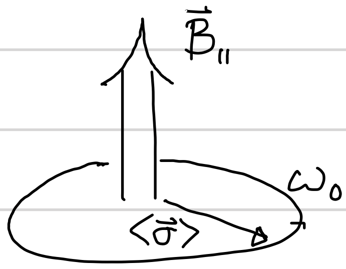
Physics 566: Quantum Optics
Problem Set #4 Solutions

Problem 1: Inhomogeneous Broadening

(a) Free induction decay by inhomogeneous broadening: Consider a macroscopic ensemble of spins in a static magnetic field in the z-direction with inhomogeneous magnitude described by a Gaussian distribution

$$P(B) = \frac{1}{\sqrt{2\pi\,\delta B^2}}\, e^{-(B-B_0)^2/2\delta B^2}$$

Suppose we apply a "hard" $\frac{\pi}{2}$-pulse that rotates the spin into the x-y plane. Each spin will precess about the z-axis at a frequency $\omega_0 = \gamma B$, depending on the local B-field. The rotating spin will radiate magnetic dipole radiation at frequency ω_0 in a long wave train.



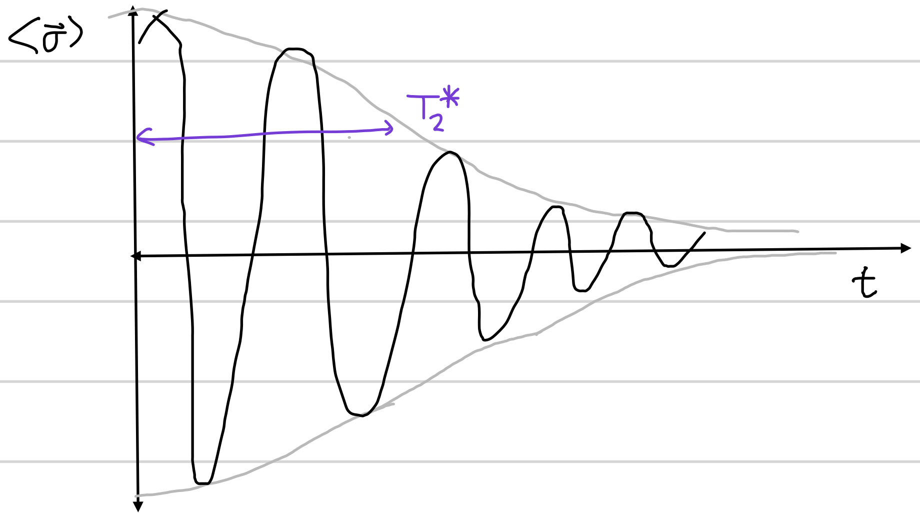
$$\langle\vec{\sigma}\rangle(t) = \langle\hat{\sigma}_x\rangle(0)\cos[\omega_0(B)t] + \langle\hat{\sigma}_y\rangle(0)\sin[\omega_0(B)t]$$
$$= \mathrm{Re}\left[\langle\hat{\sigma}_+(0)\rangle\, e^{-i\omega_0(B)t}\right]$$

Note: In order to achieve the "hard pulse", we must rotate all the spins by $\frac{\pi}{2}$. In the spin resonance $\Theta = \Omega_{tot}T = \frac{\pi}{2}$, where $\Omega_{tot} = \sqrt{\Omega^2 + \Delta^2}$. By choosing $\Omega >> |\Delta|$, for all Δ in the spread $\gamma\,\delta B$ then $\Omega_{tot} \approx \Omega$ and $\Theta \approx \frac{\pi}{2}$ for all spins in the ensemble. Thus, to achieve a hard pulse, choose a sufficient large Rabi frequency $\Omega << \frac{1}{T_2^*}$.

Because of the distribution of precession frequencies the different dipoles will radiate slightly different frequency waves. Eventually, these different waves will get out of phase with one another and so the total signal will show decay. This time scale is T_2^*. The total signal is then the average of $\langle\vec{\sigma}\rangle(t)$ over the distribution of B-fields. The spectrum of precession frequencies is equivalent to the distribution of B-fields.

$$\langle\vec{\sigma}\rangle_{tot}(t) = \mathrm{Re}\Big[\langle\hat{\sigma}_+(0)\rangle \underbrace{\int_{-\infty}^{\infty} dB\; P(B)\, e^{-i\omega_0(B)t}}_{\sim\text{ Fourier transform}}\Big] \propto \mathrm{Re}\left(\langle\hat{\sigma}_+(0)\rangle\, e^{-\frac{(\delta B t)^2}{2}}\, e^{-i\gamma B_0 t}\right)$$

$$= e^{-\frac{(\delta B t)^2}{2}}\left(\langle\hat{\sigma}_x\rangle(0)\cos(\gamma B_0 t) + \langle\hat{\sigma}_y(0)\rangle\sin(\gamma B_0 t)\right)$$



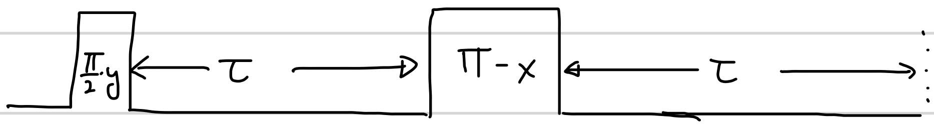
Wave train with a decaying Gaussian envelope due to inhomogeneity.

