Physics 566: Quantum Optics I
Problem Set # 1

Due: Thursday August 28, 2025

Problem 1: Trace Operations (15 points)

Let $\hat{A}$ be a linear operator on a Hilbert space of dimension d. Let $\{|e_i\rangle \mid i=1,2,..d\}$ be a basis on the space. The trace of the opertor is defined $\mathrm{Tr}(\hat{A}) = \sum_{i=1}^{d} \langle e_i|\hat{A}|e_i\rangle$.

(a) By considering another basis set $\{|f_i\rangle \mid i=1,\ldots,d\}$, show that the trace is independent of basis

(b) Prove the following properties

(i) $\mathrm{Tr}(\hat{A}^\dagger) = \mathrm{Tr}(\hat{A})^*$

(ii) $\mathrm{Tr}(\hat{A}+\hat{B}) = \mathrm{Tr}(\hat{A}) + \mathrm{Tr}(\hat{B})$

(iii) $\mathrm{Tr}(\hat{A}\hat{B}\hat{C}) = \mathrm{Tr}(\hat{C}\hat{A}\hat{B}) = \mathrm{Tr}(\hat{B}\hat{C}\hat{A})$

(iv) $\mathrm{Tr}(|\phi\rangle\langle\psi|) = \langle\psi|\phi\rangle$

(v) $\mathrm{Tr}(|\phi\rangle\langle\psi|\hat{A}) = \langle\psi|\hat{A}|\phi\rangle$

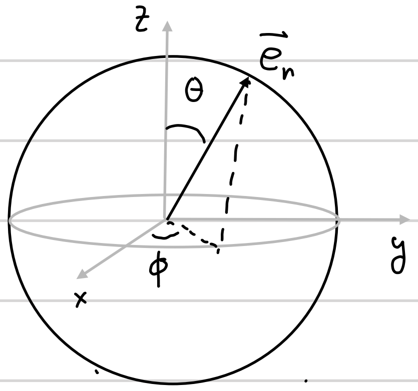
(vi) When $\hat{A}^\dagger = \hat{A}$, $\mathrm{Tr}(\hat{A}) = \sum_a a$, $\{a\}$ = eigenvalues of $\hat{A}$

Problem 2: Spin along arbitrary direction (20 points)

Consider a spin-1/2 particle (this will be one of the stars of our show this semester). The "standard basis" are spin-up and spin-down along the "quantion axis" typically chosen as the z-axis, $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$; They are defined by the eigenvalue equations

$$\hat{\sigma}_z|\uparrow_z\rangle = |\uparrow_z\rangle, \quad \hat{\sigma}_z|\downarrow_z\rangle = -|\downarrow_z\rangle;$$

where $\{\hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z\}$ are the Pauli operators. We seek to understand the properties of spin-up along an arbitrary direction in space.



Consider a vector on the unit sphere whose direction is defined by the polar angles (θ, ϕ)

$$\vec{e}_n = \sin\theta\,(\cos\phi\,\vec{e}_x + \sin\phi\,\vec{e}_y) + \cos\theta\,\vec{e}_z$$

Define $\hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}} = \sin\theta\cos\phi\,\hat{\sigma}_x + \sin\theta\sin\phi\,\hat{\sigma}_y + \cos\theta\,\hat{\sigma}_z$

Define spin-up/down along $\vec{e}_n$ $\{|\uparrow_n\rangle, |\downarrow_n\rangle\}$ by $\hat{\sigma}_n|\uparrow_n\rangle = |\uparrow_n\rangle$, $\hat{\sigma}_n|\downarrow_n\rangle = -|\downarrow_n\rangle$

Note: The antipode $-\vec{e}_n$, has polar angles $(\pi-\theta, \phi+\pi)$, and $|\uparrow_{-n}\rangle = |\downarrow_n\rangle$.

Geometrically we can obtain $\vec{e}_n$ through a series of rotations of the vector $\vec{e}_z$

$$\vec{e}_n = \mathbb{R}_z(\phi)\,\mathbb{R}_y(\theta) : \vec{e}_z$$

(Rotate z by θ about y, then rotate by ϕ about z)

Quantumly,

$$|\uparrow_n\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)\,|\uparrow_z\rangle,$$
$$|\downarrow_n\rangle = \hat{D}_z(\phi)\,\hat{D}_y(\theta)\,|\downarrow_z\rangle$$

Where $\hat{D}_m(\alpha) = e^{-i\frac{\alpha}{2}\hat{\sigma}_m}$ is an SU(2) rotation operator about axis m, by angle α.

(a) Show that up to an overall phase,

$$|\uparrow_n\rangle = \cos\tfrac{\theta}{2}\,|\uparrow_z\rangle + e^{i\phi}\sin\tfrac{\theta}{2}\,|\downarrow_z\rangle$$
$$|\downarrow_n\rangle = \sin\tfrac{\theta}{2}\,|\uparrow_z\rangle - e^{i\phi}\cos\tfrac{\theta}{2}\,|\downarrow_z\rangle$$

and show $|\downarrow_n\rangle = |\uparrow_{-n}\rangle$

(Hint: Recall $\hat{D}_m(\alpha) = \cos\frac{\alpha}{2}\,\mathbb{1} - i\sin\frac{\alpha}{2}\,\hat{\sigma}_m$)

(b) Use (a) to express the bases $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$ and $\{|\uparrow_y\rangle, |\downarrow_y\rangle\}$ in terms of $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ and show that these are the expected eigenvectors, e.g. $\hat{\sigma}_x|\uparrow_x\rangle = |\uparrow_x\rangle$, etc.

(c) Consider an arbitrary pure state of spin-½ particle expanded in the standard basis

$$|\psi\rangle = \alpha\,|\uparrow_z\rangle + \beta\,|\downarrow_z\rangle, \qquad |\alpha|^2 + |\beta|^2 = 1$$

Show that this state is spin-up along some direction $\vec{e}_n$, and find $\vec{e}_n$ in terms of (α, β).

Problem 3: Some algebra with density matrices (20 points)

Consider the state of a spin-½ particle (qubit). Supose a prepared sends you the particle and tells you with probability ⅓ she prepared $|\uparrow_z\rangle$ and with ⅔ probability she prepared $|\downarrow_z\rangle$.

(a) What density operator would you assign as the state of the system prepared. Find a matrix representati of $\hat{\rho}$ in the basis $\{|\uparrow_z\rangle, |\downarrow_z\rangle\}$ and also the basis $\{|\uparrow_x\rangle, |\downarrow_x\rangle\}$

(b) Find $\langle\hat{\sigma}_x\rangle$, $\langle\hat{\sigma}_y\rangle$, $\langle\hat{\sigma}_z\rangle$ for this state and compare this to the completely mixed state $\hat{\rho} = \frac{1}{2}\hat{\mathbb{1}}$ ← identity matrix. Please comment on your findings.

(c) Find $\langle\hat{\sigma}_n\rangle$, where $\hat{\sigma}_n = \vec{e}_n \cdot \hat{\vec{\sigma}}$ for the ⅓-⅔ mixture and also for the completely mixed state. Please comment on your findings.

Now suppose the prepared sends you $|\uparrow_z\rangle$ with probability-½ and $|\uparrow_x\rangle$ with probability-½.

(d) What is the purity of this state. Is it completely mixed? Please comment.

(e) What are the eigenvalues of this the density operator of this state? Express the density operator as a statistical mixture of its eigenvectors.