Physics 566: Quantum Optics I
Problem Set #5: Solutions

Problem 1 Light Forces on Atoms

Given at monochromatic, polarized electric field at the position of an atom

$$\vec{E}(\vec{R},t) = \vec{\epsilon}_L \underbrace{E_0(\vec{R})}_{\text{position-dependent amplitude}} \cos(\omega_L t + \underbrace{\phi(\vec{R})}_{\text{position-dependent phase}})$$

The atom-light interaction Hamiltonian, in the rotating frame, RWA is

$$\hat{H}_{AL}(\vec{R}) = \frac{\hbar\Omega(\vec{R})}{2}\left[e^{-i\phi(\vec{R})}|e\rangle\langle g| + e^{i\phi(\vec{R})}|g\rangle\langle e|\right]$$

$$\text{where } \hbar\Omega(\vec{R}) = -\langle e|\vec{d}|g\rangle\cdot\vec{\epsilon}_L E_0(\vec{R})$$

The mean force on the atom $\vec{F} = -\langle \vec{\nabla}\hat{H}_{AL}(\vec{R})\rangle$

(a) $\vec{F} = -\mathrm{Tr}(\hat{\rho}\,\vec{\nabla}H_{AL}(\vec{R})) = -\frac{\hbar}{2}\left[\vec{\nabla}\Omega\left(e^{-i\phi}\rho_{ge} + e^{i\phi}\rho_{eg}\right) - i\Omega\vec{\nabla}\phi\left(e^{-i\phi}\rho_{eg} - e^{i\phi}\rho_{ge}\right)\right]$

$$\phi(\vec{R}) = 0 \text{ at the position of the atom}$$

$$\Rightarrow \vec{F} = -\hbar\left[\vec{\nabla}\Omega\,\mathrm{Re}(\rho_{eg}) + \Omega\vec{\nabla}\phi\,\mathrm{Im}(\rho_{eg})\right] \qquad \rho_{eg} = \frac{1}{2}(u+iv)$$

Bloch vector components

$$\Rightarrow \boxed{\vec{F} = \vec{F}_{diss} + \vec{F}_{react}, \text{ where } \vec{F}_{diss} = -\frac{\hbar}{2}v\,\vec{\nabla}\Omega, \quad \vec{F}_{react} = -\frac{\hbar u}{2}\Omega\vec{\nabla}\phi}$$

(b) Following our studies of the classical Lorentz oscillator model, the rate at which the field does work on the atom is

$$\frac{dW}{dt} = \overbrace{\dot{\vec{d}}\cdot\vec{E}_L(\vec{R},t)}^{\text{time average}}, \text{ where } \dot{\vec{d}} = \frac{d}{dt}\overbrace{\vec{d}_{eg}(u\cos\omega_L t - v\sin\omega_L t)}^{\text{mean atom dipole}} = -\omega_L \vec{d}_{eg}(u\sin\omega_L t + v\cos\omega_L t)$$

$$\Rightarrow \frac{dW}{dt} = \overline{-\omega_L d_{eg}E_0(u\sin\omega_L t + v\cos\omega_L t)\cos\omega_L t} = +\hbar\omega_L\Omega_0 v/2 \quad (\text{where } \hbar\Omega_0 = -d_{eg}E_0)$$

In steady state: $\frac{dW}{dt} = +\frac{\hbar\Omega_0}{2}\omega_L v_{s.s.} = \frac{\hbar\Omega_0}{2}\omega_L\left(\frac{\Gamma}{\Omega_0}\frac{s}{1+s}\right)$

Having used the steady state solution to the Bloch equations

$$\Rightarrow \boxed{\frac{dW}{dt} = \hbar\omega_L\,\Gamma\,\underbrace{\frac{s/2}{1+s}}_{\text{steady-state population in excited state}} = \hbar\omega_L\,\Gamma\rho_{ee}^{s.s.} = \hbar\omega_L\,\gamma_s}$$

$\gamma_s = \Gamma\rho_{ee}^{s.s.}$ = probability to be in the excited state x rate of spontaneous emission
= photon scattering rate.