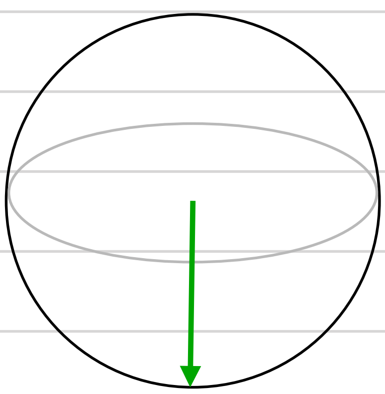
$$T_2^* = \frac{1}{\gamma(\delta B)} = \frac{1}{2\pi \times 10^{-2}} \text{ s}$$

$$T_2^* \approx 16 \, \mu s.$$

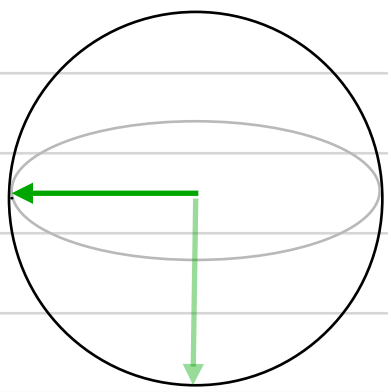
(b) Spin echo: Time reversing inhomogeneous (but coherent) evolution

Consider the following sequence: $\frac{\pi}{2}$-pulse about y – free evolution τ — π-pulse around x – free evolution τ

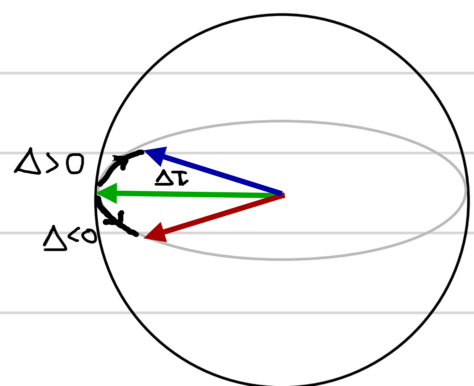




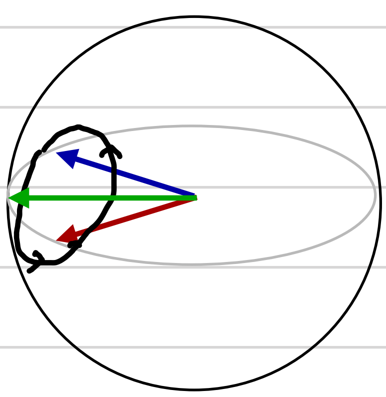
Initially all spins down



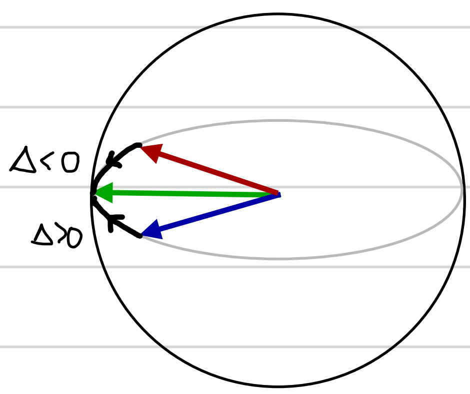
A hard $\frac{\pi}{2}x$ pulse rotates to y-axis



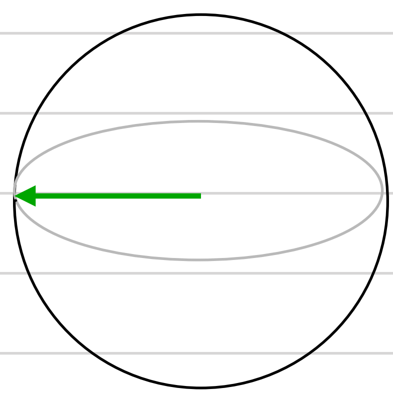
Spins freely precess spread out due to inhomogeneous detunings



