{\vec k\mu}(t') + h.c. = -2i\sum_{\vec k\mu} g_{\vec k\mu}\underbrace{\int_0^t e^{i(\omega_{eg}-\omega_k)(t'-t)}}_{\zeta^*(\omega_{eg}-\omega_k)}\,\hat\sigma_+^{free}(t)\,\hat a^{free}_{\vec k\mu}(t) + h.c.$$

where $\zeta(\omega_{eg}-\omega_k) \equiv \int_0^t e^{i(\omega_{eg}-\omega_k)(t-t')} \to \int_0^\infty e^{i(\omega_{eg}-\omega_k)\tau}d\tau = \pi\delta(\omega_{eg}-\omega_k) + i\,\mathbb{P}\left[\frac{1}{\omega_{eg}-\omega_k}\right]$

In the Markoff approximation

(c) We take as the state (constant in the Heisenberg picture): $|\psi\rangle_{AF} = |\psi\rangle_A \otimes |0\rangle_F$

Note: $\hat a^{free}_{\vec k\lambda}|0\rangle = 0\,, \qquad \langle 0|\hat a^{\dagger\, free}_{\vec k\lambda}(t) = 0$

Taking the expectation value of atomic Heisenberg equations of motion:

$$\frac{d}{dt}\langle\hat\sigma_+\rangle = i\omega_{eg}\langle\hat\sigma_+\rangle - i\sum_{\vec k\mu} g^*_{\vec k\mu}\, s\,\langle\psi_{AF}|(\hat a^{\dagger\, free}_{\vec k\mu} + \delta\hat a^\dagger_{\vec k\mu})(\hat\sigma_z^{free} + \delta\hat\sigma_z)|\psi_{AF}\rangle$$
$$- i\sum_{\vec k\mu} g^*_{\vec k\mu}(1-s)\langle\psi_{AF}|(\hat\sigma_z^{free} + \delta\hat\sigma_z^{free})(\hat a^{free}_{\vec k\mu} + \delta\hat a^\dagger_{\vec k\mu})|\psi_{AF}\rangle$$

$$\Rightarrow \frac{d}{dt}\langle\hat\sigma_+\rangle = i\omega_{eg}\langle\hat\sigma_+\rangle - i\sum_{\vec k\mu} g^*_{\vec k\mu}\Big[s\langle\psi_{AF}|\delta\hat a^\dagger_{\vec k\mu}\hat\sigma_z^{free}|\psi_{AF}\rangle + (1-s)\big(\langle\psi_{AF}|\hat\sigma_z^{free}\delta\hat a^\dagger_{\vec k\mu}|\psi_{AF}\rangle + \langle\psi_{AF}|\delta\hat\sigma_z\,\hat a^{\dagger\, free}_{\vec k\mu}|\psi_{AF}\rangle\big)\Big]$$

$$\frac{d}{dt}\langle\hat\sigma_z\rangle = 2i\sum_{\vec k\mu}\Big(g_{\vec k\mu}\Big[s\langle\psi_{AF}|(\hat\sigma_+^{free} + \delta\hat\sigma_+)(\hat a^{free}_{\vec k\mu} + \delta\hat a_{\vec k\mu})|\psi_{AF}\rangle + (1-s)\langle\psi_{AF}|(\hat a^{free}_{\vec k\mu} + \delta\hat a_{\vec k\mu})(\hat\sigma_+^{free} + \delta\hat\sigma_+)|\psi_{AF}\rangle + h.c.$$

$$\Rightarrow \frac{d}{dt}\langle\hat\sigma_z\rangle = 2i\sum_{\vec k\mu} g_{\vec k\mu}\Big[s\langle\psi_{AF}|\hat\sigma_+^{free}\,\delta\hat a_{\vec k\mu}|\psi_{AF}\rangle + (1-s)\big(\langle\psi_{AF}|\hat a^{free}_{\vec k\mu}\,\delta\hat\sigma_+|\psi_{AF}\rangle + \langle\psi_{AF}|\hat\sigma_+^{free}\,\delta\hat a_{\vec k\mu}|\psi_{AF}\rangle\big)\Big] + h.c.$$