Interpretation: The (time averaged) rate at which the field does work on the atom is equal to the rate at which the atom scatters photons times energy $\hbar\omega_L$/photon. In other words, the every photon of the laser field absorbed by the atom does work on the atom. Photons are reëmitted in random directions, and thus do no work on the atoms.

(c) For a plane wave, $\phi(\vec{R}) = -\vec{k}_L\cdot\vec{R}$, where $\vec{k}_L$ is the laser beam's wave vector

From part (b), $\vec{F}_{diss} = -\frac{1}{2}\hbar v\Omega(\vec{R})\vec{\nabla}(-\vec{k}_L\cdot\vec{R}) = \hbar\vec{k}_L\frac{\Omega}{2}v \Rightarrow \boxed{\vec{F}_{diss} = \gamma_s\hbar\vec{k}_L}$

The dissipative force is also known as the "scattering force." Each scattered photon imparts a moment $\hbar\vec{k}_L$ on the atom. The rate of photon impulses = γ_s = scattering rate.

(d) Consider the reactive force for weak saturation, $s \ll 1 \Rightarrow u \approx \frac{2\Delta}{\Omega}s = \frac{\Omega\Delta}{\Delta^2+\Gamma^2/4}$

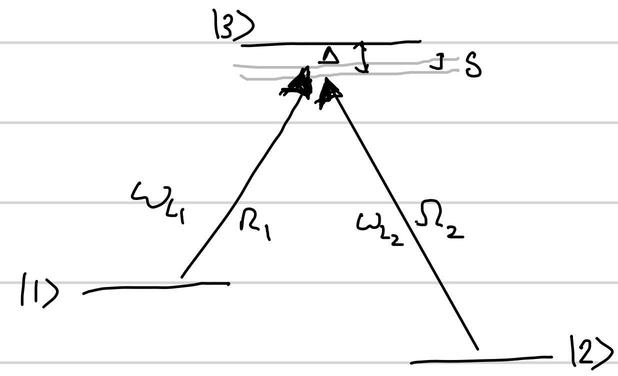
$\vec{F}_{react} = -\frac{1}{2}\hbar u\vec{\nabla}\Omega = -\frac{1}{2}\hbar\Delta\frac{\Omega\vec{\nabla}\Omega}{\Delta^2+\Gamma^2/4} = -\vec{\nabla}\left\{\hbar\Delta\frac{\Omega^2(\vec{R})/4}{\Delta^2+\Gamma^2/2}\right\} \Rightarrow \boxed{\vec{F}_{react} = -\vec{\nabla}U_{L.S.}(\vec{R})}$

The light-shift potential $\boxed{U_{L.S.} = \frac{\hbar\Delta}{2}s(\vec{R}) = \frac{\hbar\Delta}{4}\frac{d_{ge}^2E_0(\vec{R})^2/\hbar^2}{\Delta^2+\frac{\Gamma^2}{4}} = -\frac{1}{4}\mathrm{Re}(\tilde{\alpha}(\Delta))\,|E_0(\vec{R})|^2}$

Where $\tilde{\alpha}(\omega) = \frac{d_{eg}^2}{-\hbar(\Delta+i\frac{\Gamma}{2})}$.

U_{LS} is the conservative potential associated with a polarizable particle in an ac-electric field.

Problem 2: Λ-transitions and the Master Equation



$$\Delta = \Delta_1 = \omega_{L_1} - \frac{E_3 - E_1}{\hbar}$$
$$\Delta_2 = \omega_{L_2} - \frac{E_3 - E_2}{\hbar}$$
$$\delta = \Delta_1 - \Delta_2 = \omega_{L_1} - \omega_{L_2} - \frac{E_2 - E_1}{\hbar}$$

(a) The Hamiltonian $\hat{H} = \hat{H}_A - \hat{H}_{AL}$ in the Schrödinger Picture:

$$\hat{H}_A = E_1 |1\rangle\langle 1| + E_2 |2\rangle\langle 2| + E_3 |3\rangle\langle 3| , \qquad \hat{H}_{AL} = \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| e^{-i\omega_{L_1} t} + |1\rangle\langle 3| e^{+i\omega_{L_1} t}\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| e^{-i\omega_{L_2} t} + |2\rangle\langle 3| e^{+i\omega_{L_2} t}\right)$$