A π pulse around y flips the red and blue detuned

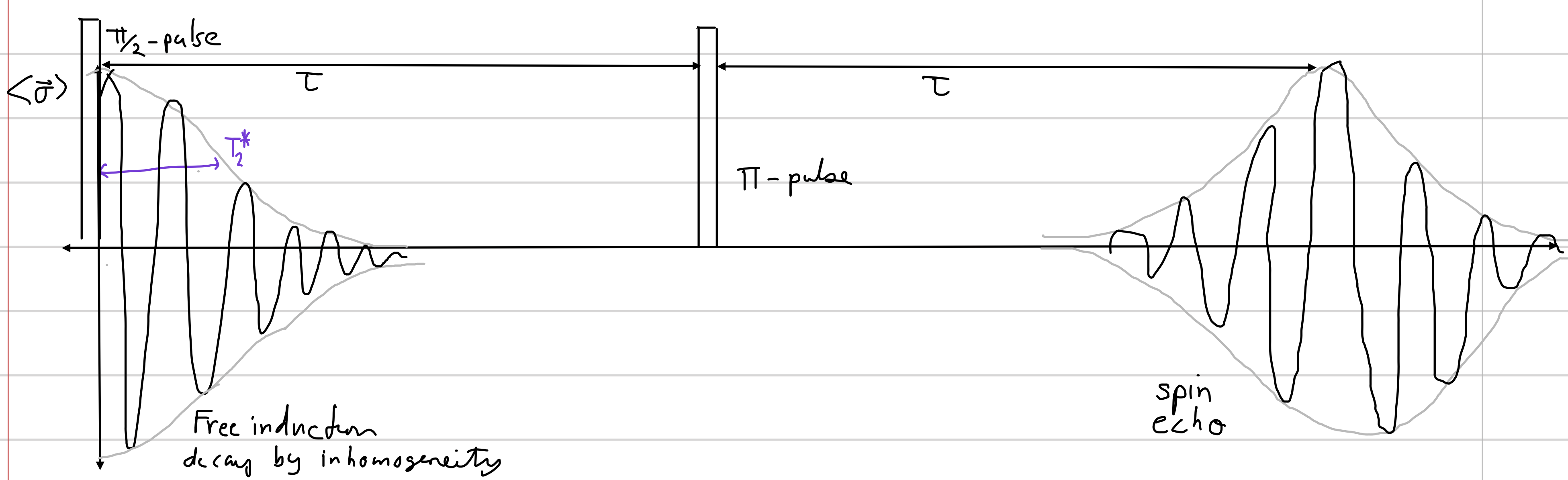


Everything is now time reversed and the spreading "refocuses"



After time τ we rephase → echo!

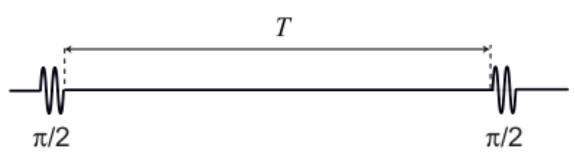
The key idea of the spin echo is that the π pulse effectively flips $z \Rightarrow -z$. The same spread in B_z leads to the same spread in precession frequencies, but in the reverse direction. We effectively time-reverse the spread of precession angles of the Bloch vectors. If a given Bloch vector precessed an angle $\Delta\tau$ then after the π pulse it precesses $-\Delta\tau$. After this time, all the Bloch vectors return to the y axis. Note, in this problem, we have chosen the $\frac{\pi}{2}$ pulse and the echo pulse around orthogonal axes. The same echo phenomenon would occur if we had chosen the sequence $\frac{\pi}{2}x - \tau - \pi_x - \tau$. In that case the Bloch vector would refocus on the $-y$-axis, but otherwise, everything is the same.



The Ramsey interferometer is intimately related to a "two-path" wave interferometer, such as a Mach-Zender interferometer. The spin-echo pulse is equivalent to flipping the two modes with mirrors. The detuning between the applied oscillating "transverse field" and the "longitudinal" resonance frequency plays the role of the frequency in the Mach-Zender interferometer. The MZ-interferometer is robust to the frequency of the light. That is, if the two arms are perfectly balanced, we achieve perfect interference, regardless of the frequency of light. Such an interferometer is known as a "white-light" interferometer, because it will shown coherent interference for a broad spectrum of frequencies.

Problem 2: Measuring T_2 times via a Ramsey interferometer

(a) The standard Ramsey "separated zone" pulse sequence provides a method for measuring the coherence time in a quantum superposition between two orthogonal states $\{|0\rangle, |1\rangle\}$.



Text

Given the qubit initially in the state $|1\rangle$. The probability to find the qubit in $|0\rangle$ after the pulse sequence was derived in lecture

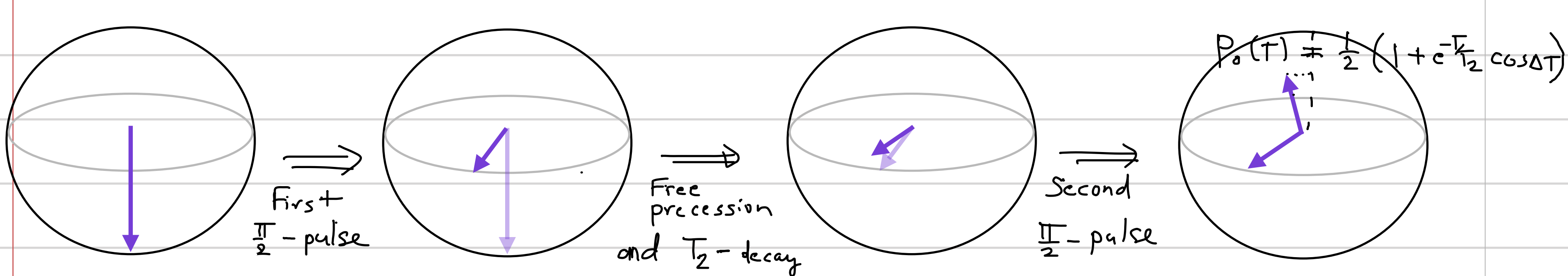
$$P_0(T) = \langle 0| e^{-i\frac{\pi}{4}\hat\sigma_x}\,\hat\rho(T)\,e^{i\frac{\pi}{4}\hat\sigma_x}|0\rangle = \left(\frac{\langle 0| - i\langle 1|}{\sqrt 2}\right)\hat\rho(T)\left(\frac{|0\rangle + i|1\rangle}{\sqrt 2}\right)$$