(d) We now put it all together:

(to lowest order)

$$\langle\psi_{AF}|\delta\hat a^\dagger_{\vec k\mu}\hat\sigma_z^{free}|\psi_{AF}\rangle = +i g_{\vec k\mu}\,\zeta^*(\omega_{eg}-\omega_k)\,\langle\overbrace{\hat\sigma_+^{free}(t)\,\hat\sigma_z^{free}(t)}^{=-\hat\sigma_+^{free}(t)}\rangle \simeq -i g_{\vec k\mu}\,\zeta^*(\omega_{eg}-\omega_k)\langle\hat\sigma_+(t)\rangle$$

$$\langle\psi_{AF}|\hat\sigma_z^{free}\,\delta\hat a^\dagger_{\vec k\mu}|\psi_{AF}\rangle = +i g_{\vec k\mu}\,\zeta^*(\omega_{eg}-\omega_k)\langle\underbrace{\hat\sigma_z^{free}\hat\sigma_+^{free}}_{=\hat\sigma_+^{free}(t)}\rangle = +i g_{\vec k\mu}\,\zeta^*(\omega_{eg}-\omega_k)\langle\hat\sigma_+(t)\rangle$$

$$\langle\psi_{AF}|\delta\hat\sigma_z\,\hat a^\dagger_{\vec k\mu}|\psi_{AF}\rangle = -2i\sum_{\vec k'\mu'} g_{\vec k'\mu'}\,\zeta^*(\omega_{eg}-\omega_k)\langle\hat\sigma_+\rangle\underbrace{\langle 0|\hat a_{\vec k'\mu'}\hat a^\dagger_{\vec k\mu}|0\rangle}_{\delta_{\vec k'\vec k}\,\delta_{\mu'\mu}} = -2i\, g_{\vec k\mu}\,\zeta^*(\omega_{eg}-\omega_k)\langle\hat\sigma_+(t)\rangle$$

$$\Rightarrow \left(\frac{d}{dt} - i\omega_{eg}\right)\langle\hat{\sigma}_+\rangle = \Big[s\Big(-\sum_{\vec{k},\mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg}-\omega_k)\Big) + (1-s)\Big(+\sum_{\vec{k}\mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg}-\omega_k) - 2\sum_{\vec{k}\mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg}-\omega_k)\Big)\langle\hat{\sigma}_+\rangle$$

Aside: $\sum_{\vec{k},\mu} |g_{\vec{k}\mu}|^2 \zeta^*(\omega_{eg}-\omega_k) = \sum_{\vec{k},\mu} |g_{\vec{k}\mu}|^2 \left(\pi\delta(\omega_{eg}-\omega_k) - i\,\mathbb{P}\left(\frac{1}{\omega_{eg}-\omega_k}\right)\right) = \frac{\Gamma}{2} + i\delta$

where $\Gamma = 2\pi \sum_{\vec{k},\mu} |g_{\vec{k}\mu}|^2 \delta(\omega_k - \omega_{eg})$, $\hbar\delta = \sum_{\vec{k}\mu} |g_{\vec{k},\mu}|^2\, \mathbb{P}\left[\frac{1}{E_g + \hbar\omega_k - E_e}\right]$

$$\Rightarrow \left(\frac{d}{dt} - i\omega_{eg}\right)\langle\hat{\sigma}_+\rangle = s\underbrace{\left(-\frac{\Gamma}{2} - i\delta\right)}_{\text{radiation reaction}}\langle\hat{\sigma}_+\rangle + (1-s)\Big(\underbrace{\left(\frac{\Gamma}{2}+i\delta\right)}_{\text{radiation reaction}} - \underbrace{2\left(\frac{\Gamma}{2}+i\delta\right)}_{\text{Vacuum fluct}}\Big)\langle\hat{\sigma}_+\rangle$$