We go to a "rotating frame" by making a unitary transformation $\hat{U} = \sum_{j=1}^{3} e^{-i\lambda_j t} |j\rangle\langle j|$

In the rotating frame: $\hat{H}_{RF} = \hat{U}^\dagger \hat{H} \hat{U} + \frac{\hbar}{-i}\frac{\partial U^\dagger}{\partial t}\hat{U} = \hat{U}^\dagger \hat{H} \hat{U} - \sum_j \hbar\lambda_j |j\rangle\langle j|$

$$\Rightarrow \hat{H}_{RF} = (E_1 - \hbar\lambda_1)|1\rangle\langle 1| + (E_2 - \hbar\lambda_2)|2\rangle\langle 2| + (E_3 - \hbar\lambda_3)|3\rangle\langle 3|$$
$$+ \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| e^{-i[\omega_{L_1} - (\lambda_3 - \lambda_1)]t} + h.c.\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| e^{-i(\omega_{L_2} - (\lambda_3 - \lambda_2))t} + h.c.\right)$$

We choose the rotating frame to make $\hat{H}_{RF}$ time independent

$$\Rightarrow \lambda_3 - \lambda_1 = \omega_{L_1} , \quad \lambda_3 - \lambda_2 = \omega_{L_2}$$

This is two equations for three unknowns. We can arbitrarily set the zero of energy.

$\Rightarrow$ Chose $\lambda_1 = E_1/\hbar \Rightarrow \lambda_3 = \omega_{L_1} + E_1/\hbar$, $\lambda_2 = -\omega_{L_2} + \lambda_3 = \omega_{L_1} - \omega_{L_2} + E_1/\hbar$

$$\boxed{\Rightarrow \hat{H}_{RF} = -\hbar\delta |2\rangle\langle 2| - \hbar\Delta |3\rangle\langle 3| + \frac{\hbar\Omega_1}{2}\left(|3\rangle\langle 1| + |1\rangle\langle 3|\right) + \frac{\hbar\Omega_2}{2}\left(|3\rangle\langle 2| + |2\rangle\langle 3|\right)}$$

(b) Given: Level $|3\rangle$ decays to $|1\rangle$ and $|2\rangle$ with rates Γ_{31} and Γ_{32} respectively. The total decay rate is $\Gamma = \Gamma_{31} + \Gamma_{32}$, so the effective (non-Hermitian) Hamiltonian that includes decay of $|3\rangle$ according to an imaginary part of the excited state eigenvalue: $\hat{H}_{eff} = \hat{H} - i\frac{\hbar\Gamma}{2}|3\rangle\langle 3|$. The Master equation that describes the evolution of the density operator:

$$\frac{d}{dt}\hat{\rho} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{\dagger}\right) + \mathcal{L}_{feed}[\hat{\rho}]$$

$$\mathcal{L}_{feed}[\hat{\rho}] = \underbrace{\Gamma_{31}\rho_{33}|1\rangle\langle 1|}_{\text{Feeding level } |1\rangle} + \underbrace{\Gamma_{32}\rho_{33}|2\rangle\langle 2|}_{\text{Feeding level } |2\rangle}$$

We can find the equations of motion for the density matrix most easily using the non-Hermitian Schrödinger equation: $\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}\hat{H}_{eff}|\psi\rangle$, which including the decay, but not feeding.

$$|\psi\rangle = c_1|1\rangle + c_2|2\rangle + c_3|3\rangle$$

$$\Rightarrow \dot{c}_1 = \frac{-i}{\hbar}\langle 1|\hat{H}_{eff}|\psi\rangle = -i\frac{\Omega_1}{2}c_3; \quad \dot{c}_2 = -\frac{i}{\hbar}\langle 2|\hat{H}_{eff}|\psi\rangle = i\delta c_2 - \frac{i}{2}\Omega_2 c_3;$$

$$\dot{c}_3 = -\frac{i}{\hbar}\langle 3|\hat{H}_{eff}|\psi\rangle = \left(i\Delta - \frac{\Gamma}{2}\right)c_3 - \frac{i}{2}\Omega_1 c_1 - \frac{i}{2}\Omega_2 c_2.$$