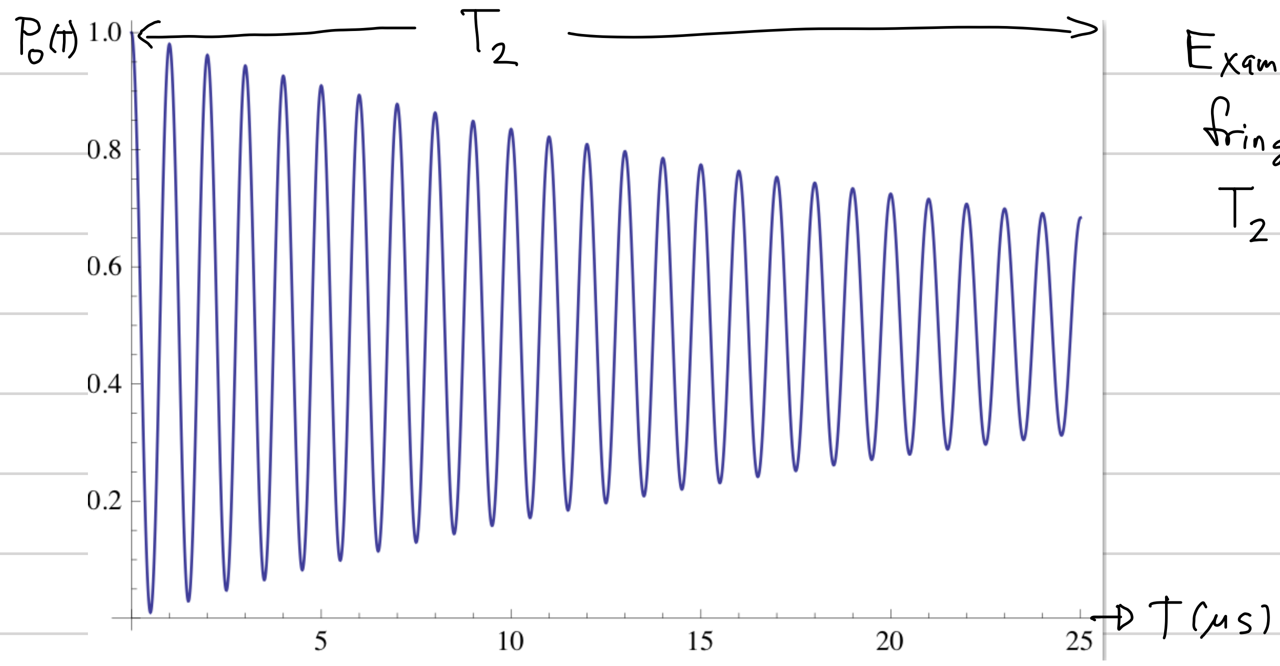
$$= \frac{1}{2}\left(\rho_{00}(T) + \rho_{11}(T) + i\left(\rho_{01}(T) - \rho_{10}(T)\right)\right) = \frac{1}{2}\left(1 + 2\,\mathrm{Im}\left(\rho_{10}(T)\right)\right)$$

Aside: $$\rho_{10}(T) = \underbrace{e^{-T/T_2}}_{T_2\text{-decay}}\underbrace{\langle 1|}_{\langle\Downarrow_z|} e^{+i\frac{\Delta T}{2}\hat\sigma_z}\underbrace{e^{-i\frac{\pi}{4}\hat\sigma_x}\hat\rho(0)\,e^{i\frac{\pi}{4}\hat\sigma_x}}_{|\Uparrow_y\rangle\langle\Uparrow_y|}\underbrace{e^{-i\frac{\Delta T}{2}\hat\sigma_z}|0\rangle}_{|\Uparrow_z\rangle} = e^{-T/T_2}\langle\Downarrow_z|e^{i\frac{\Delta T}{2}\hat\sigma_z}|\Uparrow_y\rangle\langle\Uparrow_y|e^{-i\frac{\Delta T}{2}\hat\sigma_z}|\Uparrow_z\rangle$$

$$\Rightarrow \rho_{10}(T) = \frac{i}{2}\,e^{-T/T_2}\,e^{-i\Delta T}$$

$$\Rightarrow \boxed{P_0(T) = \frac{1}{2}\left(1 + e^{-T/T_2}\cos(\Delta T)\right)}$$



