Physics 566: Quantum Optics I
Problem Set 5

Due: September 30, 2025

Problem 1: Light forces on atoms (20 Points)

Electromagnetic fields can exert forces on atoms. This is force can be dissipative (the basis of laser cooling) or conservative (the basis for optical trapping, such as optical lattices). Suppose we are given a monochromatic, uniformly polarized laser field of the form $\mathbf{E}(\mathbf{x},t) = \vec{\varepsilon}_L E_0(\mathbf{x})\cos(\omega_L t + \phi(\mathbf{x}))$. The interaction of this field with a two-level atom is described by the Hamiltonian in the rotating frame,

$$\hat{H}_{AL}(\mathbf{R}) = \frac{\hbar\Omega(\mathbf{R})}{2}\left(e^{-i\phi(\mathbf{R})}\left|e\right\rangle\left\langle g\right| + e^{i\phi(\mathbf{R})}\left|g\right\rangle\left\langle e\right|\right),$$

where $\mathbf{R}$ is the center of mass position of the atom, and $\hbar\Omega(\mathbf{R}) = -\left\langle e\right|\hat{\mathbf{d}}\cdot\vec{\varepsilon}_L\left|g\right\rangle E_0(\mathbf{R})$.

Assuming the internal state of the atom relaxes to its steady state much faster than the atom moves, we can neglect the quantum mechanics of the atom's center of mass, and treat its motion as a classical point particle (this is know as the "semiclassical model"). The force on the atom is the defined by the expectation value

$$\mathbf{F} = -\left\langle \nabla\hat{H}_{AL}(\mathbf{R})\right\rangle = -Tr\left(\nabla\hat{H}_{AL}(\mathbf{R})\,\hat{\rho}(t)\right),$$

where $\hat{\rho}(t)$ is the “internal state” of the atoms according to the optical Bloch equations.

(a) Under these condition show that the mean force on the atom is, $\mathbf{F} = \mathbf{F}_{diss} + \mathbf{F}_{react}$, where

$\mathbf{F}_{diss} = -\frac{1}{2}\hbar\,\mathrm{v}(t)\;\Omega(\mathbf{R})\nabla\phi(\mathbf{R})$ is the "dissipative force" and

$\mathbf{F}_{react} = -\frac{1}{2}\hbar u(t)\nabla\Omega(\mathbf{R})$ is the "reactive force",

with u and v the components of the Bloch vector in the rotating frame relative to the incident phase $\rho_{ge}e^{i\phi(\mathbf{R})} = (u + i\mathrm{v})/2$.

(b) Show that in steady state, the rate at which that laser does work on the atom, averaged over an optical period is:

$$\left\langle \frac{dW}{dt} \right\rangle_{s.s} = \frac{\hbar\Omega\omega_L}{2}\,\mathrm{v}_{s.s} = \gamma_s \hbar\omega_L \text{, where } \gamma_s = \Gamma\rho_{ee}^{s.s.} \text{ is the photon scattering rate.}$$

Interpret this result.

(c) For the case of a plane wave $\mathbf{E}(\mathbf{R},t) = \vec{\varepsilon}_L E_0 \cos(\omega_L t - \mathbf{k}\cdot\mathbf{R})$, show that in steady-state:

$\mathbf{F}_{diss} = \gamma_s \hbar \mathbf{k}_L$. This is known as "radiation pressure" or the "scattering force" - interpret.

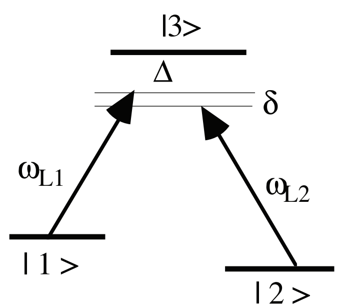
(d) For the "reactive force" consider the case of weak saturation, s<<1. Show that

$$\mathbf{F}_{react} = -\nabla U(\mathbf{R}),$$

where the optical "dipole force" is $U(\mathbf{R}) = \hbar\Delta(\mathbf{R})\frac{s}{2} = -\frac{1}{4}\mathrm{Re}(\tilde{\alpha})\left|E_0(\mathbf{R})\right|^2$ -- interpreted the physical meaning of $U(\mathbf{R})$.