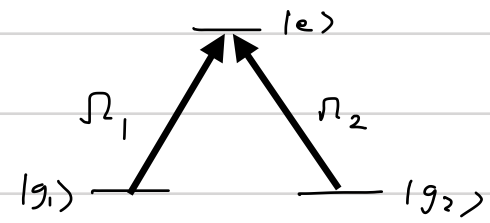
$$\rho_{\alpha\beta} = c_\alpha c_\beta^* \qquad \dot{\rho}_{\alpha\beta} = \dot{c}_\alpha c_\beta^* + c_\alpha \dot{c}_\beta^* + \langle\alpha|\mathcal{L}_{feed}[\hat{\rho}]|\beta\rangle$$

$$\boxed{\begin{aligned}
\frac{d}{dt}\rho_{11} &= \dot{c}_1 c_1^* + c_1\dot{c}_1^* + \frac{d}{dt}\rho_{11}\Big|_{feed} = -i\frac{\Omega_1}{2}(\rho_{31} - \rho_{13}) + \Gamma_{31}\rho_{33} \\
\frac{d}{dt}\rho_{22} &= \dot{c}_2 c_2^* + c_2\dot{c}_2^* + \frac{d}{dt}\rho_{22}\Big|_{feed} = -\frac{i}{2}\Omega_2(\rho_{32} - \rho_{23}) + \Gamma_{32}\rho_{33} \\
\frac{d}{dt}\rho_{33} &= \dot{c}_3 c_3^* + c_3\dot{c}_3^* = -\Gamma\rho_{33} - \frac{i}{2}\Omega_1(\rho_{13} - \rho_{31}) - \frac{i}{2}\Omega_2(\rho_{23} - \rho_{32}) \\
\frac{d}{dt}\rho_{23} &= \dot{c}_2 c_3^* + c_2\dot{c}_3^* = i\left(\delta - \Delta + i\frac{\Gamma}{2}\right)\rho_{23} - \frac{i}{2}\Omega_2(\rho_{33} - \rho_{22}) + \frac{i}{2}\Omega_1\rho_{21} \\
\frac{d}{dt}\rho_{13} &= \dot{c}_1 c_3^* + c_1\dot{c}_3^* = \left(-i\Delta - \frac{\Gamma}{2}\right)\rho_{13} - \frac{i}{2}\Omega_1(\rho_{33} - \rho_{11}) + \frac{i}{2}\Omega_2\rho_{12} \\
\frac{d}{dt}\rho_{12} &= \dot{c}_1 c_2^* + c_1\dot{c}_2^* = -i\delta\rho_{12} - i\frac{\Omega_1}{2}\rho_{32} + \frac{i}{2}\Omega_2\rho_{13}
\end{aligned}}$$

These equations determine the full dynamics of the three-level system, including decay and refeeding of population by optical pumping. However, when the saturation parameter is low, often one can obtain the important physics solely from the non-Hermitian Schrödinger equation.

Problem 2: Dark States

We consider the 3-level "lambda system" with two ground states resonantly coupled to an excited state.



In the rotating frame: $\hat{H}_{RF} = \frac{\hbar\Omega_1}{2}\left(|g_1\rangle\langle e| + |e\rangle\langle g_1|\right) + \frac{\hbar\Omega_2}{2}\left(|g_2\rangle\langle e| + |e\rangle\langle g_2|\right)$

(a) The "dressed states" are the eigenstates of the coupled atom-laser system

The Hamiltonian matrix: $\hat{H}_{RF} = \frac{\hbar}{2}\begin{bmatrix} 0 & 0 & \Omega_1 \\ 0 & 0 & \Omega_2 \\ \Omega_1 & \Omega_2 & 0 \end{bmatrix}$ In the ordered basis $\{|g_1\rangle, |g_2\rangle, |e\rangle\}$

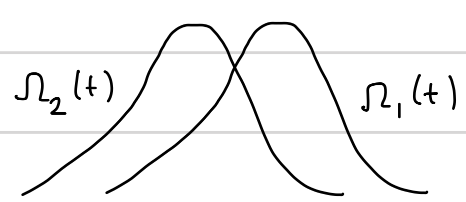
The eigenvectors / eigenvalues of this matrix are:

$$|\psi_{Dark}\rangle = \frac{-\Omega_2|1\rangle + \Omega_1|2\rangle}{\sqrt{\Omega_1^2 + \Omega_2^2}}, \quad E_{Dark} = 0 \qquad \text{(Dark state)}$$

$$|\psi_{Coup,\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\psi_{Bright}\rangle \pm |e\rangle\right), \quad E_{coup,\pm} = \pm\frac{\hbar}{2}\sqrt{\Omega_1^2+\Omega_2^2} \qquad \text{(Coupled states)}$$