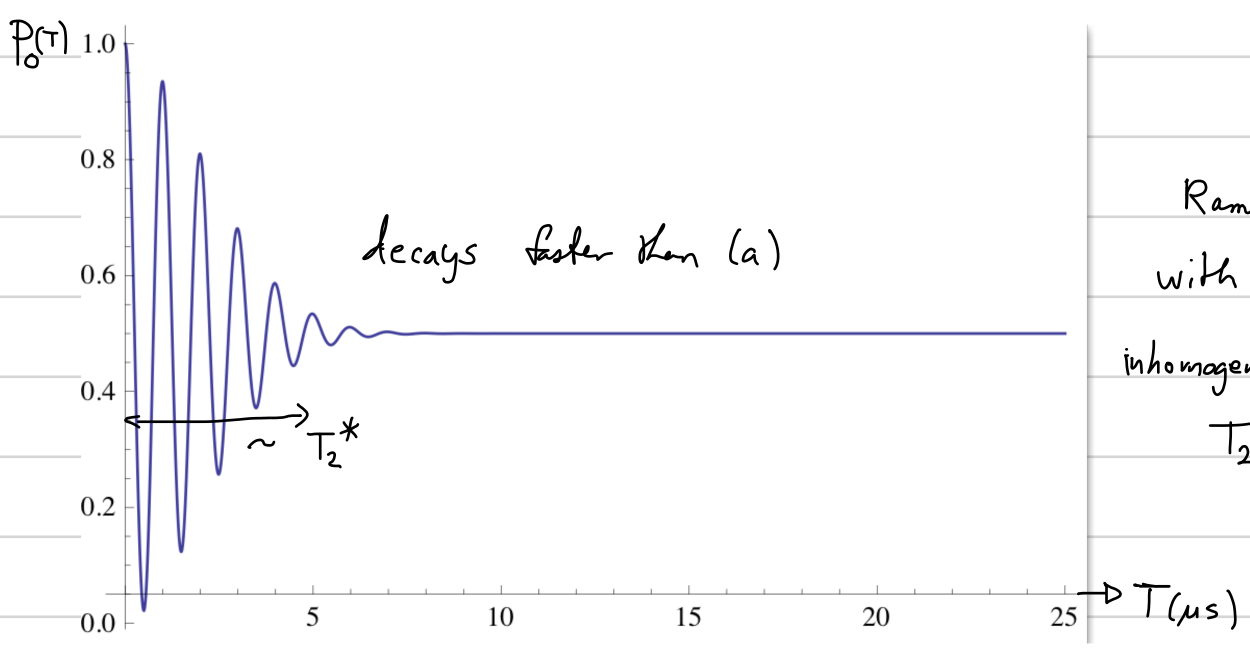


Example of decay Ramsey fringes for $\Delta/2\pi = 1\ \text{MHz}$, $T_2 = 25\,\mu s$.

(b) Now we add inhomogeneous broadening – a distribution of detunings $p(\Delta) = \dfrac{e^{-\frac{(\Delta-\Delta_0)^2}{2\delta^2}}}{\sqrt{2\pi\delta^2}}$

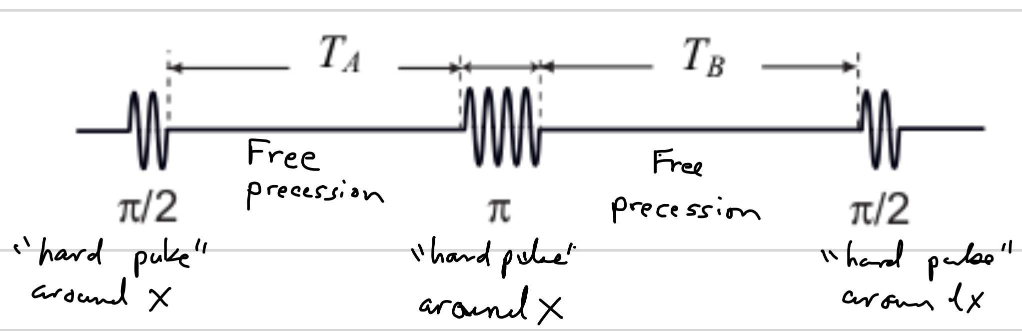
$$\Rightarrow P_0(T) = \int d\Delta\, p(\Delta)\, P_0(T,\Delta) = \int d\Delta\, p(\Delta) \left(\frac{1 + e^{-T/T_2} \cos(\Delta T)}{2} \right)$$

$$P_0(T) = \tfrac{1}{2}\left(1 + e^{-T/T_2}\, e^{-\frac{T^2}{2T_2^{*2}}} \cos(\Delta_0 T)\right) \qquad \text{where} \quad T_2^* = \frac{1}{\delta}$$



Ramsey Two-pulse sequence with same mean detuning but inhomogeneously broadened, with a $T_2^* = 5\,\mu s < T_2 = 25\,\mu s$

(c) The Hahn spin echo sequence removes the inhomogenous brodening in Δ to first order.



The spin-echo sequence is isomorphic to a Mach-Zehnder interferometer, as we have seen. For a given detuning, and without decay, the probability of transitioning $|1\rangle \to |0\rangle$

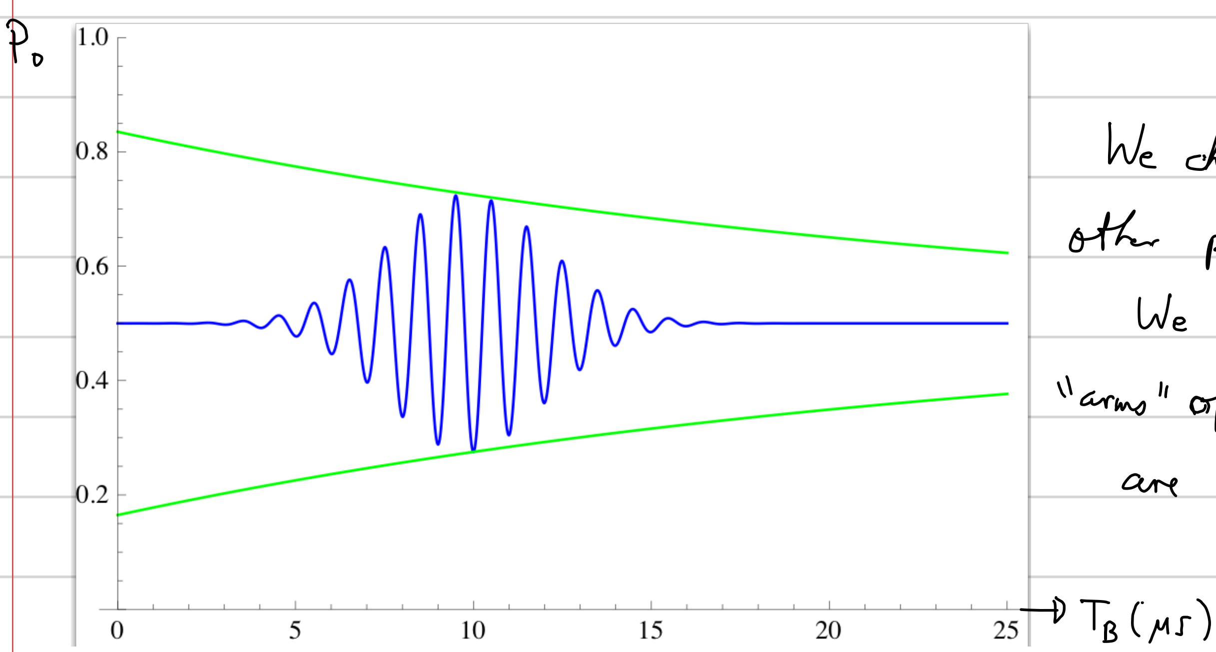
$$P_0(T_A, T_B, \Delta) = \left|\langle 0| e^{-\frac{i\Delta}{2}(T_A - T_B)\hat{\sigma}_y} |1\rangle\right|^2 = \sin^2\left(\frac{\Delta}{2}(T_A - T_B)\right) = \frac{1}{2}\left(1 - \cos(\Delta(T_A - T_B))\right)$$