Where the vacuum terms arise from $\hat{a}^{free}_{\vec{k}\mu}(t)$ and the radiation reaction from $\delta\hat{a}(t)$

Note: In normal order $(s=1)$, the entire level shift arises from radiation reaction.
In antinormal order, $(s=0)$, by vacuum fluctuation and radiation reaction contribute to δ.
In Symmetric order $(s=\frac{1}{2})$, the entire level shift arises from vacuum fluctuations.

Now let's find the remaining correlation functions:

$$\langle \hat{\sigma}_+^{free}\,\delta\hat{a}_{\vec{k}\mu}\rangle = -i g^*_{\vec{k}\mu}\,\zeta(\omega_{eg}-\omega_k)\langle\hat{\sigma}_+^{free}\hat{\sigma}_-^{free}\rangle = -i g^*_{\vec{k}\mu}\,\zeta(\omega_{eg}-\omega_k)\langle |e\rangle\langle e|\rangle$$
$$= -i g^*_{\vec{k}\mu}\,\zeta(\omega_{eg}-\omega_k)\left\langle \frac{1+\hat{\sigma}_z}{2}\right\rangle$$

$$\langle \delta\hat{a}_{\vec{k}\mu}\,\hat{\sigma}_+^{free}\rangle = -i g^*_{\vec{k}\mu}\,\zeta(\omega_{eg}-\omega_k)\langle |g\rangle\langle g|\rangle = -i g^*_{\vec{k}\mu}\,\zeta(\omega_{eg}-\omega_k)\left\langle \frac{1-\hat{\sigma}_z}{2}\right\rangle$$

$$\langle \hat{a}^{free}_{\vec{k}\mu}\,\delta\hat{\sigma}_+\rangle = -i\sum_{\vec{k}',\mu'} g^*_{\vec{k}'\mu'}\,\zeta(\omega_k-\omega_{eg})\langle 0|\hat{a}^{free}_{\vec{k}\mu}\,\hat{a}^{\dagger free}_{\vec{k}'\mu'}|0\rangle\langle\hat{\sigma}_z\rangle = -i g^*_{\vec{k}\mu}\,\zeta(\omega_k-\omega_{eg})\langle\hat{\sigma}_z\rangle$$

$$\Rightarrow \frac{d}{dt}\langle\hat{\sigma}_z\rangle = s\Big[\sum_{\vec{k}\mu}|g_{\vec{k}\mu}|^2\left[\zeta(\omega_{eg}-\omega_k) + c.c.\right](1+\langle\sigma_z\rangle)\Big] + (1-s)\Big[-2\sum_{\vec{k}\mu}|g_{\vec{k}\mu}|^2\left(\zeta(\omega_k-\omega_{eg}) + c.c.\right)\langle\hat{\sigma}_z\rangle$$
$$- \sum_{\vec{k}\mu}|g_{\vec{k}\mu}|^2\left(\zeta(\omega_{eg}-\omega_k) + c.c.\right)(1-\langle\hat{\sigma}_z\rangle)\Big]$$

$$\Rightarrow \frac{d}{dt}\langle\hat{\sigma}_z\rangle = s\underbrace{\left(-\Gamma\langle\hat{\sigma}_z\rangle - \Gamma\right)}_{\text{radiation reaction}} + (1-s)\Big[\underbrace{-2\Gamma\langle\hat{\sigma}_z\rangle}_{\text{Vacuum fluct.}} + \underbrace{(\Gamma\langle\hat{\sigma}_z\rangle - \Gamma)}_{\text{radiation reaction}}\Big]$$

- In normal order, entire decay rate attributed to radiation reaction
- In anti-normal order both contribute
- In symmetric order, the vacuum "stabilizes" the ground state