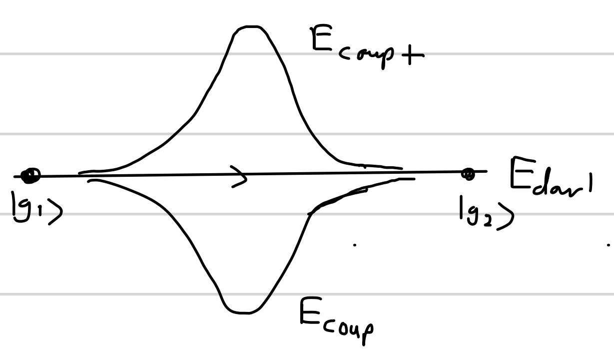
$$\left(\Omega_1|1\rangle + \Omega_2|2\rangle\right)/\sqrt{\Omega_1^2+\Omega_2^2}$$

(b) One can transfer population from $|g_1\rangle$ to $|g_2\rangle$ using the "counter-intuitive pulse sequence".



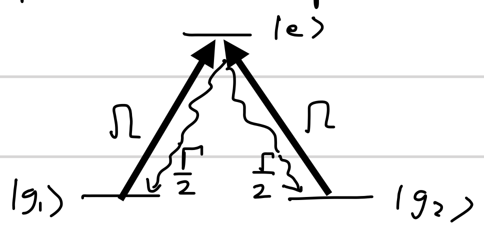
First apply field that couples $|2\rangle \to |3\rangle$.
As we turn that field off turn on $|1\rangle \to |3\rangle$

The counter-intuitive pulse sequence achieves the transfer from $|g_1\rangle \to |g_2\rangle$ by adiabatic following of the dark state. The dressed eigenvalues as a function of time appear as:



At $t=0$ $|\psi\rangle = |g_1\rangle$. As we slowly turn on Ω_2, $|\psi\rangle$ is in the dark state (the eigenstate of $\hat{H}$ with $\Omega_1 = 0$)). If we turn on and off the fields slowly, so the transition is adiabatic, then $|\psi(t)\rangle$ follows $|\psi_{Dark}(t)\rangle = \frac{\Omega_2(t)|2\rangle - \Omega_1(t)|1\rangle}{\sqrt{\Omega_1^2(t)+\Omega_2^2(t)}}$. Intermediately, when $\Omega_1=\Omega_2$ $|\psi_{Dark}\rangle = \frac{|1\rangle - |2\rangle}{\sqrt{2}}$. At the end of pulses $|\psi(t)\rangle = |\psi_{Dark}(t\to\infty)\rangle = |2\rangle$.

(c) We now consider spontaneous decay. We simplify to the case $\Omega_1=\Omega_2=\Omega$; $\Gamma_{31}=\Gamma_{32}=\frac{\Gamma}{2}$



In the basis $\{|D\rangle, |B\rangle, |e\rangle\}$ where $|D\rangle = \frac{|1\rangle-|2\rangle}{\sqrt{2}}$, $|B\rangle = \frac{|1\rangle+|2\rangle}{\sqrt{2}}$, $|e\rangle$

$|1\rangle = \frac{1}{\sqrt{2}}(|D\rangle + |B\rangle)$; $|2\rangle = \frac{1}{\sqrt{2}}(|B\rangle - |D\rangle)$

$$\Rightarrow \hat{H} = \frac{\sqrt{2}\hbar\Omega}{2}\left(|B\rangle\langle e| + |e\rangle\langle B|\right)$$

(effective 2-level coupling between $|B\rangle$ & $|e\rangle$ w/ Rabi frequency $\sqrt{2}\Omega$. The Dark State is decoupled)

the effective Hamiltonian $\hat{H}_{eff} = \hat{H} - i\frac{\hbar}{2}\Gamma\, |e\rangle\langle e|$