Including homogeneous T_2 decay: $P_0(T, \Delta, T_2) = \frac{1}{2}\left[1 - \underbrace{e^{-\frac{T_A+T_B}{T_2}}}_{\text{decay during } T_A \,\&\, T} \cos\{\Delta(T_A - T_B)\}\right]$

Including both T_2 and inhomogeneous T_2^*:

$$P_0(T_A, T_B, \Delta_0, T_2, T_2^*) = \int d\Delta \; p(\Delta, T_2^*, \Delta_0)\, P_0(T_A, T_B, \Delta, T_2)$$

$$\Rightarrow \boxed{P_0(T_A, T_B, \Delta_0, T_2, T_2^*) = \frac{1}{2}\left[1 - e^{\frac{-(T_A+T_B)}{T_2}} e^{-\frac{1}{2}\left(\frac{T_A - T_B}{T_2^*}\right)^2} \cos\{\Delta_0 (T_A - T_B)\}\right]}$$



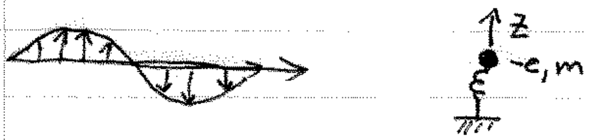
We chose here $T_A = 10\,\mu s$ – all other parameters are as in part (b). We see the "echo" we the two "arms" of the Ramsey interferometer are near equal $T_A \sim T_B$

The Hahn echo allows us to measure the true decay time T_2 (shown here as the green exponential decay curves), removing the dephasing due to inhomogeneity.

Problem 2: The ac-Stark effect

Interaction of an induced oscillating dipole with an oscillating field $\vec{E} = \vec{E}_0 \cos\omega_L t$ $\quad \vec{\mathcal{E}}_0 = \mathcal{E}_z \vec{e}_z$

(i) Lorentz oscillator model:



(a) The incident field will drive oscillations of the charge at frequency ω_L. The eq. of motion

$$\ddot{z} + \omega_0^2 z = -e/m\, \mathcal{E}_z \cos\omega_L t$$

Go to complex amplitude $\quad z = Re\left(Z_0 e^{-i\omega_L t}\right)$

$$\Rightarrow \left(-\omega_L^2 + \omega_0^2\right) Z_0 = -\frac{e}{m} \mathcal{E}_z$$

$$\Rightarrow \quad Z_0 = \left(\frac{-e/m}{\omega_0^2 - \omega_L^2}\right) \mathcal{E}_z$$

Induced dipole moment oscillating at drive-frequency ω_L

$$d_{induced}(t) = Re\left(-e Z_0\, e^{-i\omega_L t}\right)$$

$$= \frac{+e^2/m}{\omega_0^2 - \omega_L^2}\, \mathcal{E}_z \cos\omega_L t$$

$$\Rightarrow \quad \boxed{\vec{d}_{ind}(t) = \alpha\, \vec{\mathcal{E}}(t)} \qquad \boxed{\alpha = \frac{+e^2/m}{\omega_0^2 - \omega_L^2}}$$

(Next page)

In the near resonance approximation:

Let $\Delta \equiv \omega_L - \omega_0$ ("detuning") $\quad \Delta << \omega_0 \sim |\omega_L|$

$$\Rightarrow \omega_0^2 - \omega_L^2 = (\omega_0+\omega_L)(\omega_0-\omega_L) = (2\omega_0+\Delta)(-\Delta)$$

$$\approx -2\omega_0\Delta \quad \text{(to first order in } \Delta << \omega_0\text{)}$$

$$\therefore \quad \boxed{\alpha \approx \frac{-e^2}{2m\omega_0\Delta}}$$

(b) Total energy $\quad H = \frac{1}{2}m\dot{z}^2 + \frac{1}{2}m\omega_0^2 z^2 - \vec{d}\cdot\vec{E}$

$$\Rightarrow H = \frac{1}{2}m(\omega_0^2 - \omega_L^2)z^2 - \vec{d}\cdot\vec{E}$$

$$= \frac{1}{2}m(\omega_0^2-\omega_L^2)\frac{e^2/m^2}{(\omega_0^2-\omega_L^2)^2}E^2 - \alpha E^2$$

$$= \frac{1}{2}\alpha E^2 - \alpha E^2 = -\frac{1}{2}\alpha E^2$$

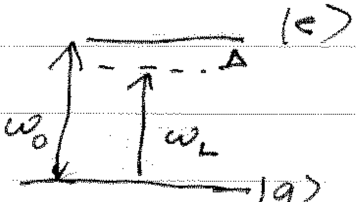
$$\Rightarrow \boxed{H = -\frac{1}{2}\alpha E^2(t) = -\frac{1}{2}\vec{d}_{ind}(t)\cdot\vec{E}(t)}$$