Problem 2: Λ−Transitions and the master equation (25 Points)

Consider a three-level atom in the so-called "lambda" configuration (because it looks like the Greek letter Λ):



Levels $|1\rangle$ and $|2\rangle$ are connected to level $|3\rangle$ on two dipole-allowed transitions driven by lasers at frequencies ω_{L1} and ω_{L2} respectively. Laser-1 is detuned from resonance by Δ. Difference between the detunings of lasers 1 and 2 is $\delta = (\omega_{L1} - \omega_{L2}) - (E_2 - E_1)/\hbar$.

(a) The Hamiltonian for this system (in the RWA) is $H = H_A + H_{AL}$, where

$$H_A = E_1|1\rangle\langle 1| + E_2|2\rangle\langle 2| + E_3|3\rangle\langle 3|,$$

$$H_{AL} = \frac{\hbar\Omega_1}{2}\left(e^{-i\omega_{L1}t}|3\rangle\langle 1| + e^{i\omega_{L1}t}|1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(e^{-i\omega_{L2}t}|3\rangle\langle 2| + e^{i\omega_{L2}t}|2\rangle\langle 3|\right)$$

where $\Omega_{1,2}$ are the two Rabi frequencies. Because there are two laser frequencies, the usual unitary transformation to the frame rotating at ω_L does not apply. However, one can perform a unitary transformation that makes H time independent. Define a "rotating frame": $|\psi\rangle_{RF} = U^{\dagger}|\psi\rangle$, $H_{RF} = U^{\dagger}HU + i\hbar\frac{\partial U^{\dagger}}{\partial t}U$, where $U = \sum_{j=1}^{3} e^{-i\lambda_j t}|j\rangle\langle j|$.

Show that for appropriate choice of λ_j we can transform H to,

$$\hat{H}_{RF} = -\hbar\delta|2\rangle\langle 2| - \hbar\Delta|3\rangle\langle 3| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right).$$

(b) Suppose that level-3 decays to level-1 at a rate Γ_{31} and level-2 with rate Γ_{32}, and the total decay rate from level-3 is $\Gamma = \Gamma_{31} + \Gamma_{32}$. The effective non-Hermitian Hamiltonian is $\hat{H}_{eff} = \hat{H} - i\hbar\frac{\Gamma}{2}|3\rangle\langle 3|$. The (trace preserving) dynamics of the density operator for the system is described by the master equation,

$$\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{\dagger}\right) + \mathcal{L}_{feed}[\hat{\rho}], \quad \text{where}$$

$$\mathcal{L}_{feed}[\hat{\rho}] = \Gamma_{31}|1\rangle\langle 3|\hat{\rho}|3\rangle\langle 1| + \Gamma_{32}|2\rangle\langle 3|\hat{\rho}|3\rangle\langle 2|.$$

Show that the matrix elements evolve according to:

$$\dot{\rho}_{11} = \Gamma_{31}\rho_{33} - i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right),$$

$$\dot{\rho}_{22} = \Gamma_{32}\rho_{33} - i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right),$$

$$\dot{\rho}_{33} = -\Gamma\rho_{33} + i\frac{\Omega_1}{2}\left(\rho_{31} - \rho_{13}\right) + i\frac{\Omega_2}{2}\left(\rho_{32} - \rho_{23}\right),$$

$$\dot{\rho}_{23} = -i\left(\Delta - \delta - i\frac{\Gamma}{2}\right)\rho_{23} - i\frac{\Omega_2}{2}\left(\rho_{33} - \rho_{22}\right) + i\frac{\Omega_1}{2}\rho_{21},$$

$$\dot{\rho}_{13} = -i\left(\Delta - i\frac{\Gamma}{2}\right)\rho_{13} - i\frac{\Omega_1}{2}\left(\rho_{33} - \rho_{11}\right) + i\frac{\Omega_2}{2}\rho_{12},$$

$$\dot{\rho}_{12} = -i\delta\,\rho_{12} + i\frac{\Omega_2}{2}\rho_{13} - i\frac{\Omega_1}{2}\rho_{32}$$

These equations describe the full dynamics, including optical pumping (refeeding) and saturation. They're pretty complicated to solve. Often, we can obtain good approximation and physical insight by using solely non-Hermitian Schrödinger evolution, $\frac{\partial}{\partial t}|\psi\rangle = -\frac{i}{\hbar}\hat{H}_{eff}|\psi\rangle$, as we studied in class. When possible, one should do this.