The "Feeding term" in the Master Equation: $\mathcal{L}_{feed}[\hat{\rho}] = \frac{\Gamma}{2}\rho_{ee}\left(|1\rangle\langle 1| + |2\rangle\langle 2|\right) = \frac{\Gamma}{2}\rho_{ee}\left(|B\rangle\langle B| + |D\rangle\langle D|\right)$

(d) The Master equation: $\frac{d\hat{\rho}}{dt} = \frac{-i}{\hbar}\left(\hat{H}_{eff}\hat{\rho} - \hat{\rho}\hat{H}_{eff}^{+}\right) + \mathcal{L}_{feed}[\hat{\rho}]$

Non-Hermitian Schrödinger Eqn: $\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}\hat{H}_{eff}|\psi\rangle$

$$\Rightarrow \dot{c}_e = -\frac{\Gamma}{2}c_e - i\frac{\sqrt{2}\Omega}{2}c_B, \quad \dot{c}_B = -i\frac{\sqrt{2}\Omega}{2}c_e, \quad \dot{c}_D = 0$$

$$\begin{aligned}
\Rightarrow \dot{\rho}_{ee} &= \dot{c}_e c_e^* + c_e \dot{c}_e^* = -\Gamma\rho_{ee} - i\sqrt{2}\frac{\Omega}{2}(\rho_{Be} - \rho_{eB}) \\
\dot{\rho}_{BB} &= \dot{c}_B c_B^* + \dot{c}_B^* c_B + \langle B|\mathcal{L}_{feed}[\hat{\rho}]|B\rangle = +\frac{\Gamma}{2}\rho_{ee} - i\frac{\sqrt{2}\Omega}{2}(\rho_{eB} - \rho_{Be}) \\
\dot{\rho}_{DD} &= \dot{c}_D c_D^* + c_D \dot{c}_D^* + \langle D|\mathcal{L}_{feed}[\hat{\rho}]|D\rangle = \frac{\Gamma}{2}\rho_{ee} \\
\dot{\rho}_{eB} &= \dot{c}_e c_B^* + c_e \dot{c}_B^* = -\frac{\Gamma}{2}\rho_{eB} + i\frac{\sqrt{2}\Omega}{2}(\rho_{ee} - \rho_{BB}) \\
\dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e \dot{c}_D^* = -\frac{\Gamma}{2}\rho_{eD} - i\frac{\sqrt{2}\Omega}{2}\rho_{BD} \\
\dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B \dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2}\rho_{eD}
\end{aligned}$$

The steady-state solution: $\dot{\rho}_{BB} = \frac{\Gamma}{2} \rho_{ee} = 0 \Rightarrow \rho_{ee} = 0$

$$\Rightarrow \dot{\rho}_{ee} = i\frac{\sqrt{2}\Omega}{2}(\rho_{eB} - \rho_{Be}) = 0 \Rightarrow \mathrm{Im}(\rho_{eB}) = 0$$

$\Rightarrow \dot{\rho}_{eB} = -\frac{\Gamma}{2}\rho_{eB} - i\frac{\sqrt{2}\Omega}{2}\rho_{BB} = 0 \Rightarrow \mathrm{Re}(\rho_{eB}) = 0$ and $\rho_{BB} = 0$ (since Real & Imag $= 0$)

$$\left.\begin{aligned} \dot{\rho}_{eD} &= \dot{c}_e c_D^* + c_e \dot{c}_D^* = -\frac{\Gamma}{2}\rho_{eD} + i\frac{\sqrt{2}\Omega}{2}\rho_{BD} = 0 \\ \dot{\rho}_{BD} &= \dot{c}_B c_D^* + c_B \dot{c}_D^* = i\frac{\sqrt{2}\Omega}{2}\rho_{eD} = 0 \end{aligned}\right\} \Rightarrow \rho_{eD} = 0,\ \rho_{BD} = 0$$

$\Rightarrow$ Dynamically ALL elements of $\hat{\rho} \to 0$ in steady state except ρ_{DD}

$\Rightarrow$ Since the equation is trace preserving, in steady state, $\hat{\rho}_{s.s.} = |D\rangle\langle D|$