Time averaging $\quad \overline{\cos^2\omega_L t} = 1/2$

$$\Rightarrow \boxed{\overline{H} = -\frac{1}{4}\alpha E_z^2}$$

(ii) Quantum picture

(a) Given two level atom



$\Delta \ll \omega_0 \quad \Delta \ll |\omega_L|$

(ignore all other levels)

$\Rightarrow$ Effective Hamiltonian (we will derive later)

$$\hat{H} = \hat{H}_0 + \hat{H}_1$$

$$\hat{H}_0 = \hat{H}_{Atom} = -\hbar\Delta |e\rangle\langle e| \qquad (\text{"unperturbed atom"})$$

$$\hat{H}_1 = \hat{H}_{int} = -\frac{\hbar\Omega}{2}\left(|e\rangle\langle g| + |g\rangle\langle e|\right) \qquad (\text{"laser interaction"})$$

$$\Omega \equiv \frac{\langle e|\vec{d}|g\rangle \cdot \vec{E}}{\hbar} \qquad \text{"Rabi frequency"}$$

(a) This simple 2-dimensional problem can be solve exactly: Matrix representation in basis $\{|e\rangle, |g\rangle\}$

$$H \doteq -\hbar\begin{bmatrix} \Delta & \Omega/2 \\ \frac{\Omega}{2} & 0 \end{bmatrix} = -\hbar\left(\frac{\Delta}{2}\hat{\mathbb{1}} + \frac{\Delta}{2}\hat{\sigma}_z + \frac{\Omega}{2}\hat{\sigma}_x\right)$$

$$= -\frac{\hbar\Delta}{2}\hat{\mathbb{1}} - \frac{\hbar\vec{\tilde{\Omega}}}{2}\cdot\vec{\sigma}$$

where $\vec{\tilde{\Omega}} = \Delta\vec{e}_z + \Omega\vec{e}_x$ (Generalized Rabi frequency)

$$\tilde{\Omega} \equiv |\vec{\tilde{\Omega}}| = \sqrt{\Omega^2 + \Delta^2} \qquad \frac{\vec{\tilde{\Omega}}}{\tilde{\Omega}} \equiv \cos\theta\,\vec{e}_z + \sin\theta\,\vec{e}_x$$

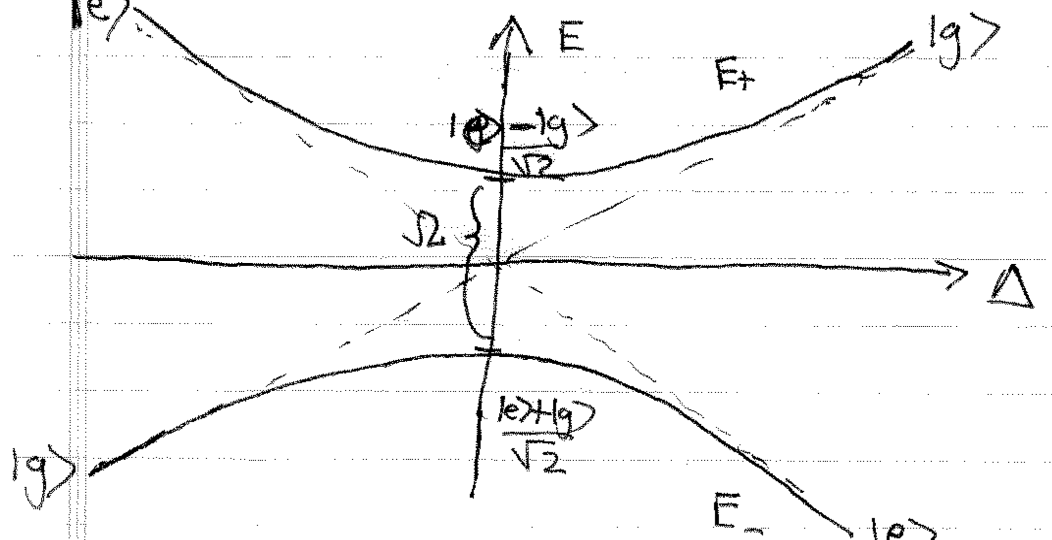
$$\tan\theta = \frac{\Omega}{\Delta}$$

$\Rightarrow$ Eigenvalues:

$$E_\pm = -\left(\frac{\hbar\Delta}{2} \pm \frac{\hbar\tilde{\Omega}}{2}\right) = -\frac{\hbar}{2}\left(\Delta \pm \sqrt{\Omega^2+\Delta^2}\right)$$

Eigenvectors: $|\pm\rangle = \cos\frac{\theta}{2}\,|e\rangle \pm \sin\frac{\theta}{2}\,|g\rangle$

$$\tan\theta = \frac{\Omega}{\Delta}$$



Typical anti-crossing behavior. At $\Delta=0$ $|e\rangle$ and $|g\rangle$ are "degenerate" in the absence of coupling. The field breaks the degeneracy into symmetric and anti-symmetric superpositions

Note for $\Delta<0$ ("red detuning") $|e\rangle$ is shifted up and $|g\rangle$ down (level repulsion)

For $\Delta>0$ ("blue detuning") the reverse occurs $\Rightarrow$ level attraction

__Note:__ In class we ~~saw~~ saw that in static perturbation the levels __always__ repell due to perturbation. This agrees with our result, since in the d.c. limit Δ is __negative__.

(b) Expansion for $\frac{\Omega}{\Delta} \ll 1$ $\left(\frac{\text{coupling matrix element}}{\text{energy level difference}}\right)$

$$E_{\pm} = -\frac{\hbar}{2}\left(\Delta \pm \Delta\sqrt{1+\frac{\Omega^2}{\Delta^2}}\right) \approx -\frac{\hbar}{2}\left(\Delta \pm \Delta\left(1+\frac{\Omega^2}{2\Delta^2}\right)\right)$$

$$\Rightarrow \boxed{E_+ \approx -\hbar\Delta \cdot \frac{\hbar\Omega^2}{4\Delta} \qquad E_- \approx -\frac{\hbar\Omega^2}{4\Delta}}$$

Lowest non-vanishing perturbation is __second order__ in Ω

(c) Using perturbation theory $\hat{H}_1 = -\frac{\hbar\Omega}{2}\left(|e\rangle\langle g| + h.c.\right)$

__0^{th} order__ $E_e^{(0)} = -\hbar\Delta$ $\qquad E_g^{(0)} = 0$

$|e\rangle \qquad |g\rangle$

__1^{st} order__ $E_e^{(1)} = \langle e|\hat{H}_1|e\rangle = 0 \qquad E_g^{(1)} = \langle g|\hat{H}_1|g\rangle = 0$

__2^{nd} order__ $$E_e^{(2)} = \frac{|\langle g|\hat{H}_1|e\rangle|^2}{E_e^{(0)} - E_g^{(0)}} = \frac{\frac{\hbar^2\Omega^2}{4}}{-\hbar\Delta} = -\frac{\hbar\Omega^2}{4\Delta}$$

$$E_g^{(2)} = \frac{|\langle e|\hat{H}_1|g\rangle|^2}{E^{(0)} - E^{(0)}} = \frac{\frac{\hbar\Omega^2}{4}}{\ } = \frac{\hbar\Omega^2}{4\Delta}$$

Thus to second order

$$E_e = E_e^{(0)} + E_e^{(2)} = -\hbar\Delta - \frac{\hbar\Omega^2}{4\Delta} \checkmark$$

$$E_g = E_g^{(0)} + E_g^{(2)} = +\frac{\hbar\Omega^2}{4\Delta} \checkmark$$

as in (6)

<u>Mean dipole:</u> (Assume atom starts in ground state)

To first order: $$|\tilde{\Phi}_g\rangle = \underbrace{|g\rangle} + \frac{|e\rangle\langle e|\hat{H}_1|g\rangle}{E_g^{(0)} - E_e^{(1)}}$$
$$= |\phi_g^{(0)}\rangle + |\phi_g^{(1)}\rangle$$

$$\Rightarrow |\tilde{\Phi}_g\rangle = |g\rangle - \frac{\Omega}{2\Delta}|e\rangle \quad \text{(unnormalized)}$$

$$\langle \vec{d}\rangle = \frac{\langle\tilde{\Phi}_g|\vec{d}|\tilde{\Phi}_g\rangle}{\langle\tilde{\Phi}_g|\tilde{\Phi}_g\rangle} = \frac{-\frac{1}{2\Delta}\left(\Omega^*\langle e|\vec{d}|g\rangle + \Omega\langle g|\vec{d}|e\rangle\right)}{1 + \underbrace{\frac{\Omega^2}{4\Delta^2}}_{\text{neglect}}}$$

$\Rightarrow$ To lowest order in $\Omega = \frac{\langle e|\vec{d}|g\rangle}{\hbar}\cdot\vec{\mathcal{E}}$

$$\boxed{\langle\vec{d}\rangle = -\frac{|\langle e|\vec{d}|g\rangle|^2}{\hbar\Delta}\,\vec{\mathcal{E}} = \alpha\,\vec{\mathcal{E}}}$$

Now

$$\boxed{E_g^{(2)} = \frac{\hbar\Omega^2}{4\Delta} = \frac{|\langle e|\vec{d}|g\rangle|^2}{4\hbar\Delta}|\vec{\mathcal{E}}|^2 = -\frac{1}{4}\alpha|\vec{\mathcal{E}}|^2}$$

as in the classical calculation (6)

Oscillator strength

$$f \equiv \frac{\alpha_q}{\alpha_c} = \left(\frac{|\langle e|\vec{d}|g\rangle|^2}{\hbar\Delta}\right)\left(\frac{2m\omega_0\Delta}{e^2}\right)$$

$$= |\langle e|z|g\rangle|^2\left(\frac{2m\omega_0}{\hbar}\right)$$

$$\boxed{f = \frac{|\langle e|z|g\rangle|^2}{(\Delta z)^2_{SHO}}} \quad \text{where} \quad (\Delta z)_{SHO} = \frac{\hbar}{2m\omega_0}$$

For a multi-level atom with resonances $\{\omega_i\}$

$$\alpha = \sum_i f(\omega_i)\,\alpha(\omega_i)$$

The oscillator strength satifies the "sum rule"

$$\sum_i f(\omega_i) = Z \text{ (atomic \#)}$$

For Hydrogen and the alkalies, the majority of the oscillator strength lies in the ~~[illegible]~~ first $s \to p$ transition. Thus, if the perturbation is far from any resonance, this transition will dominate.